Well-posedness of a system of SDEs driven by jump random measures

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Abstract

We study the well-posedness of a system of multi-dimensional SDEs which are correlated through non-homogeneous mean-field drifts and also by driving Brownian motions and jump random measures. Supposing the drift coefficients are non-Lipschitz, we prove for the system the existence of strong, L^1-integrable, càdlàg solution which can be obtained as monotone limit of solutions to some approximating system of SDEs, extending existing results for one-dimensional jump SDE with non-Lipschitz coefficient. We show in addition the positivity and the pathwise uniqueness of the solution.

1 Introduction

We consider a multi-dimensional generalisation of the following one-dimensional stochastic differential equation (SDE)

$$d\lambda_t = a\,(b-\lambda_t)\,dt + \sigma\sqrt{\lambda_t}dB_t + \sigma_Z \lambda_{t-}^{1/\alpha} dZ_t, \quad t \geq 0 \tag{1.1}$$

where $a, b, \sigma, \sigma_Z \geq 0$, $B = (B_t, t \geq 0)$ is a Browinan motion and $Z = (Z_t, t \geq 0)$ is an independent spectrally positive α-stable compensated Lévy process with parameter $\alpha \in (1, 2]$. The existence of unique strong solutions to (1.1) is obtained by Fu and Li [9], see also Li and Mytnik [16]. Dawson and Li [5] consider and prove in the framework of CBI processes (continuous state branching processes with immigration) a more general integral representation

$$\lambda_t = \lambda_0 + a\int_0^t (b-\lambda_s)\,ds + \sigma\int_0^t\int_0^{\lambda_s} W(ds,du) + \sigma_Z\int_0^t\int_0^{\lambda_{s-}}\int_{\mathbb{R}^+} \zeta \widetilde{N}(ds,dv,d\zeta), \tag{1.2}$$

where $W(ds, du)$ is a white noise on $\mathbb{R}_+^2$ with intensity $dsdu$, $\widetilde{N}(ds, dv, d\zeta)$ is an independent compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $dsdv\mu(d\zeta)$ with $\mu(d\zeta)$ being a Lévy measure on $\mathbb{R}_+$ and satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < \infty$.

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The process given by (1.1) generalises the well-known Cox–Ingersoll–Ross (CIR) process and its applications in mathematical finance are studied by Jiao et al. [14, 15]. The link between general CBI processes and the affine modeling framework is established by Filipović [6]. In a recent paper, Frikha and Li [8] study the well-posedness and numerical approximation of a time-inhomogeneous jump SDE with generally non-Lipschitz coefficients which, as a one-dimensional generalisation of (1.1), has a drift term involving the law of the solution and can be viewed as a mean-field limit of an individual particle evolving within a system. The consequences of assuming generally non-Lipschitz coefficients in these many settings make the well-posedness become challenging since the classic iteration method fails to apply (see [9, 5, 8]).

In this paper, we focus on a system of finite number of jump SDEs where the drift term of each equation is given by a mean-field function depending on other components of the system and characterizing their interactions. Each equation contains a jump part driven by general random measures which allows to include a large class of jump processes such as Poisson, compound Poisson processes or Lévy processes. We impose mild conditions on the dependence among components. In particular there is no need for the driving processes, that is, Brownian motions and jump random measures, of the associated SDEs to be independent. So the system can admit a flexible structure of dependence which could be useful for potential modelling of correlated inhomogeneous system such as credit portfolio with CIR-like stochastic volatility, see Hambly and Kolliopoulos [12, 13], or systemic risks with mean-field drift functions, see e.g. Bo and Capponi [2], Fouque and Ichiba [7] and Giesecke et al. [11].

To prove the strong well-posedness of the multi-dimensional system, we construct a sequence of approximating solutions whose drifts are defined by a piecewise projection of the minimal drift processes of all the components. We show that the approximating systems are monotone by using a comparison theorem from Gal'chuk [10], see also Abdelghani and Melnikov [1], who considered SDEs with respect to continuous martingales and jump random measures where the coefficients of the semimartingale are not Lipschitz. We then use the monotone convergence to establish that the family of limit processes solves our system of SDEs. The key element is a technical lemma on one-dimensional SDEs with a general drift coefficient. This result is essential to deal with the approximating solutions since their drifts are defined by conditional expectations so that standard assumptions in literature fail to hold. We show in addition that the solutions are positive, which is similar to CIR-like processes such as the one-dimensional SDE (1.1). Finally, we prove the pathwise uniqueness for the solution of the system by following similar ideas as in [9].

The rest of the paper is organized as follows. In Section 2, we present the system of SDEs and the assumptions on the coefficients. The main existence and pathwise uniqueness results and their proofs are given in Section 3. The proof of the technical lemma are left to Appendix in Section 4.

2 System of SDEs and assumptions

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions. Let U_0 and U_1 be two locally compact and separable metric spaces. For $N \in \mathbb{N}$, we study the system of the following SDEs of the following form

$$\begin{aligned}\lambda_t^i = \lambda_0^i \quad &+ \quad a_i \int_0^t \left(b_i\left(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N\right) - \lambda_s^i\right) ds + \int_0^t \sigma_i(\lambda_s^i) dW_s^i \\ &+ \quad \int_0^t \int_{U_0} g_{i,0}\left(\lambda_{s-}^i,\, u\right) \tilde{N}_{i,0}\left(ds,\, du\right) + \int_0^t \int_{U_1} g_{i,1}\left(\lambda_{s-}^i,\, u\right) N_{i,1}\left(ds,\, du\right)\end{aligned} \tag{2.1}$$

for all $i \in \{1, 2, ..., N\}$, where $\lambda_0^i \geq 0$, $a_i \geq 0$, $W^i = (W_t^i)_{t \geq 0}$ is an $\mathbb{F}$-adapted Brownian motion, $N_{i,0}\,(ds,\, du)$ and $N_{i,1}\,(ds,\, du)$ are Poisson random measures associated to two $\mathbb{F}$-adapted point processes $p_{i,0} : \Omega \times \mathbb{R}_+ \longrightarrow U_0$ and $p_{i,1} : \Omega \times \mathbb{R}_+ \longrightarrow U_1$ with compensator measures $\mu_{i,0}(du)dt$ and $\mu_{i,1}(du)dt$ respectively. Let $\tilde{N}_{i,0}\,(ds,\, du) = N_{i,0}\,(ds,\, du) - \mu_{i,0}(du)dt$ be the compensated measure of $p_{i,0}\,(\cdot)$. For every $i \in \{0, 1, ..., N\}$, we suppose that W^i, $p_{i,0}$ and $p_{i,1}$ are mutually independent but we do not require the triplet to be independent for different $i, j \in \{0, 1, ..., N\}$.

Example 2.1. A typical example of the drift function is $b_i(t, x_1, \cdots, x_N) = \frac{1}{N}\sum_{k=1}^N x_k$, which is the same for every $i \in \{1, \cdots, N\}$. Let $Z^0 = (Z_t^0)_{t \geq 0}$ be an α_0-stable compensated Lévy process and $Z^i = (Z_t^i)_{t \geq 0}$ be an independent α_i-stable compensated Lévy process with $\alpha_0, \alpha_i \in (1, 2]$. Let $U_0 = \mathbb{R}^2$, $\overline{Z}^i = (Z^0, Z^i)$ with compensated measure $\widetilde{N}_{i,0}(dt, du)$ with $u = (u_0, u_i) \in U_0$, and the diffusion coefficient functions be given as

$$\sigma_i(x) = \sigma_i \cdot x^{1/2} \quad \text{and} \quad g_{i,0}(x, u) = \sigma_{Z,0} u_0 \cdot x^{1/\alpha_0} + \sigma_{Z,i} u_i \cdot x^{1/\alpha_i}$$

where $\sigma_i \geq 0$ and $\sigma_{Z,0}, \sigma_{Z,i} \geq 0$. In this example, the process Z^0 represents a common external factor which affects significantly the whole market such as financial crisis or pandemics (like the recent Covid pandemic crisis), or can also include more common events which lead to more frequent and smaller jumps. The processes Z^i are associated to idiosyncratic factors which lead to individual shocks.

For the coefficients appearing in the diffusion terms, we assume the following conditions are satisfied for all the components of the system (2.1). The regularity conditions of Assumption 2.2 are motivated by the one-dimensional case in [9].

