Dynamic Bivariate Mortality Modelling*

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Abstract

The dependence structure of the life statuses plays an important role in the valuation of life insurance products involving multiple lives. Although the mortality of individuals is well studied in the literature, their dependence remains a challenging field. In this paper, the main objective is to introduce a new approach for analyzing the mortality dependence between two individuals in a couple. It is intended to describe in a dynamic framework the joint mortality of married couples in terms of marginal mortality rates. The proposed framework is general and aims to capture, by adjusting some parametric form, the desired effect such as the "broken-heart syndrome". To this end, we use a well-suited multiplicative decomposition, which will serve as a building block for the framework and thus will be used to separate the dependence structure from the marginals. We make the link with the existing practice of affine mortality models. Finally, given that the framework is general, we propose some illustrative examples and show how the underlying model captures the main stylized facts of bivariate mortality dynamics.

Keywords: *Bivariate Mortality; Dependence; Conditional Survival Probability; Copula; Broken-Heart Syndrome.*

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1 Introduction

Mortality modeling has been an active field of research in actuarial science. The evolution of mortality is of paramount importance to a life insurance company as the liabilities depend to a large extent on the evolution of the number of deaths occurrence. Although the mortality of individuals is well documented in the literature, the dependence between lives remains a challenging field. Indeed, future lifetimes within a group of people like married couples, for example, can exhibit dependencies due to similar lifestyles or to the effect of exposure to common risk factors. Such a dependence structure of the life statuses is not only a key element in the evaluation of insurance products involving multiple lives but also in a better understanding of the elderly longevity. The main objective of this paper is to analyze the mortality dependence between individuals in a couple by introducing a new approach, which is intended to describe the joint mortality of married couples in terms of marginal mortality rates under different life statuses. In fact, it has long been documented that the death of a spouse does impact the mortality of the surviving spouse, in particular it may accelerate the death of the latter. This causal effect is known as the "broken-heart syndrome". In addition, the spouses are also impacted by their common lifestyle, which is also a source of dependence.

In the literature, the common methods to handle such a dependency are based on copulas or Markovian approaches [11, 12, 14, 19, 29]. As noted in literature, the attractive advantage of a copula-based approach is that it allows the correlation structure of the remaining lifetime variables to be estimated separately from their marginal distributions [15], [31]. A Markovian approach, on the other hand, can show clearly the change of state in couple's lifetime, but does, generally, fail at showing the dependence structure between the spouses.

Also, the broken-heart syndrome is not accommodated. In fact, the death of a spouse introduces a jump in the mortality of the survivor. This phenomenon is now widely recognized. For instance, Gourieroux and Lu [12] introduced jumps in mortality intensity using a Freund model with an unobservable common static factor representing the shared socioeconomic conditions. Such a behavior was empirically investigated in Lu [18] using data on joint annuities. The analysis shows that the effect of losing one's spouse is persistent, and asymmetric for the spouses. However, even if the recent literature considered these different stylized facts, the common approach is still based on a static development and does not tackle the dynamic aspect either of the dependency between the spouse neither the broken-heart syndrome. In Blanchet-Scalliet *et al.* [5], a first approach is developed to handle these aspects and propose a framework with a dependence structure governed by a Farlie-Gumbel-Morgenstern copula in a dynamic setting. The proposed approach is based on dynamic characterization of the joint density, which convenient for modeling the dependency of deaths among a population not only from a theoretical point of view but also for a practical use. This amounts to saying that one individual's intensity will have a jump when the other individual is deceased. However, in [5] the framework assumes a symmetric reaction the considered individuals. In fact, the investigation of the dependence as well as the broken-heart syndrome exhibit an asymmetric behavior between the coupled lifetimes [18].