Assumption 2.2. We assume the conjunction of the following conditions for the parameters $(\sigma, g_0, g_1, N_0, N_1)$:

(1) $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\sigma(x) = 0$ for $x \leq 0$. Moreover, there exists a non-negative and increasing function $\rho(\cdot)$ on $\mathbb{R}_+$ such that

$$\int_0^x \frac{dz}{\rho^2(z)} = +\infty \tag{2.2}$$

for any $x > 0$ and that $|\sigma(x) - \sigma(y)| \leq \rho(|x - y|)$ for all $x, y \geq 0$.

(2) N_0 is the Poisson random measure of an $\mathbb{F}$-adapted point process with compensator measure μ_0 and $g_0 : \mathbb{R} \times U_0 \to \mathbb{R}$ is a Borel function, such that

   (i) for each fixed $u \in U_0$, the function $g_0(\cdot, u) : x \mapsto g_0(x, u)$ is increasing, and satisfies the inequality $g_0(x, u) + x \geq 0$ when $x \geq 0$ and the equality $g_0(x, u) = 0$ when $x \leq 0$,

(ii) for each fixed $x \in \mathbb{R}$, the function $u \mapsto g_0(x,u)$ is locally integrable with respect to the measure μ_0,

(iii) the function $x \longmapsto \int_{U_0} |g_0(x,u)| \wedge |g_0(x,u)|^2 \, \mu_0(du)$ is locally bounded,

(iv) for any $m \in \mathbb{N}$, there exists a non-negative and increasing function $x \longrightarrow \rho_m(x)$ on $\mathbb{R}_+$ such that

$$\int_0^x \frac{dz}{\rho_m^2(z)} = +\infty \tag{2.3}$$

for any $x > 0$ and

$$\int_{U_0} |g_0(x,u) \wedge m - g_0(y,u) \wedge m|^2 \, \mu_0(du) \leq \rho_m^2(|x-y|) \tag{2.4}$$

for all $0 \leq x,\, y \leq m$.

(3) N_1 is the Poisson random measure of an $\mathbb{F}$-adapted point process with compensator measure μ_1, and $g_1 : \mathbb{R} \times U_1 \to \mathbb{R}$ is a Borel function, such that

(i) for any $(x,u) \in \mathbb{R} \times U_1$, $g_1(x,u) + x \geq 0$,

(ii) the function

$$x \longrightarrow \int_{U_1} |g_1(x,u)| \, \mu_1(du) \tag{2.5}$$

is locally bounded and has at most a linear growth when $x \to +\infty$,

(iii) there exists a Borel set $U_2 \subset U_1$ with $\mu_1(U_1 \backslash U_2) < +\infty$, and for any $m \in \mathbb{N}$, a concave and increasing function $x \longrightarrow r_m(x)$ on $\mathbb{R}_+$ such that

$$\int_0^x \frac{dz}{r_m(z)} = +\infty \tag{2.6}$$

for all $x > 0$ and

$$\int_{U_2} |g_1(x,u) \wedge m - g_1(y,u) \wedge m| \, \mu_1(du) \leq r_m(|x-y|) \tag{2.7}$$

for all $0 \leq x,\, y \leq m$.

In particular, the inequality (2.2) can be compared to Hölder condition in [8]. It is satisfied if $\sigma(x)$ are α-Hölder continuous in x for some $\alpha \in [1/2, 1]$.

In order to construct appropriate monotone approximations in the multi-dimensional case, extra monotonicity and continuity conditions are required in Assumption 2.3.

Assumption 2.3. The function σ is either bounded or increasing on $\mathbb{R}_+$, the functions g_0 and g_1 are left continuous in $x \in \mathbb{R}$, and the function g_1 is either increasing in $x \in \mathbb{R}$ or bounded by some function $(x,\, u) \longrightarrow G(u)$ with

$$\int_{U_1} |G(u)| \, \mu_1(du) \vee \int_{U_1} G^2(u) \, \mu_1(du) < +\infty. \tag{2.8}$$

Note that Assumption 2.3 allows to admit some discontinuity for g_0 and g_1. For example, following (1.2), g_0 can take the form $u = (v, \zeta) \in \mathbb{R}_+^2$ and $g_0(x, v, \zeta) = 1_{\{v<x\}} \zeta$.

3 Existence and pathwise uniqueness of the solution

The main result of this paper is given in the following Theorem.

Theorem 3.1. *Consider the system of SDEs* (2.1) *and suppose for all* $i \in \{1, \cdots, N\}$ *that*

(1) *the parameter* a_i *is non-negative and the mean-field function* $b_i : \mathbb{R}_+ \times \mathbb{R}^N \longrightarrow \mathbb{R}$ *is non-negative, increasing and Lipschitz continuous in each of its last* N *variables,*

(2) *the coefficients* $(\sigma_i, g_{i,0}, g_{i,1}, N_{i,0}, N_{i,1})$ *satisfy Assumption 2.2 and 2.3.*

Then (2.1) *has a càdlàg* $\mathbb{F}$*-adapted solution* $(\lambda^1_t, \cdots, \lambda^N_t)_{t\geq 0}$*, with* $\lambda^i_\cdot$ *non-negative and* $\mathbb{E}[\int_0^T \lambda^i_t\, dt] < +\infty$ *for any* $T \geq 0$.

Before proving the main result, we need the following technical key lemma on the auxiliary one-dimensional SDE with a more general drift coefficient.

Lemma 3.2. *Let* $T > 0$. *Consider the SDE*

$$\begin{aligned} Y_t = Y_0 + a \int_0^t (b_s - Y_s)\, ds + \int_0^t \sigma(Y_s) dW_s \\ + \int_0^t \int_{U_1} g_1(Y_{s-}, u)\, N_1(ds, du) + \int_0^t \int_{U_0} g_0(Y_{s-}, u)\, \tilde{N}_0(ds, du),\ t \in [0, T] \end{aligned} \tag{3.1$_b$}$$

where $a > 0$ *and* $b = (b_t)_{t\in[0,T]}$ *is a non-negative* $\mathbb{F}$*-adapted càdlàg process. If Assumption 2.2 and 2.3 hold for the coefficients* $(\sigma, g_0, g_1, N_0, N_1)$ *in* $(3.1)_b$*, then, the above SDE has a non-negative* $\mathbb{F}$*-adapted càlàg solution* $Y = (Y_t)_{t\in[0,T]}$.

Note that in the above equation $(3.1)_b$, the symbol "b" is attached as a subscript to its label in order to emphasize the dependence of the equation on the drift coefficient process b. The process b could be replaced by some auxiliary processes in the following and the subscript will be changed accordingly.

We now provide the proof of Theorem 3.1 on the existence of a strong solution to the system of SDEs (2.1). The idea is to construct a sequence of approximating solutions whose drift contains a piecewise conditional expectation with respect to the minimal of all pre-determined drift processes. The previous lemma allows to prove the existence of solutions for the approximating system. We then use monotone convergence to establish that the limit processes solves our system of SDEs.

Proof. Without loss of generality, we show that the equation admits a solution $(\lambda^1_t, \cdots, \lambda^N_t)$ for $t \in [0, T]$, with λ^i_t non-negative and $\mathbb{E}[\int_0^T \lambda^i_t\, dt] < +\infty$ for a given $T > 0$. Then the solution can be extended to $\mathbb{R}_+$ without difficulty.

Step 1. *Construction of the approximating systems and monotonicity.* For $n \in \mathbb{N}$, we construct a partition $0 = t^n_0 < t^n_1 < \ldots < t^n_{2^{n-1}} = T$ of $[0, T]$ as follows: We start with $t^1_0 = 0$ and $t^1_1 = T$ and, for any integer n, define inductively $t^{n+1}_{2j} = t^n_j$ for all $j \in \{0, \ldots, 2^{n-1}\}$ and $t^{n+1}_{2j+1} = (t^n_j + t^n_{j+1})/2$ for all $j \in \{0, \ldots, 2^{n-1} - 1\}$. Next, for each $i \in \{1, 2, ..., N\}$, let $\lambda^{i,1}_\cdot$ to be the solution to the SDE

$$\lambda^{i,1}_t \quad = \lambda^i_0 - a_i \int_0^t \lambda^{i,1}_s ds + \int_0^t \sigma_i(\lambda^{i,1}_s) dW^i_s$$

$$+\int_0^t \int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,1},\, u\right) N_{i,1}\left(ds,\, du\right) + \int_0^t \int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,1},\, u\right) \tilde{N}_{i,0}\left(ds,\, du\right) \quad (3.2)$$

which exists and is unique as shown in [9]. Then, having $\lambda_{\cdot}^{i,n}$ defined for some $n \geq 1$ and all $i \in \{1, 2, ..., N\}$, we define:

$$b_k^{i,n} = \inf_{s \in \left[t_k^n,\, t_{k+1}^n\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \quad (3.3)$$

and $\lambda_{\cdot}^{i,n+1}$ in $\left[t_k^n,\, t_{k+1}^n\right]$ for any $k \in \{0, 1, ..., 2^{n-1} - 1\}$ by solving the SDE

$$\lambda_t^{i,n+1} = \lambda_{t_k^n}^{i,n+1} + a_i \int_{t_k^n}^t \left(\mathbb{E}\left[b_k^{i,n} | \mathcal{F}_s\right] - \lambda_s^{i,n+1}\right) ds + \int_{t_k^n}^t \sigma_i(\lambda_s^{i,n+1}) dW_s^i$$
$$+ \int_{t_k^n}^t \int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\, u\right) N_{i,1}\left(ds,\, du\right) + \int_{t_k^n}^t \int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+1},\, u\right) \tilde{N}_{i,0}\left(ds,\, du\right) \quad (3.4)$$

for $t \in \left[t_k^n,\, t_{k+1}^n\right]$, which also has a solution by Lemma 3.2.

We will show now that for any $n \geq 1$ we have $\lambda_t^{i,n+1} \geq \lambda_t^{i,n}$ for all $i \in \{1, 2, ..., N\}$ and all $t \in [0,\, T]$ by induction on n. For the initial case, that is, $\lambda_t^{i,2} \geq \lambda_t^{i,1}$, we only need to recall that $\mathbb{E}[b_k^{i,n} | \mathcal{F}_s] \geq 0$ since each b_i in (3.3) is a non-negative function, and then use the comparison theorem from [10]. Suppose now that for some $n \geq 1$ we have $\lambda_t^{i,n+1} \geq \lambda_t^{i,n}$ for all $i \in \{1, 2, ..., N\}$ and $t \in [0,\, T]$. Then, by the monotonicity of each b_i we have

$$\begin{aligned}
b_{2k}^{i,n+1} &= \inf_{s \in \left[t_{2k}^{n+1},\, t_{2k+1}^{n+1}\right]} b_i(s,\, \lambda_s^{1,n+1},\, \lambda_s^{2,n+1},\, ...,\, \lambda_s^{N,n+1}) \\
&\geq \inf_{s \in \left[t_{2k}^{n+1},\, t_{2k+1}^{n+1}\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \\
&\geq \inf_{s \in \left[t_{2k}^{n+1},\, t_{2(k+1)}^{n+1}\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \\
&= \inf_{s \in \left[t_k^n,\, t_{k+1}^n\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) = b_k^{i,n} \qquad (3.5)
\end{aligned}$$

for $n \geq 1$, all $i \in \{1, 2, ..., N\}$ and all $k \in \{0, 1, ..., 2^{n-1} - 1\}$, and also

$$\begin{aligned}
b_{2k+1}^{i,n+1} &= \inf_{s \in \left[t_{2k+1}^{n+1},\, t_{2k+2}^{n+1}\right]} b_i(s,\, \lambda_s^{1,n+1},\, \lambda_s^{2,n+1},\, ...,\, \lambda_s^{N,n+1}) \\
&\geq \inf_{s \in \left[t_{2k+1}^{n+1},\, t_{2k+2}^{n+1}\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \\
&\geq \inf_{s \in \left[t_{2k}^{n+1},\, t_{2k+2}^{n+1}\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \\
&= \inf_{s \in \left[t_k^n,\, t_{k+1}^n\right]} b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) = b_k^{i,n} \qquad (3.6)
\end{aligned}$$

for $n \geq 1$, all $i \in \{1, 2, ..., N\}$ and all $k \in \{0, 1, ..., 2^{n-1} - 1\}$. We will use these two inequalities to show that $\lambda_t^{i,n+2} \geq \lambda_t^{i,n+1}$ for all $i \in \{1, 2, ..., N\}$ and $t \in [0,\, T]$. This is done by applying a second induction as follows: For $t \in [t_0^{n+1},\, t_1^{n+1}] = [0,\, t_1^{n+1}] \subset [0,\, t_1^n]$ we have

$$\lambda_t^{i,n+2} = \lambda_0^{i,n+2} + a_i \int_0^t \left(\mathbb{E}\left[b_0^{i,n+1} | \mathcal{F}_s\right] - \lambda_s^{i,n+2}\right) ds + \int_0^t \sigma_i(\lambda_s^{i,n+2}) dW_s^i$$

$$+\int_0^t\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+2},\,u\right)N_{i,1}\left(ds,\,du\right)+\int_0^t\int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+2},\,u\right)\tilde{N}_{i,0}\left(ds,\,du\right)\qquad(3.7)$$

and

$$\begin{aligned}\lambda_t^{i,n+1} &= \lambda_0^{i,n+1}+a_i\int_0^t\left(\mathbb{E}\left[b_0^{i,n}|\mathcal{F}_s\right]-\lambda_s^{i,n+1}\right)ds+\int_0^t\sigma_i(\lambda_s^{i,n+1})dW_s^i\\&\quad+\int_0^t\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\,u\right)N_{i,1}\left(ds,\,du\right)+\int_0^t\int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+1},\,u\right)\tilde{N}_{i,0}\left(ds,\,du\right)\end{aligned}\qquad(3.8)$$

and since $\mathbb{E}[b_0^{i,n+1}|\mathcal{F}_s]\geq\mathbb{E}[b_0^{i,n}|\mathcal{F}_s]$ (by taking conditional expectations in (3.5) for $k=0$ and n replaced by $n+1$) the comparison theorem implies that $\lambda_t^{i,n+2}\geq\lambda_t^{i,n+1}$ for all $t\in[t_0^{n+1},\,t_1^{n+1}]=[0,\,t_1^{n+1}]$. Suppose now that for some $k'\in\{0,1,\ldots,2^n-1\}$ we have $\lambda_t^{i,n+2}=\lambda_t^{i,n+1}$ for all $t\in[t_0^{n+1},\,t_{k'}^{n+1}]=\left[0,\,t_{k'}^{n+1}\right]$. Then for $k'=2k$ with $k\in\{0,1,...,2^{n-1}-1\}$ we have $t_{k'}^{n+1}=t_k^n$ and $t_{k'+1}^{n+1}=(t_k^n+t_{k+1}^n)/2$, while for $k'=2k+1$ with $k\in\{0,1,...,2^{n-1}-1\}$ we have $t_{k'}^{n+1}=(t_k^n+t_{k+1}^n)/2$ and $t_{k'+1}^{n+1}=t_{k+1}^n$, so in both cases it holds that $[t_{k'}^{n+1},t_{k'+1}^{n+1}]\subset[t_k^n,t_{k+1}^n]$ and for any $t\in[t_{k'}^{n+1},t_{k'+1}^{n+1}]$ we have both

$$\begin{aligned}\lambda_t^{n+2} &= \lambda_{t_{k'}^{n+1}}^{i,n+2}+a_i\int_{t_{k'}^{n+1}}^t\left(\mathbb{E}\left[b_{k'}^{i,n+1}|\mathcal{F}_s\right]-\lambda_s^{i,n+2}\right)ds+\int_{t_{k'}^{n+1}}^t\sigma_i(\lambda_s^{i,n+2})dW_s^i\\&\quad+\int_{t_{k'}^{n+1}}^t\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+2},\,u\right)N_{i,1}\left(ds,\,du\right)\\&\quad+\int_{t_{k'}^{n+1}}^t\int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+2},\,u\right)\tilde{N}_{i,0}\left(ds,\,du\right)\end{aligned}\qquad(3.9)$$

and

$$\begin{aligned}\lambda_t^{n+1} &= \lambda_{t_{k'}^{n+1}}^{i,n+1}+a_i\int_{t_{k'}^{n+1}}^t\left(\mathbb{E}\left[b_k^{i,n}|\mathcal{F}_s\right]-\lambda_s^{i,n+1}\right)ds+\int_{t_{k'}^{n+1}}^t\sigma_i(\lambda_s^{i,n+1})dW_s^i\\&\quad+\int_{t_{k'}^{n+1}}^t\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\,u\right)N_{i,1}\left(ds,\,du\right)\\&\quad+\int_{t_{k'}^{n+1}}^t\int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+1},\,u\right)\tilde{N}_{i,0}\left(ds,\,du\right)\end{aligned}\qquad(3.10)$$

with $\mathbb{E}\left[b_{k'}^{i,n+1}|\mathcal{F}_s\right]\geq\mathbb{E}\left[b_k^{i,n}|\mathcal{F}_s\right]$ (by taking expectations given $\mathcal{F}_s$ in (3.5) and (3.6)). Thus, the comparison theorem implies that $\lambda_t^{i,n+2}\geq\lambda_t^{i,n+1}$ for all $t\in\left[t_{k'}^{n+1},\,t_{k'+1}^{n+1}\right]$, which means that the same inequality holds for all $t\in\left[t_0^{n+1},\,t_{k'+1}^{n+1}\right]\equiv\left[0,\,t_{k'+1}^{n+1}\right]$. This completes the second induction and gives $\lambda_t^{i,n+2}\geq\lambda_t^{i,n+1}$ for all $t\in[0,\,T]$, and the last completes the initial induction giving $\lambda_t^{i,n+1}\geq\lambda_t^{i,n}$ for all $t\in[0,\,T]$ and all $n\geq1$.