The model developed in this paper aims to reconcile the copula and the Markovian approaches while taking into account most of the aforementioned stylized facts. First, we consider a forward mortality rate model by taking into consideration a common life-status of the married couple as background information. Then using a multiplicative decomposition will serve as a building block of the joint survival probability in order to identify the dependency structure from the marginals. The proposed framework is general and aims by adjusting some parametric form to capture the desired effect. In fact, we write the conditional joint probability as the product of two marginal probabilities, and a random variable which represents the dependence between the two life statuses. The latter, contrary to the linear correlation parameter which takes value in $[-1, 1]$, can be viewed as a random extension of Sibuya's function [28] and take any strictly positive real value, which allows for a large choice of parametrisation. Such a framework allows for an inherent structure for the dependency and can be adapted to various approaches used in the literature. Indeed, the class of affine term structure which is often used to describe the individual's intensity of mortality [4, 13, 20, 23, 26], fits naturally in our model when the couple shares a common factor. Then the joint (conditional) survival probability can be derived explicitly. By doing so, we will focus on the impact of the dependency structure separately from the individual marginals as the former plays an important role in the evaluation. Second, for the purpose of studying the "broken-heart syndrome"which can be described as "an elevated level of hazard to the life of a husband in the period of time directly following the death of the wife and vice versa", we consider the (conditional) joint survival probability under different scenarios of life statuses of the couple similar as in Norberg [21], Denuit and Cornet [6] where the life status of the couple are distinguished. The study, which can be compared to that in credit risk (see e.g. El Karoui *et al.* [9, 10]) for the before-default and after-default analysis, is based on the events of "before the first death" and "after the first death" and allows us to examine in detail the impact of one death event on the surviving individual. For each case, we characterize the dynamics of the involved intensities and quantify the impact of the first death on the intensity of the survivor. In particular, we will investigate models that take into account the main feature that drives the broken-heart syndrome. To this end, the dynamic approach together with the multiplicative decomposition of the joint survival function provide a new vision on the life statuses dependence problems.

The remaining of the paper is organized as follows. Section 2 has still an introductory purpose. It introduces the mathematical setting and the notation used throughout the paper. We also consider the problem of pricing joint life insurance contracts and discuss the main contracts of interest, which involve a guarantee as a function of the joint life statuses. In Section 3 we introduce the modelling framework well suited for bivariate mortality. We rely on a characterization of the joint survival function using multiplicative decomposition and isolating the marginal effect and the dependence structure. This section makes the parallel to classic intensity models commonly used in single life mortality modelling. The price of joint life contracts is derived. Section 4 is dedicated to the broken-heart syndrome, which introduces the conditional survival probability as means of capturing the impact of the death of the first individual on the survivor. We discuss the impact of the dependency as well as the broken-heart syndrome on main contracts, i.e. first-to-die and last-survivor. Under the theoretical framework, in Section 5, some explicit models are studied in more detail for further practical use and we complete the paper by numerical illustrations.

2 Joint-Life Insurance Contracts and Symmetric Statuses

2.1 Mathematical Setting and Notation

We start by introducing the elementary notions of bivariate life insurance contracts and the underlying mortality setting. Let us assume that all uncertainty is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. On this space, we consider a couple whose initial ages are x_1 and x_2 and denote τ_1 and τ_2 to be positive random variables representing the future lifetimes of each spouse. For any $i \in \{1, 2\}$, let $D^i(t) = \mathbb{1}_{\{\tau_i \leq t\}}$. The filtration $\mathbb{D}^i = (\mathcal{D}^i_t)_{t\geq 0}$ generated by the process $(D^i(t))_{t\geq 0}$, i.e., $\mathcal{D}^i_t := \sigma(\mathbb{1}_{\{\tau_i \leq s\}}, s \leq t)$, describes the information on the life status of the i^{th} individual, that is, whether he or she is still alive, and if not, the death time. The information on the bivariate life statuses is then defined by $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0} := \mathbb{D}^1 \vee \mathbb{D}^2$.