Step 2. *Finiteness of the monotone limits.* We have shown in the previous step that the family of processes $\{\lambda_t^{i,n}\}_{t\in[0,T]}$ is pointwise increasing in n, we will show that almost surely, $\lim_{n\longrightarrow+\infty}\lambda_t^{i,n}$ is finite for almost all $t\in[0,\,T]$ and every $i\in\{1,\,2,\,\ldots,\,N\}$. This will follow by Fatou's lemma if we can show that

$$\sup_{1\le i\le N} \mathbb{E}\left[\int_0^T \lambda_t^{i,n} dt\right] \tag{3.11}$$

is bounded in $n \in \mathbb{N}$. For the last, we recall that by the Lipschitz property of each b_i, there exist constants B, $L > 0$ such that

$$b_i\left(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}\right) \le B + L\sum_{i=1}^N \lambda_s^{i,n} \tag{3.12}$$

for every $i \in \{1,\, 2,\, \dots,\, N\}$, so taking infimum on the LHS for $s \in \left[t_k^n,\, t_{k+1}^n\right]$ and then conditioning on $\mathcal{F}_s$ we obtain

$$\mathbb{E}\left[b_k^{i,n} \,|\, \mathcal{F}_s\right] \le B + L\sum_{i=1}^N \lambda_s^{i,n} \tag{3.13}$$

for all $s \in \left[t_k^n,\, t_{k+1}^n\right]$ and $i \in \{1,\, 2,\, ...,\, N\}$. Plugging the above in (3.4), localizing if needed, taking expectations and then supremum in i and finally using (2.5), we can easily get

$$\begin{aligned}\sup_{1\le i\le N} \mathbb{E}\left[\lambda_t^{i,n+1}\right] \le \sup_{1\le i\le N} \mathbb{E}\left[\lambda_{t_k^n}^{i,n+1}\right] \quad &+ \quad \bar{a}B(t-t_k^n) + \bar{a}LN\int_{t_k^n}^t \sup_{1\le i\le N} \mathbb{E}\left[\lambda_s^{i,n}\right] ds \\ &+ \quad K\int_{t_k^n}^t \left(\sup_{1\le i\le N} \mathbb{E}\left[\lambda_s^{i,n}\right] + 1\right) ds \end{aligned} \tag{3.14}$$

for $\bar{a} := \sup_{1\le i\le N} a_i$, which can be written as

$$\sup_{1\le i\le N} \mathbb{E}\left[\lambda_t^{i,n+1}\right] \le \sup_{1\le i\le N} \mathbb{E}\left[\lambda_{t_k^n}^{i,n+1}\right] \quad + \quad B'(t-t_k^n) + L'\int_{t_k^n}^t \sup_{1\le i\le N} \mathbb{E}\left[\lambda_s^{i,n}\right] ds \tag{3.15}$$

for $B' = \bar{a}B + K$ and $L' = \bar{a}LN + K$, so replacing k with $k' < k$ and taking $t = t_{k'+1}^n$ we get also

$$\begin{aligned}\sup_{1\le i\le N} \mathbb{E}\left[\lambda_{t_{k'+1}^n}^{i,n+1}\right] \quad &\le \quad \sup_{1\le i\le N} \mathbb{E}\left[\lambda_{t_{k'}^n}^{i,n+1}\right] \\ &\quad + B'(t_{k'+1}^n - t_{k'}^n) + L'\int_{t_{k'}^n}^{t_{k'+1}^n} \sup_{1\le i\le N} \mathbb{E}\left[\lambda_s^{i,n}\right] ds. \end{aligned} \tag{3.16}$$

Summing (3.15) with (3.16) for $k' \in \{0,\, 1,\, ...,\, k-1\}$ we obtain

$$\sup_{1\le i\le N} \mathbb{E}\left[\lambda_t^{i,n+1}\right] \le \sup_{1\le i\le N} \mathbb{E}\left[\lambda_0^i\right] + B't + L'\int_0^t \sup_{1\le i\le N} \mathbb{E}\left[\lambda_s^{i,n}\right] ds \tag{3.17}$$

and since k was arbitrary, the above holds for any $t \in [0,\, T]$. Take now a constant $M > 0$ such that $\sup_{1\le i\le N} \mathbb{E}\left[\lambda_t^{i,1}\right] \le M$ for all $t \in [0,\, T]$, which is possible by recalling the estimate (2.5). Then, provided that M is large enough, we will show by induction on n that

$$\sup_{1\le i\le N} \mathbb{E}\left[\lambda_t^{i,n}\right] \le Me^{L't} \tag{3.18}$$

for all $n \in \mathbb{N}$ and $t \in [0,\, T]$. The base case is trivial, and if M is large enough such that $M > \sup_{1\le i\le N} \mathbb{E}\left[\lambda_0^i\right] + B'T$, plugging $\sup_{1\le i\le N} \mathbb{E}\left[\lambda_s^{i,n}\right] \le Me^{L's}$ in (3.17) we find that

$$\begin{aligned} \sup_{1\le i\le N} \mathbb{E}\left[\lambda_t^{i,n+1}\right] &\le \sup_{1\le i\le N} \mathbb{E}\left[\lambda_0^i\right] + B't + L'M\int_0^t e^{L's}ds \\ &= \sup_{1\le i\le N} \mathbb{E}\left[\lambda_0^i\right] + B't + Me^{L't} - M \le Me^{L't} \end{aligned} \tag{3.19}$$

which completes the induction. Integrating then (3.18) for $t \in [0,\, T]$ we obtain the desired boundedness.

Step 3. *Limit processes as solution to the system* (2.1) *and positivity.* Now that we have the pointwise monotone convergence of $\left\{\lambda_t^{i,n}\right\}_{t\in[0,\,T]}$ to a finite process $\left\{\lambda_t^i\right\}_{t\in[0,\,T]}$ for all $i \in \{1,\, 2,\, ...,\, N\}$, we will show that these limiting processes solve our system of SDEs. The first step is to fix an $i \in \{1,\, 2,\, ...,\, N\}$, and for each $n \in \mathbb{N}$ and $s \in [0,\, T]$ take $k_n(s) \in \{1,\, 2,\, ...,\, 2^{n-1}-1\}$ such that $s \in \left[t^n_{k_n(s)},\, t^n_{k_n(s)+1}\right]$. Obviously, if we take $s_n \in \left[t^n_{k_n(s)},\, t^n_{k_n(s)+1}\right]$ for all $n \in \mathbb{N}$, we will have $s_n \longrightarrow s$ as $n \longrightarrow +\infty$ since $|s_n - s| \le |t^n_{k_n(s)} - t^n_{k_n(s)+1}| = \mathcal{O}\left(2^{-n}\right)$. Taking $s \in D$ with D denoting the set of points where $\lambda_\cdot^{j,n}$ is continuous for all j and n, for an arbitrary $\epsilon > 0$ we have

$$b_i(s_n,\, \lambda_{s_n}^{1,n},\, \lambda_{s_n}^{2,n},\, ...,\, \lambda_{s_n}^{N,n}) - \epsilon \le b^{i,n}_{k_n(s)} \le b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \tag{3.20}$$

for some $s_n \in \left[t^n_{k_n(s)},\, t^n_{k_n(s)+1}\right]$ (by the definition of infimum). For an $m \in \mathbb{N}$, recalling the pointwise monotonicity of each $\lambda_\cdot^{i,n}$ in $n \in \mathbb{N}$ and the monotonicity of each mean-field function b_i in each of its arguments, the previous double inequality easily gives

$$b_i(s_n,\, \lambda_{s_n}^{1,m},\, \lambda_{s_n}^{2,m},\, ...,\, \lambda_{s_n}^{N,m}) - \epsilon \le b^{i,n}_{k_n(s)} \le b_i(s,\, \lambda_s^{1,n},\, \lambda_s^{2,n},\, ...,\, \lambda_s^{N,n}) \tag{3.21}$$

for all $n \ge m$. Since $s \in D$, taking $n \longrightarrow +\infty$ in the above and recalling that each b_i is continuous, we obtain

$$b_i(s,\, \lambda_s^{1,m},\, \lambda_s^{2,m},\, ...,\, \lambda_s^{N,m}) - \epsilon \le \liminf_{n\longrightarrow+\infty} b^{i,n}_{k_n(s)} \le \limsup_{n\longrightarrow+\infty} b^{i,n}_{k_n(s)} \le b_i(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N) \tag{3.22}$$

Taking now $m \longrightarrow +\infty$ we get

$$b_i(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N) - \epsilon \le \liminf_{n\longrightarrow+\infty} b^{i,n}_{k_n(s)} \le \limsup_{n\longrightarrow+\infty} b^{i,n}_{k_n(s)} \le b_i(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N). \tag{3.23}$$

and since $\epsilon > 0$ was arbitrary, the above implies that $\lim_{n\longrightarrow+\infty} b^{i,n}_{k_n(s)} = b_i(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N)$, where the convergence is obviously monotone. Next, for any $t \in [0,\, T]$, recalling (3.4) and that for all $k \in \{1,\, 2,\, ...,\, 2^{n-1}-1\}$ we have $k = k_n(s)$ for all $s \in \left[t^n_k,\, t^n_{k+1}\right]$, for any $i \in \{1,\, 2,\, ...,\, N\}$ we can write

$$\lambda_t^{i,n+1} = \lambda_0^i + \sum_{k=0}^{k_n(t)-1} \left(\lambda^{i,n+1}_{t^n_{k+1}} - \lambda^{i,n+1}_{t^n_k}\right) + \left(\lambda^{i,n+1}_t - \lambda^{i,n+1}_{t^n_{k_n(t)}}\right)$$

$$
\begin{aligned}
&= \lambda_0^i + a_i \sum_{k=0}^{k_n(t)-1} \int_{t_k^n}^{t_{k+1}^n} \left(\mathbb{E}\left[b_{k_n(s)}^{i,n} | \mathcal{F}_s \right] - \lambda_s^{i,n+1} \right) ds \\
&\qquad\qquad + a_i \int_{t_{k_n(t)}^n}^{t} \left(\mathbb{E}\left[b_{k_n(s)}^{i,n} | \mathcal{F}_s \right] - \lambda_s^{i,n+1} \right) ds \\
&\qquad + \sum_{k=0}^{k_n(t)-1} \int_{t_k^n}^{t_{k+1}^n} \sigma_i(\lambda_s^{i,n+1}) dW_s^i + \int_{t_{k_n(t)}^n}^{t} \sigma_i(\lambda_s^{i,n+1}) dW_s^i \\
&\qquad + \sum_{k=0}^{k_n(t)-1} \int_{t_k^n}^{t_{k+1}^n} \int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\, u \right) N_{i,1}(ds,\, du) \\
&\qquad\qquad + \int_{t_{k_n(t)}^n}^{t} \int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\, u \right) N_{i,1}(ds,\, du) \\
&\qquad + \sum_{k=0}^{k_n(t)-1} \int_{t_k^n}^{t_{k+1}^n} \int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+1},\, u \right) \tilde{N}_{i,0}(ds,\, du) \\
&\qquad\qquad + \int_{t_{k_n(t)}^n}^{t} \int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+1},\, u \right) \tilde{N}_{i,0}(ds,\, du) \\
&= \lambda_0^i + a_i \int_0^t \left(\mathbb{E}\left[b_{k_n(s)}^{i,n} | \mathcal{F}_s \right] - \lambda_s^{i,n+1} \right) ds + \int_0^t \sigma_i(\lambda_s^{i,n+1}) dW_s^i \\
&\qquad + \int_0^t \int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\, u \right) N_{i,1}(ds,\, du) \\
&\qquad + \int_0^t \int_{U_0} g_{i,0}\left(\lambda_{s-}^{i,n+1},\, u \right) \tilde{N}_{i,0}(ds,\, du)
\end{aligned}
\tag{3.24}
$$

and taking $n \longrightarrow +\infty$ in the above for all i we derive the desired system of SDEs satisfied by the limiting processes $\left\{ \lambda_\cdot^i : i \in \{1,\, 2,\, ...,\, N\} \right\}$. Indeed, since $[0,\, T]/D$ is obviously a countable random subset of $[0,\, T]$, the monotone convergence theorem gives

$$
\begin{aligned}
\int_0^t \left(\mathbb{E}\left[b_{k_n(s)}^{i,n} | \mathcal{F}_s \right] - \lambda_s^{i,n+1} \right) ds &= \int_0^t \mathbb{E}\left[b_{k_n(s)}^{i,n} | \mathcal{F}_s \right] ds - \int_0^t \lambda_s^{i,n+1} ds \\
&\longrightarrow \int_0^t \mathbb{E}\left[b_i(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N) | \mathcal{F}_s \right] ds - \int_0^t \lambda_s^i ds \\
&= \int_0^t \left(b_i(s,\, \lambda_s^1,\, \lambda_s^2,\, ...,\, \lambda_s^N) - \lambda_s^i \right) ds,
\end{aligned}
\tag{3.25}
$$

and then we have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(\sup_{t \in [0,T]} \Big| \int_0^{t \wedge \tau^m} \sigma_i\left(\lambda_s^{i,n+1} \right) dW_s^i - \int_0^{t \wedge \tau^m} \sigma_i\left(\lambda_s^i \right) dW_s^i \Big| \Big)^2 \Big] \\
&= \mathbb{E}\Big[\Big(\sup_{t \in [0,T]} \Big| \int_0^{t \wedge \tau^m} \left(\sigma_i\left(\lambda_s^{i,n+1} \right) - \sigma_i\left(\lambda_s^i \right) \right) dW_s^i \Big| \Big)^2 \Big] \\
&\le C\mathbb{E}\Big[\int_0^{T \wedge \tau^m} \left(\sigma_i\left(\lambda_s^{i,n+1} \right) - \sigma_i\left(\lambda_s^i \right) \right)^2 ds \Big]
\end{aligned}
$$

and for $j \in \{0, 1\}$ also

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,\,T]}\Big|\int_0^{t\wedge\tau^m}\int_{U_j} g_{i,j}\left(\lambda_{s-}^{i,n+1},\,u\right)\tilde{N}_{i,j}\,(ds,\,du) - \int_0^{t\wedge\tau^m}\int_{U_j} g_{i,j}\left(\lambda_{s-}^{i},\,u\right)\tilde{N}_{i,j}\,(ds,\,du)\Big|\Big)^2\Big]$$

$$= \mathbb{E}\Big[\Big(\sup_{t\in[0,\,T]}\Big|\int_0^{t\wedge\tau^m}\int_{U_j}\left(g_{i,j}\left(\lambda_{s-}^{i,n+1},\,u\right) - g_{i,j}\left(\lambda_{s-}^{i},\,u\right)\right)\tilde{N}_{i,j}\,(ds,\,du)\Big|\Big)^2\Big]$$

$$\leq C\mathbb{E}\Big[\int_0^{T\wedge\tau^m}\int_{U_j}\left(g_{i,j}\left(\lambda_{s-}^{i,n+1},\,u\right) - g_{i,j}\left(\lambda_{s-}^{i},\,u\right)\right)^2\mu_{i,j}\,(du)\,ds\Big]$$

by the Burkholder-Davis-Gundy inequality (see [4]), with the sequence $\{\tau^m\}_{m\in\mathbb{N}}$ of stopping times selected as in the proof of Lemma 3.2 to ensure that the RHS in the last two estimates is finite for all $n \in \mathbb{N}$, and these RHS tending to zero by the monotone pointwise convergence of $\lambda_{\cdot}^{i,n}$ to $\lambda_{\cdot}^{i}$, the continuity of σ_i and $g_{i,j}$, the monotonicity or boundedness of these functions and the corresponding convergence theorem. Finally, a similar argument shows that

$$\int_0^{t\wedge\tau^m}\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\,u\right)\mu_{i,1}\,(du)\,ds \longrightarrow \int_0^{t\wedge\tau^m}\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i},\,u\right)\mu_{i,1}\,(du)\,ds \tag{3.26}$$

surely for all $t \in [0,\,T]$ as $n \longrightarrow +\infty$, and combining this with the previous convergence result for the integral with respect to $\tilde{N}_{i,1}$ we deduce that almost surely we have

$$\int_0^{t\wedge\tau^m}\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i,n+1},\,u\right)N_{i,1}\,(ds,\,du) \longrightarrow \int_0^{t\wedge\tau^m}\int_{U_1} g_{i,1}\left(\lambda_{s-}^{i},\,u\right)N_{i,1}\,(ds,\,du) \tag{3.27}$$

as $n \longrightarrow +\infty$ for any $t \in [0,\,T]$. The desired system of SDEs is obtained by observing that almost surely we have $t = t \wedge \tau^m$ for large enough m. The proof is now complete since for every i and all $t \geq 0$ we have almost surely $\lambda_t^i \geq \lambda_t^{i,1}$ with $\lambda_t^{i,1}$ being non-negative in the one-dimensional case, and since we can integrate (3.18) and use Fatou's lemma to deduce that $\lambda_{\cdot}^{i}$ is L^1 - integrable for each i. ☐