Many multiple life insurance contracts concern the order of the deaths occurrences such as the minimal-survivor or the last-survivor status of the couple. For this purpose, we consider the order statistic of the lifetime variables. Let $\tau_{(1)}$ and $\tau_{(2)}$ be the order statistic of τ_1 and τ_2, i.e., $\tau_{(1)} = \min\{\tau_1, \tau_2\}$ and $\tau_{(2)} = \max\{\tau_1, \tau_2\}$. We introduce the information stemming from the ordered lifetime statuses in a similar way and let $\mathcal{D}^{(i)}_t := \sigma(\mathbb{1}_{\{\tau_{(i)} \leq s\}}, s \leq t)$ and $\widetilde{\mathcal{D}}_t := \mathcal{D}^{(1)}_t \vee \mathcal{D}^{(2)}_t$. The filtration $\widetilde{\mathbb{D}} = (\widetilde{\mathcal{D}}_t)_{t\geq 0}$ is a sub-filtration of $\mathbb{D}$, which gives the occurrence times of the deaths but does not precise the correspondence to which member in the couple. In general, if the vector (τ_1, τ_2) is exchangeable, that is, if $(\tau_1, \tau_2) \overset{\text{d}}{=} (\tau_2, \tau_1)$ where "$\overset{\text{d}}{=}$" signifies the equality in distribution, then we can use $\widetilde{D}_t$ instead of $\mathcal{D}_t$. Besides the life status of the couple, we also take into account the environmental information, which is modeled by an auxiliary filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ on $(\Omega, \mathcal{F})$ satisfying the usual conditions. In this setting, the filtration $\mathbb{F}$ gathers information about the likelihood of death events, but not their actual occurrence. We may think of $\mathbb{F}$ as carrying out information stemming from medical or demographical factors. It may also gather relevant non-demographic information related, for example, to financial markets or economic indicators. Such information can have both positive or negative impact on the vector (τ_1, τ_2) or $(\tau_{(1)}, \tau_{(2)})$ and we shall consider their dependence later on. By combining the above two sources of information, the global information $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$ or $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{D}}_t$.

2.2 Joint-Life Insurance Contracts

In order to introduce life insurance policies depending on the joint lives (τ_1, τ_2), we define, for $t \geq 0$, the positive random variable $Y_t(\tau_1, \tau_2)$ which characterizes the guaranteed payment of the contracts. More precisely, $Y_t(\tau_1, \tau_2)$ indicates whether the guarantee is triggered or not at time t. For example, the function $Y_t(\cdot)$ can depend on the indicators of the life statuses, i.e. $t \to \mathbf{1}_{[0,t]}(\tau_i)$ for $i = 1, 2$. In this case, $Y_t(\tau_1, \tau_2)$ is simply called the joint-life status of the spouses. We are also interested in a class of joint-life contracts with regard to the events triggering the payment which are characterized by using the definition of symmetric statuses [16]. More formally, we say that the joint-life status $Y_t(\tau_1, \tau_2)$ is symmetric if $Y_t(\cdot)$ is a symmetric function, i.e., for any $(u_1, u_2) \in \mathbb{R}^2_+$ $Y_t(u_1, u_2) = Y_t(u_2, u_1)$.

Various insurance products providing joint-life benefits falls into the definition of symmetric statuses:

(i) the first-to-die contracts paying out a monetary amount at the first death of a spouse, i.e. $Y_t(\tau_1, \tau_2) = \mathbf{1}_{\{\tau_{(1)} \leq t\}}$

(ii) the last-to-die (last-survivor) contracts triggering the guarantee payment upon the death of the second spouse and thus $Y_t(\tau_1, \tau_2) = \mathbf{1}_{\{\tau_{(2)} \leq t\}}$.

Clearly, these contracts depend, respectively, on the smallest and the largest order statistics of the two lives $\tau_{(1)}$ and $\tau_{(2)}$. Moreover, we should note that the contracts described above cover both whole-life and term insurance contracts. In the first type, the guarantee remains in force until the payment is triggered. For term insurance, guarantees are only provided for a limited period of time. Thus the payment of the guaranteed amount is made when the triggering event occurs before the maturity of the contract. These contracts are similar to some well-known credit portfolio derivatives: the k^{th}-to-default swaps, with $k = \{1, 2\}$.

On the other hand, some contracts fail to fulfill the symmetric statuses definition. An example is the status $Y_t(\tau_1, \tau_2) = \mathbf{1}_{\{\tau_2 < \tau_1\}}$, related to reversionary annuities that are payable in full for lifetime of the annuitant upon whose death the pension is paid to the spouse for whole life. Therefore, only the death of the policyholder triggers the guarantee and thus no payment is made when the spouse dies before the policyholder.