Remark 3.3. Observe that the Lipschitz continuity of each b_i is only used to obtain the inequality (3.12), which leads then to the almost surely upper boundedness of the pointwise increasing sequence of processes and thus to the existence of a finite limit. It would suffice to have any condition on the b_i that can imply an inequality of the form (3.12) (e.g simple boundedness), or lead to the same uniform upper boundedness in a different way.

We finally show that the solution to our multi-dimensional system is unique by following ideas in [9].

Theorem 3.4 (Pathwise Uniqueness). *There is at most one solution* $\left(\lambda_{\cdot}^{1},\,\lambda_{\cdot}^{2},\,...,\,\lambda_{\cdot}^{N}\right)$ *to the system of equations* (2.1).

Proof. Suppose that that there exist two different solutions $\left(\lambda_{\cdot}^{1,1},\,\lambda_{\cdot}^{1,2},\,...,\,\lambda_{\cdot}^{1,N}\right)$ and $\left(\lambda_{\cdot}^{2,1},\,\lambda_{\cdot}^{2,2},\,...,\,\lambda_{\cdot}^{2,N}\right)$ to the system (2.1). Let $\lambda_{\cdot}^{i} = \lambda_{\cdot}^{1,i} - \lambda_{\cdot}^{2,i}$ for each $i \in \{1,\,2,\,...,\,N\}$.

The idea is to show that each $\lambda^i_\cdot$ is identically zero for each equation. So for each $m \in \mathbb{N}$, we need to construct, similar as for Theorem 3.1 in [9], a sequence $\{\phi_{m,k}\}_{k\in\mathbb{N}}$ of non-negative, twice continuously differentiable functions satisfying:

1. $\phi_{m,k}(x) \longrightarrow |x|$ increasingly as $k \longrightarrow +\infty$.
2. $0 \leq \phi'_{m,k}(x) \leq 1$ for $x \geq 0$ and $-1 \leq \phi'_{m,k}(x) \leq 0$ for $x \leq 0$.
3. $\phi''_{m,k}(x) \geq 0$ for all $x \in \mathbb{R}$.
4. As $k \longrightarrow +\infty$ we have the following two convergences for all $i \in \{1, 2, ..., N\}$, uniformly in $0 \leq x,\, y \leq m$:

   (a)
   $$\phi''_{m,k}(x-y)\left[\sigma_i(x) - \sigma_i(y)\right]^2 \longrightarrow 0$$

   (b)
   $$\int_{U_0} D_{g_{i,0}(x,u)-g_{i,0}(y,u)}\phi_{m,k}(x-y)\mu_{i,0}(du) \longrightarrow 0$$

   where $D_z f(x) := f(x+z) - f(x) - zf'(x)$ for any $x, z \in \mathbb{R}$ and any function f defined on a domain containing $x, x+z$ and differentiable at x.

The construction of the above sequence of functions is similar to that in the proof of Theorem 3.2 in [9]. First, we set $a_0 = 1$ and for each $k \geq 1$ we take $0 < a_k < a_{k-1}$ such that $\int_{a_k}^{a_{k-1}} \min\left\{\frac{1}{\rho^2(x)},\, \frac{1}{\rho_m^2(x)}\right\} dx = k$. Next, for each k, we take a smooth function $x \longrightarrow \psi_{m,k}(x)$ supported in $(a_k,\, a_{k-1})$ such that:

$$0 \leq \psi_{m,k}(x) \leq \frac{2}{k}\min\left\{\frac{1}{\rho^2(x)},\, \frac{1}{\rho_m^2(x)}\right\} \tag{3.28}$$

with $\int_{a_k}^{a_{k-1}} \psi_{m,k}(x)dx = 1$, and we define

$$\phi_{m,k}(z) = \int_0^{|z|}\int_0^y \psi_{m,k}(x)dx \tag{3.29}$$

for all $z \in \mathbb{R}$. The difference compared to the construction in [9] is that $\frac{1}{\rho_m^2(x)}$ is replaced by $\min\left\{\frac{1}{\rho^2(x)},\, \frac{1}{\rho_m^2(x)}\right\}$, but it can be verified in the same way that the functions $\phi_{m,k}$ are non-negative, twice continuously differentiable, and satisfy the the first three of the four required properties. $\square$

4 Appendix: proof of Lemma 3.2

We finally provide the proof of the technical key lemma. The idea is to approximate the drift coefficient b from below by a pointwise increasing sequence of adapted, piecewise constant processes, and use the comparison theorem from Gal'chuk [10], together with the monotone convergence theorem.

Proof. **Step 1:** *Discretization in time of the process b.* For each $n \in \mathbb{N}_+$ we define $t_0^n = 0$, $b_0^n = b_0 - \frac{1}{n}$, and recursively for $k \in \mathbb{N}$:

$$
\begin{aligned}
t_{k+1}^n &= \inf\{t > t_k^n : b_{t_k^n} - \frac{1}{n} > b_t\} \wedge \left(t_k^n + \frac{1}{n}\right) \wedge T, \\
b_t^n &= b_{t_k^n} - \frac{1}{n}, \qquad t \in \left[t_k^n,\, t_{k+1}^n\right).
\end{aligned} \tag{4.1}
$$

Obviously, we have $b_t^n \leq b_t$ for all $t \in [0, T]$ and $n \in \mathbb{N}$. We also define $b_t^0 = 0$ and $\bar{b}_t^n = \max\{b_t^m : 0 \leq m \leq n\}$ for all $0 \leq t \leq T$. By definition, for any fixed positive integer n and $\omega \in \Omega$, $t_k^n(\omega)$ is increasing in k. We have in addition the following assertion.

Claim A. For any $\omega \in \Omega$, one has $t_k^n(\omega) = T$ for sufficiently large k.

Proof of the Claim A. We prove by contradiction. Suppose that $t_k^n(\omega)$ takes infinitely many values, then

$$
t_{k+1}^n(\omega) = \inf\{t > t_k^n(\omega) : b_{t_k^n(\omega)}(\omega) - \frac{1}{n} > b_t(\omega)\}
$$

for all large enough k. By the right continuity of the process b, we also have

$$
b_{t_k^n(\omega)}(\omega) - \frac{1}{n} \geq b_{t_{k+1}^n(\omega)}(\omega)
$$

for all such k. Moreover, $t_k^n(\omega)$ increase to a finite limit $t^n(\omega)$ as $k \to +\infty$, and since the function $t \mapsto b_t(\omega)$ has a left limit $\ell^n(\omega)$ at $t^n(\omega)$, we have

$$
\ell^n(\omega) = \lim_{k \to +\infty} b_{t_{k+1}^n(\omega)}(\omega) \leq \lim_{k \to +\infty} b_{t_k^n(\omega)}(\omega) - \frac{1}{n} = \ell^n(\omega) - \frac{1}{n}
$$

which is a contradiction. Therefore, $t_k^n(\omega)$ only takes finitely many values in $[0, T]$ when k varies. In particular, there exists $\ell^n(\omega) \in [0, T]$ and $k_0 \in \mathbb{N}$ such that $t_k^n(\omega) = \ell^n(\omega)$ for any $k \in \mathbb{N}$ with $k \geq k_0$. Note that $\ell^n(\omega)$ should equal T since otherwise by the right continuity of the process b we would have $t_{k_0+1}^n(\omega) > t_{k_0}^n(\omega)$, which leads again to a contradiction. □

Step 2. *Resolution of the equation with discretized drift coefficients.* Note that t_k^n is a stopping time for each n and each k, and if we define $\bar{t}_k^n$ to be the k^{th} smallest element of the set $\{t_k^m : k \in \mathbb{N}, m \in \{1, 2, \ldots, n\}\}$, then for any n, $\{\bar{t}_k^n\}_{k \in \mathbb{N}}$ is an increasing sequence of stopping times, with $\bar{b}_t^n$ being constant on each stochastic interval of the form $[\![\bar{t}_k^n, \bar{t}_{k+1}^n[\![$. Moreover, we obtain by Claim A that, for each fixed $\omega \in \Omega$, $\bar{t}_k^n(\omega) = T$ for all large enough k. Assuming that we can find a non-negative semimartingale $(Y_t^n)_{t \in [\![0, \bar{t}_k^n]\!]}$ satisfying the SDE

$$
\begin{aligned}
Y_t^n = Y_0 &+ a \int_0^t \left(\bar{b}_s^n - Y_s^n\right) ds + \int_0^t \sigma(Y_s^n) dW_s \\
&+ \int_0^t \int_{U_1} g_1\left(Y_{s-}^n, u\right) N_1(ds, du) + \int_0^t \int_{U_0} g_0\left(Y_{s-}^n, u\right) \tilde{N}_0(ds, du)
\end{aligned} \tag{4.2}
$$

on the stochastic interval $[\![0, \bar{t}^n_k]\!]$, we claim that we can extend the solution to the stochastic interval $[\![0, \bar{t}^n_{k+1}]\!]$. Indeed, we only need to find a non-negative solution to the SDE

$$\begin{aligned} Y^n_t = Y^n_{\bar{t}^n_k} + a\int_{\bar{t}^n_k}^t \left(\bar{b}^n_s - Y^n_s\right) ds + \int_{\bar{t}^n_k}^t \sigma(Y^n_s) dW_s \\ + \int_{\bar{t}^n_k}^t \int_{U_1} g_1\left(Y^n_{s-},\, u\right) N_1\left(ds,\, du\right) + \int_{\bar{t}^n_k}^t \int_{U_0} g_0\left(Y^n_{s-},\, u\right) \tilde{N}_0\left(ds,\, du\right) \end{aligned} \tag{4.3}$$

on $]\!]\bar{t}^n_k, \bar{t}^n_{k+1}]\!]$ given $\mathcal{F}_{\bar{t}^n_k}$, in which case $\bar{t}^n_k$, $Y^n_{\bar{t}^n_k}$ and $\bar{b}^n_s = \bar{b}^n_{\bar{t}^n_k} \geq 0$ are known constants and $\bar{t}^n_{k+1}$ is a stopping time. This is possible by recalling the results of [9] to solve

$$\begin{aligned} Y^n_t = Y^n_{\bar{t}^n_k} + a\int_{\bar{t}^n_k}^t \left(\bar{b}^n_{\bar{t}^n_k} - Y^n_s\right) ds + \int_{\bar{t}^n_k}^t \sigma(Y^n_s) dW_s \\ + \int_{\bar{t}^n_k}^t \int_{U_1} g_1\left(Y^n_{s-},\, u\right) N_1\left(ds,\, du\right) + \int_{\bar{t}^n_k}^t \int_{U_0} g_0\left(Y^n_{s-},\, u\right) \tilde{N}_0\left(ds,\, du\right) \end{aligned} \tag{4.4}$$

on $]\!]\bar{t}^n_k, T]\!]$ given $\mathcal{F}_{\bar{t}^n_k}$, and then stopping at time $\bar{t}^n_{k+1}$. This inductive argument defines a non-negative càdlàg semimartingale $Y^n_\cdot$ which solves the equation $(3.1)_{\bar{b}^n}$ on $[0, T]$. By construction we have $b_t \geq \bar{b}^{n+1}_t \geq \bar{b}^n_t \geq 0$ for all $t \in [0,\, T]$ and $n \in \mathbb{N}$. Therefore, by the comparison theorem from Gal'chuk [10, Theorem 1], we have $Y^{n+1}_t \geq Y^n_t$ for all $t \in [0,\, T]$ and $n \in \mathbb{N}$.

Step 3. *Convergence of the drift coefficients and associated solutions.* We begin with the following claim.

Claim B. The sequence $(Y^n)_{n\in\mathbb{N}}$ defined in Step 2 converges pointwise from below to an $\mathbb{F}$-adapted process Y.

Proof of Claim B. We first show that the sequence is pointwisely bounded from above. For this purpose, we apply the construction of Step 1 to the process $-b$ as follows. We define $s_0 = 0$, $\tilde{b}_0 = b_0 + 1$, and recursively on $k \in \mathbb{N}$,

$$\begin{aligned} s_{k+1} &= \inf\{t > s_k \,:\, b_{s_k} + 1 < b_t\} \wedge (s_k + 1) \wedge T, \\ \tilde{b}_t &= b_{s_k} + 1, \quad t \in [\![s_k, s_{k+1}[\![. \end{aligned}$$

By definition, one has $\tilde{b}_t \geq b_t$ for all $t \in [0, 1]$. Similarly to Claim A, for any fixed $\omega \in \Omega$, $s_k(\omega)$ is increasing in k and $s_k(\omega) = T$ for sufficiently large k. By the same argument as in **Step 2**, we obtain that the equation $(3.1)_{\tilde{b}}$ admits a solution, which we denote by $\tilde{Y}$. Still by the comparison theorem of [10], we deduce from the relations $\tilde{b} \geq b \geq \bar{b}^{n+1} \geq \bar{b}^n \geq 0$ the inequalities $\tilde{Y}_t \geq Y^{n+1}_t \geq Y^n_t \geq 0$ for all $t \in [0, T]$ and $n \in \mathbb{N}$. Therefore, the sequence $(Y^n)_{n\in\mathbb{N},\, n\geq 1}$ converges pointwise to a limite process Y, which is clearly $\mathbb{F}$-adapted. □

We now show that, for any $\omega \in \Omega$, and any point of continuity t of the function $s \mapsto b_s(\omega)$, the sequence $\bar{b}^n_t(\omega)$ converges from below to $b_t(\omega)$ as $n \to +\infty$. Indeed, for any any positive integer n, there exists a $k(n) \in \mathbb{N}$ such that $t \in [\![t^n_{k(n)}(\omega), t^n_{k(n)+1}(\omega)[\![$ and thus

$$|t - t^n_{k(n)}(\omega)| \leq |t^n_{k(n)+1}(\omega) - t^n_{k(n)}(\omega)| \leq \frac{1}{n}, \tag{4.5}$$

which means that $t^n_{k(n)}(\omega) \to t$ from below as $n \to +\infty$. Hence, by the continuity of $b_\cdot(\omega)$ at t and the definition of $b^n_\cdot$, we have

$$b^n_t(\omega) = b_{t^n_{k(n)}(\omega)}(\omega) - \frac{1}{n} \longrightarrow b_t(\omega) \text{ as } n \to +\infty.$$

Recalling then that $b_t(\omega) \geq \bar{b}^n_t(\omega) \geq b^n_t(\omega)$ for all $n \in \mathbb{N}$, we deduce that $\bar{b}^n_t(\omega) \to b_t(\omega)$ as $n \to +\infty$.

Step 4. *Resolution of the initial equation.* Finally, we will show that a càdlàg version of the process Y solves (3.1) in $[0, T]$ by taking $n \to +\infty$ on (4.2) and by exploiting the convergence results we have just obtained. For $s \in [0, T]$, we denote by Y_{s-} the limit of the increasing sequence $(Y^n_{s-})_{n \in \mathbb{N}_+}$. First, we recall the monotone convergence theorem which gives

$$\begin{aligned} \int_0^t \left(\bar{b}^n_s - Y^n_s\right) ds &= \int_0^t \bar{b}^n_s ds - \int_0^t Y^n_s ds \\ &\longrightarrow \int_0^t b_s ds - \int_0^t Y_s ds = \int_0^t (b_s - Y_s)\, ds \end{aligned} \tag{4.6}$$

as $n \longrightarrow +\infty$, for any $t \in [0, T]$. Next, for every $n \in \mathbb{N}$, we consider a sequence $\{\tau^{m,n}\}_{m \in \mathbb{N}}$ of $\mathbb{F}$-stopping times such that $\lim_{m \longrightarrow +\infty} \tau^{m,n} = +\infty$ and also