To study the multiple life premium for the corresponding insurance contract, we are interested in the following quantities under the probability measure $\mathbb{P}$[1]:

(1) $\mathbb{E}[Y_T(\tau_1, \tau_2)]$, representing the expected payoff of the multiple life insurance contract. This is a key milestone in actuarial mathematics and can be related to the so-called *best-estimate* value of the contract incurred liability.

(2) $\mathbb{E}[Y_T(\tau_1, \tau_2)|\mathcal{H}_t]$, which is the value of the contract given the evolution of the relevant information at time t represented by $\mathcal{H}_t$. Here $\mathcal{H}_t$ can denote the pure information about the life status $\mathcal{D}_t$ or $\widetilde{\mathcal{D}}_t$ and can also represent the global information, i.e., $\mathcal{H}_t = \mathcal{G}_t$. This quantity can be interpreted as the prospective reserve on the single contract. Moreover, this conditional expectation is of paramount importance in the new European regulation, namely Solvency II, which requires the derivation of the one-year-time best estimate value of the liability. In the case of a single contract, the latter coincides with conditional expected value taken at time $t = 1$.

3 Joint Forward Mortality Rate Modelling for Dependent Lives

3.1 Model Setup

We now consider the bivariate mortality modelling of couples by choosing the forward modelling approach as in [2]. We focus on the conditional information set and its impact on the joint survival distribution. To this end, for a couple whose future lifetimes and current ages are denoted by $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and $\boldsymbol{x} = (x_1, x_2)$, we consider the survival probability of each spouse of the couple given the following information. First, instead of the life status of the single spouse, the first death-survival status is taken into account. As we are concerned with life insurance contracts for couples, it is relevant to include the life status of both spouses. Second, the environmental information contained in $\mathbb{F}$ is included, which allows to describe the external environmental impact or the common lifestyle factors on the life status of the couple.

[1]The probability measure $\mathbb{P}$ can be either interpreted as the historical measure or can refer to a pricing measure depending on the considered context.

More precisely, we suppose that, for any $t \geq 0$ and $T \geq t$, the conditional survival probability of each remaining life is given, by using a forward mortality rate before the first death, as

$$\mathbb{P}_{\boldsymbol{x}}(\tau_i > T | \{\tau_{(1)} > t\} \vee \mathcal{F}_{T^*}) = e^{-\int_t^T \mu^i_{\boldsymbol{x},T^*}(t,s)ds}, \quad i \in \{1,2\}, \tag{1}$$

where $T^* > 0$ is a large enough horizon time, $\mu^i_{\boldsymbol{x},T^*}(t,s)$ is $\mathcal{F}_{T^*}$-measurable which represents the forward mortality rate with the age vector $\boldsymbol{x} = (x_1, x_2)$ regard as a fixed parameter. Compared to the typical forward intensity in the literature (e.g. Bauer *et al.* [2]), the expression (1) is non standard due to the fact that T^* is a future time. This implies that events in the future, such as the long term impact of pandemics for example, can influence the mortality intensities. Such impact, although not observable directly at current time, will be projected when computing conditional joint probabilities (see Section 3.2). The marginal conditional survival probability is then

$$\mathbb{P}_{\boldsymbol{x}}(\tau_i > T | \mathcal{F}_{T^*}) = e^{-\int_0^T \mu^i_{\boldsymbol{x},T^*}(0,s)ds}, \quad i \in \{1,2\}.$$

To show the relationship between the joint and the marginal probabilities, we write

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2 | \mathcal{F}_{T^*}) = \mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1 | \mathcal{F}_{T^*}) \mathbb{P}_{\boldsymbol{x}}(\tau_2 > t_2 | \mathcal{F}_{T^*}) \rho_{\boldsymbol{x},T^*}(t_1, t_2) \tag{2}$$