$$\begin{aligned} &\int_0^{T \wedge \tau^{m,n}} \left(\sigma\left(Y^n_s\right) - \sigma\left(Y_s\right)\right)^2 ds \leq m, \\ &\int_0^{T \wedge \tau^{m,n}} \int_{U_0} \left(g_0\left(Y^n_{s-},\, u\right) - g_0\left(Y_{s-},\, u\right)\right)^2 \mu_0\left(du\right) ds \leq m, \\ &\int_0^{T \wedge \tau^{m,n}} \int_{U_1} \left(g_1\left(Y^n_{s-},\, u\right) - g_1\left(Y_{s-},\, u\right)\right)^2 \mu_1\left(du\right) ds \leq m \end{aligned} \tag{4.7}$$

and

$$\int_0^{T \wedge \tau^{m,n}} \int_{U_1} \left|g_1\left(Y^n_{s-},\, u\right) - g_1\left(Y_{s-},\, u\right)\right| \mu_1\left(du\right) ds \leq m \tag{4.8}$$

for each $m \in \mathbb{N}$. Since Y^n is increasing in n, by the monotonicity of g_0 and Assumption 2.3, we can choose, for each $m \in \mathbb{N}$, the $\mathbb{F}$-stopping times $\tau^{m,n} = \tau^m$ to be independent of n. Then, by using the Burkholder-Davis-Gundy inequality (see [4]) we have

$$\begin{aligned} &\mathbb{E}\Big[\Big(\sup_{t \in [0,\, T]} \Big| \int_0^{t \wedge \tau^m} \sigma\left(Y^n_s\right) dW_s - \int_0^{t \wedge \tau^m} \sigma\left(Y_s\right) dW_s \Big|\Big)^2\Big] \\ &= \mathbb{E}\Big[\Big(\sup_{t \in [0,\, T]} \Big| \int_0^{t \wedge \tau^m} \left(\sigma\left(Y^n_s\right) - \sigma\left(Y_s\right)\right) dW_s \Big|\Big)^2\Big] \leq C \mathbb{E}\left[\int_0^{T \wedge \tau^m} \left(\sigma\left(Y^n_s\right) - \sigma\left(Y_s\right)\right)^2 ds\right] \end{aligned}$$

where we can recall the continuity of σ and either the monotone convergence theorem or the dominated convergence theorem (depending on whether σ is bounded or increasing) to deduce that the RHS tends to zero as $n \longrightarrow +\infty$. Next, writing $\tilde{N}_1\left(ds, du\right)$ for the compensated measure $N_1(ds, du) - \mu_1(du)\,ds$, where $\mu_1(du)\,ds$ is the compensator of $N_1(ds, du)$, by using the Burkholder-Davis-Gundy inequality once more we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\sup_{t\in[0,\,T]}\Big|\int_0^{t\wedge\tau^m}\int_{U_1} g_1\left(Y^n_{s-},\, u\right)\tilde{N}_1(ds,\, du) - \int_0^{t\wedge\tau^m}\int_{U_1} g_1\left(Y_{s-},\, u\right)\tilde{N}_1(ds,\, du)\Big|\Big)^2\Big] \\
&= \mathbb{E}\Big[\Big(\sup_{t\in[0,\,T]}\Big|\int_0^{t\wedge\tau^m}\int_{U_1}\left(g_1\left(Y^n_{s-},\, u\right) - g_1\left(Y_{s-},\, u\right)\right)\tilde{N}_1(ds,\, du)\Big|\Big)^2\Big] \\
&\leq C\mathbb{E}\Big[\int_0^{T\wedge\tau^m}\int_{U_1}\left(g_1\left(Y^n_{s-},\, u\right) - g_1\left(Y_{s-},\, u\right)\right)^2\mu_1(du)\,ds\Big]
\end{aligned}$$

where the quantity $\left(g_1\left(Y^n_{s-},\, u\right) - g_1\left(Y_{s-},\, u\right)\right)^2$ is either monotone or bounded by $4G_1^2(u)$, with the last being integrable due to (2.8), so by monotone or dominated convergence and by the continuity of g_1, the RHS of the above tends also to zero as $n \longrightarrow +\infty$. Moreover, by a similar argument we have always

$$\int_0^{t\wedge\tau^m}\int_{U_1} g_1\left(Y^n_{s-},\, u\right)\mu_1(du)\,ds \longrightarrow \int_0^{t\wedge\tau^m}\int_{U_1} g_1\left(Y_{s-},\, u\right)\mu_1(du)\,ds \tag{4.9}$$

for all $t \in [0,\, T]$ as $n \longrightarrow +\infty$, and combining this with the previous convergence result we deduce that almost surely we have

$$\int_0^{t\wedge\tau^m}\int_{U_1} g_1\left(Y^n_{s-},\, u\right)N_1(ds,\, du) \longrightarrow \int_0^{t\wedge\tau^m}\int_{U_1} g_1\left(Y_{s-},\, u\right)N_1(ds,\, du) \tag{4.10}$$

for all $t \in [0,\, T]$ (in a subsequence). Finally, using the Burkholder-Davis-Gundy inequality and the monotone convergence theorem as we did for the integral with respect to $\tilde{N}_1(ds,\, du)$, we find that

$$\mathbb{E}\left[\left(\sup_{t\in[0,\,T]}\left|\int_0^{t\wedge\tau^m}\int_{U_0} g_0\left(Y^n_{s-},\, u\right)\tilde{N}_0(ds,\, du) - \int_0^{t\wedge\tau^m}\int_{U_0} g_0\left(Y_{s-},\, u\right)\tilde{N}_0(ds,\, du)\right|\right)^2\right]$$

tends also to zero as $n \longrightarrow +\infty$. It follows that almost surely, we can take limits on both sides of (4.2) and obtain (3.1) when t is replaced by $t \wedge \tau^m$, for any $t \in [0,\, T]$. Then, we can finish the proof by letting $m \longrightarrow +\infty$. $\square$

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References

[1] Abdelghani, M. and Melnikov, A. A comparison theorem for stochastic equations of optional semimartingales, *Stochastics and Dynamics*, **18(4)**, (2018), 1850029, 21.

[2] Bo, L. and Capponi, A. Systemic risk in interbanking networks. *SIAM Journal on Financial Mathematics*, **6(1)** (2015), 386–424

[3] Bass, F. R. Stochastic differential equations driven by symmetric stable processes. *Séminaire de probabilités de Strasbourg*, **36** (2002), 302–313

[4] Cohen, S. N. and Elliot, R. J. *Stochastic Calculus and Applications*. Second edition. Probability and Its Applications. Birkhauser, New York, NY (2015).

[5] Dawson, D. A. and Li, Z. Skew convolution semigroups and affine Markov processes. *The Annals of Probability*, **34(3)** (2006), 1103–1142

[6] Filipović, D. A general characterization of one factor affine term structure models. *Finance and Stochastics* **5(3)** (2001), 389–412.

[7] Fouque, J. P. and Ichiba, T. Stability in a model of interbank lending. *SIAM Journal on Financial Mathematics*, **4(1)** (2013), 784–803.

[8] Frikha, N. and Li, L. Well-posedness and approximation of some one-dimensional Lévy-driven non-linear SDEs. *Stochastic Processes and their Applications*, **132** (2021), 76–107.

[9] Fu, Z. and Li, Z. Stochastic equations of non-negative processes with jumps. *Stochastic Processes and their Applications*, **120(3)** (2010), 306–330.

[10] Gal'chuk, L. I. A Comparison Theorem for Stochastic Equations with Integrals with Respect to Martingales and Random Measures. *Theory of Probability and Its Applications*, **27(3)** (1983), 450–460.

[11] Giesecke, K.; Spiliopoulos, K and Sowers, R. B. Default clustering in large portfolios: typical events. *The Annals of Applied Probability*, **23(1)** (2013), 348–385.

[12] Hambly, B. and Kolliopoulos, N. Stochastic evolution equations for large portfolios of stochastic volatility models. *SIAM Journal on Financial Mathematics*, **8(1)** (2017), 962–1014.

[13] Hambly, B. and Kolliopoulos, N. Erratum: Stochastic evolution equations for large portfolios of stochastic volatility models. *SIAM Journal on Financial Mathematics*, **10** (2019), 857–876.

[14] Jiao, Y.; Ma, C. and Scotti, S. Alpha-CIR model with branching processes in sovereign interest rate modeling. *Finance and Stochastics*, **21(3)** (2017), 789–813.

[15] Jiao, Y.; Ma, C.; Scotti, S. and Zhou, C. The Alpha-Heston stochastic volatility model. *Mathematical Finance*, **31(3)** (2021), 943–978.

[16] Li, Z. and Mytnik, L. Strong solutions for stochastic differential equations with jumps. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, **47(4)** (2011), 1055–1067.