where $\rho_{\boldsymbol{x},T^*}(\cdot,\cdot)$ is an $\mathcal{F}_{T^*}$-measurable random variable which characterizes the dependence and can be seen as a random extension of Sibuya's function [28]. Contrary to the linear correlation parameter which takes value in $[-1,1]$, this quantity $\rho_{\boldsymbol{x},T^*}(t_1,t_2)$ can take any strictly positive real value. In particular, for any $t_1, t_2 > 0$, if $\rho_{\boldsymbol{x},T^*}(t_1,t_2) = 1$, then there is conditional independence between τ_1 and τ_2 given the environmental information. The case where $\rho_{\boldsymbol{x},T^*}(t_1,t_2)$ is greater than 1 corresponds to the "positive quadrant dependence" between the spouses and the case smaller than 1 means a "negative quadrant dependence" (see Lehmann [17]). In addition, we have $\rho_{\boldsymbol{x},T^*}(0,t) = \rho_{\boldsymbol{x},T^*}(t,0) = 1$ for any $t \geq 0$.

Under (1), we get the joint conditional suvival probability as below.

Proposition 1. *Suppose that $\rho_{\boldsymbol{x},T^*}(t_1,t_2)$ is of class $C^{1,1}$ with respect to (t_1,t_2), then*

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2 | \mathcal{F}_{T^*}) = \exp\left\{ -\int_0^{t_1} \mu^1_{\boldsymbol{x},T^*}(s \wedge t_2, s)ds - \int_0^{t_2} \mu^2_{\boldsymbol{x},T^*}(s \wedge t_1, s)ds \right\}.$$

Proof. From the expression (1), for any $T > t$, we have

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > T | \{\tau_{(1)} > t\} \vee \mathcal{F}_{T^*}) = \frac{\mathbb{P}_{\boldsymbol{x}}(\tau_1 > T, \tau_2 > t | \mathcal{F}_{T^*})}{\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t, \tau_2 > t | \mathcal{F}_{T^*})} = e^{-\int_t^T \mu^1_{\boldsymbol{x},T^*}(t,s)ds},$$

then

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > T, \tau_2 > t | \mathcal{F}_{T^*}) = \mathbb{P}_{\boldsymbol{x}}(\tau_1 > t, \tau_2 > t | \mathcal{F}_{T^*}) e^{-\int_t^T \mu^1_{\boldsymbol{x},T^*}(t,s)ds}.$$

This combined to the definition (2) allows to write

$$\frac{\rho_{\boldsymbol{x},T^*}(T,t)}{\rho_{\boldsymbol{x},T^*}(t,t)} = \exp\left\{-\int_t^T \left[\mu^1_{\boldsymbol{x},T^*}(t,s) - \mu^1_{\boldsymbol{x},T^*}(0,s)\right] ds\right\}.$$

Hence, $\partial_T \ln \rho_{\boldsymbol{x},T^*}(T,t) = -\left(\mu^1_{\boldsymbol{x},T^*}(t,T) - \mu^1_{\boldsymbol{x},T^*}(0,T)\right)$. By symmetry, we also have

$$\partial_T \ln \rho_{\boldsymbol{x},T^*}(t,T) = -\left(\mu^2_{\boldsymbol{x},T^*}(t,T) - \mu^2_{\boldsymbol{x},T^*}(0,T)\right).$$

Taking the sum, we can write

$$\frac{d}{dt} \ln \rho_{\boldsymbol{x},T^*}(t,t) = -\left(\mu^1_{\boldsymbol{x},T^*}(t,t) - \mu^1_{\boldsymbol{x},T^*}(0,t)\right) - \left(\mu^2_{\boldsymbol{x},T^*}(t,t) - \mu^2_{\boldsymbol{x},T^*}(0,t)\right),$$

and then

$$\ln \rho_{\boldsymbol{x},T^*}(t,t) = -\int_0^t \left[\left(\mu^1_{\boldsymbol{x},T^*}(s,s) - \mu^1_{\boldsymbol{x},T^*}(0,s)\right) + \left(\mu^2_{\boldsymbol{x},T^*}(s,s) - \mu^2_{\boldsymbol{x},T^*}(0,s)\right)\right] ds.$$

Therefore,

$$\begin{aligned}
&\ln \rho_{\boldsymbol{x},T^*}(T,t) = \ln \rho_{\boldsymbol{x},T^*}(t,t) - \int_t^T (\mu^1_{\boldsymbol{x},T^*}(t,s) - \mu^1_{\boldsymbol{x},T^*}(0,s))ds,\\
&= -\int_0^t \left[\left(\mu^1_{\boldsymbol{x},T^*}(s,s) - \mu^1_{\boldsymbol{x},T^*}(0,s)\right) + \left(\mu^2_{\boldsymbol{x},T^*}(s,s) - \mu^2_{\boldsymbol{x},T^*}(0,s)\right)\right] ds - \int_t^T (\mu^1_{\boldsymbol{x},T^*}(t,s) - \mu^1_{\boldsymbol{x},T^*}(0,s))ds\\
&= -\int_0^T \left(\mu^1_{\boldsymbol{x},T^*}(t\wedge s,s) - \mu^1_{\boldsymbol{x},T^*}(0,s)\right)ds - \int_0^t \left(\mu^2_{\boldsymbol{x},T^*}(t\wedge s,s) - \mu^2_{\boldsymbol{x},T^*}(0,s)\right)ds,
\end{aligned}$$

and $\ln \rho_{\boldsymbol{x},T^*}(t,T)$ is obtained by symmetry. We then obtain, combining with (1) and (2), the required result. □

The above result characterizes the conditional joint probability by specifying the forward mortality rates and avoiding to focus on the generalized Sibuya's function $\rho_{\boldsymbol{x}}$ since the dependence structure is implicitly embedded in $\mu^1_{\boldsymbol{x}}$ and $\mu^2_{\boldsymbol{x}}$. This allows for convenient and flexible choices of mortality models which can take into account common lifestyle and risk factors of the couple. For instance, we can use the class of affine processes which have attracted interest in applied research for finance as well as for insurance.

3.2 Affine Factor Model for Forward Mortality

In this section, we present the forward intensities of the couple by using a factor model where the homogeneous and heterogeneous factors are respectively described by affine processes. We suppose that each forward mortality is given by the following form

$$\mu^i_{\boldsymbol{x},t}(u,s) = \mu^i_{\boldsymbol{x}}(u,s) + Z^i_t,\ \ t\in[0,T^*] \quad i=1,2, \tag{3}$$

where $\mu_{\boldsymbol{x}}^i(t,s)$ is a deterministic function specified by an available mortality assumption[2], when doing realistic mortality projections of a population of insureds [24, 25]. Here, the processes $Z^i = (Z_t^i, t \geq 0)$, $i = 1, 2$, represent random departures from the initially chosen baseline which capture random fluctuations as well as systematic deviations.

The affine processes have been adopted in mortality modeling, for instance see [4, 13, 20, 23, 26]. Let

$$Z_t^i = \rho_i Y_t^0 + Y_t^i, \ t \in [0, T^*], \quad i = 1, 2, \tag{4}$$

where $\rho_i \in [-1, 1]$, and $Y^i = (Y_t^i, t \geq 0)$, $i = 1, 2, 3$, are independent affine processes. The process Y^0 represents the common environmental random factor for the couple, whereas Y^1 and Y^2 are individual factors.

Proposition 2. *Suppose that the factor processes are given by the SDEs*

$$dY_t^i = b_i(Y_t^i)dt + \sigma_i(Y_t^i)dW_t^i, \quad Y_0^i = y_i, \ i = 0, 1, 2,$$

where $W^i = (W_t^i, t \geq 0)$, $i = 0, 1, 2$, are independent Brownian motions and the coefficients b_i and σ_i are given by

$$\begin{cases} b_i(x) = \bar{K}_i + \hat{K}_i x, \\ \sigma_i^2(x) = \bar{H}_i + \hat{H}_i x. \end{cases}$$

Then the marginal survival probability is

$$\mathbb{P}_{\boldsymbol{x}}(\tau_i > t_i) = \exp\left(-\big(\gamma_i(0) + \gamma_0(0)\big) - \big(\beta_i(0) + \beta_0(0)\big) y_i - \int_0^{t_i} \mu_{\boldsymbol{x}}^i(0,s)ds \right), \quad i = 1, 2, \tag{5}$$

and the joint survival probability is

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2) = \exp\left(-\sum_{i=0}^{2} \big(\gamma_i(0) + \beta_i(0) y_i\big) - \int_0^{t_1} \mu_{\boldsymbol{x}}^1(s \wedge t_2, s)ds - \int_0^{t_2} \mu_{\boldsymbol{x}}^2(s \wedge t_2, s)ds \right) \tag{6}$$

where β_i and γ_i satisfy the ODEs

$$\begin{cases} \dot{\beta}_i(t) &= -\hat{K}_i \beta_i(t) - \frac{1}{2}\hat{H}_i \beta_i(t)^2 \\ \dot{\gamma}_i(t) &= -\bar{K}_i \beta_i(t) - \frac{1}{2}\bar{H}_i \beta_i(t)^2 \end{cases} \tag{7}$$

with boundary conditions $\beta_0(T^*) = \rho_1 t_1 + \rho_2 t_2$, $\beta_1(T^*) = t_1$, $\beta_2(T^*) = t_2$ and $\gamma_i(T^*) = 0$ for $i = 0, 1, 2$.

Proof. By Proposition 1, together with (3) and (4), we have the joint survival probability conditional on $\mathcal{F}_t$, for

[2] This can refer to a *best estimate* assumption on the evolution of mortality or a reference mortality [1].

$t \le t_1 \wedge t_2$ as follows

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2 | \mathcal{F}_t) = \mathbb{E}\left[\exp\left\{-Z^1_{T^*}t_1 - Z^2_{T^*}t_2 - \int_0^{t_1}\mu^1_{\boldsymbol{x}}(s\wedge t_2, s)ds - \int_0^{t_2}\mu^2_{\boldsymbol{x}}(s\wedge t_1, s)ds\right\}\Big|\mathcal{F}_t\right],$$

$$= \exp\left\{-\int_0^{t_1}\mu^1_{\boldsymbol{x}}(s\wedge t_2, s)ds - \int_0^{t_2}\mu^2_{\boldsymbol{x}}(s\wedge t_1, s)ds\right\}\mathbb{E}\left[\exp\left\{-Z^1_{T^*}t_1 - Z^2_{T^*}t_2\right\}\Big|\mathcal{F}_t\right], \tag{8}$$

$$= \exp\left\{-\int_0^{t_1}\mu^1_{\boldsymbol{x}}(s\wedge t_2, s)ds - \int_0^{t_2}\mu^2_{\boldsymbol{x}}(s\wedge t_1, s)ds\right\}\mathbb{E}\left[\exp\left\{-(\rho_1 Y^0_{T^*} + Y^1_{T^*})t_1 - (\rho_2 Y^0_{T^*} + Y^2_{T^*})t_2\right\}\Big|\mathcal{F}_t\right],$$

Then using classic results for affine processes (see for example [7]), we obtain the explicit formula for the conditional probability

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2|\mathcal{F}_t) = \exp\left(-\int_0^{t_1}\mu^1_{\boldsymbol{x}}(s\wedge t_2, s)ds - \int_0^{t_2}\mu^2_{\boldsymbol{x}}(s\wedge t_1, s)ds + \sum_{i=0}^{2}\Big(-\gamma_i(t) - \beta_i(t)Y^i_t\Big)\right)$$

which then implies the joint survival probability (6) and the marginal ones (5). □

When the processes Y^i_t follow specific affine diffusions such as in the Cox-Ingersoll-Ross (CIR) and Vasicek models, more explicit results can be obtained for the joint survival probability. We postpone the computations in Appendix A.

3.3 Life Status Scenarios and Joint-life Insurance Contracts

The dependence structure of the life status plays an important role in the evaluation. In the literature, the life status of the couple are often distinguished, see for example Norberg [21], Denuit and Cornet [6] who consider the four states according to the number and the spouses who are alive or dead. In the following, we pay a special attention to the conditional survival probability as well as the impact of the first death on the surviving spouse.

Recall that the bivariate life status at time t is described by $\mathcal{D}_t = \mathcal{D}^1_t \vee \mathcal{D}^2_t$ where $\mathcal{D}^i_t$ provides information about whether τ_i occurs before t and if so, the value of τ_i. The following result gives the survival probability of a spouse conditionally on different bivariate life status scenarios and the proof is based on [9, 10].

Proposition 3. *Denote by $S_{\boldsymbol{x},t}(t_1,t_2) := \mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2|\mathcal{F}_t)$ and suppose that $S_{\boldsymbol{x},t}(t_1,t_2)$ is of class $C^{1,1}$ w.r.t. (t_1,t_2) for all (ω,t). For any $t \le T$, we have*

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau_1>t,\tau_2>t\}}\frac{S_t(T,t)}{S_t(t,t)} + \mathbb{1}_{\{\tau_1>t,\tau_2\le t\}}\frac{\partial_2 S_t(T,\tau_2)}{\partial_2 S_t(t,\tau_2)}$$

and similarly

$$\mathbb{P}_{\boldsymbol{x}}(\tau_2 > T|\mathcal{G}_t) = \mathbb{1}_{\{\tau_1>t,\tau_2>t\}}\frac{S_t(t,T)}{S_t(t,t)} + \mathbb{1}_{\{\tau_1\le t,\tau_2>t\}}\frac{\partial_1 S_t(\tau_1,T)}{\partial_1 S_t(\tau_1,t)}.$$

Using the symmetry property of the life statuses, the above proposition implies immediately the survival probability of the last survivor of the couple conditionally on the first death occurrence.

We now consider the joint lives contracts introduced in Subsection 2.2, namely the first-to-die and last-to-die contracts.

(i) For the first-to-die contracts paying out a monetary amount at the first death of a spouse, i.e. $Y_T(\tau_1,\tau_2) = \mathbf{1}_{\{\tau_{(1)}\leq T\}}$. For at any $t<T$, we have

$$\mathbb{E}\left[\mathbf{1}_{\{\tau_{(1)}\leq T\}} \mid \mathcal{G}_t\right] = 1-\mathbf{1}_{\{\tau_{(1)}>t\}}\frac{S_{\boldsymbol{x},t}(T,T)}{S_{\boldsymbol{x},t}(t,t)}.$$

(ii) For the last-to-die contracts paying out a monetary amount at the first death of a spouse, i.e. $Y_T(\tau_1,\tau_2) = \mathbf{1}_{\{\tau_{(2)}\leq T\}}$. For any $t<T$, we have

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\{\tau_{(2)}\leq T\}} \mid \mathcal{G}_t\right] &= \mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\left(1-\frac{S_{\boldsymbol{x},t}(t,T)+S_{\boldsymbol{x},t}(T,t)-S_{\boldsymbol{x},t}(T,T)}{S_t(t,t)}\right)+\mathbf{1}_{\{\tau_1\leq t,\tau_2\leq t\}}\\ &+\mathbf{1}_{\{\tau_1>t,\tau_2\leq t\}}\left(1-\frac{\partial_2 S_{\boldsymbol{x},t}(t,\tau_2)}{\partial_1 S_{\boldsymbol{x},t}(t,\tau_2)}\right)+\mathbf{1}_{\{\tau_1\leq t,\tau_2>t\}}\left(1-\frac{\partial_1 S_{\boldsymbol{x},t}(\tau_1,t)}{\partial_1 S_{\boldsymbol{x},t}(\tau_1,t)}\right).\end{aligned}$$

4 Dependence Structure Between Spouses and Broken-Heart Syndrome

In life insurance, it is of common practice to rely on some *best estimate* assumptions on the individual mortality intensities. In such a context, the forward mortality rate is a deterministic function instead of a stochastic process. This corresponds, in our framework in Section 3, to the absence of random affine processes Z_t^1 and Z_t^2 driving the intensities.

In this section, we consider the deterministic mortality without influence of environmental information. Recall that the random vector $\boldsymbol{\tau}=(\tau_1,\tau_2)$ value in $\mathbb{R}_+^2$ describes the future lifetime of a married couple with $\boldsymbol{x}=(x_1,x_2)$, and $\tau_{(1)}=\min\{\tau_1,\tau_2\}$, $\tau_{(2)}=\max\{\tau_1,\tau_2\}$. We suppose that the available information is upon the occurrence of the first death and the deterministic assumption simply means that we relax the dependency on the filtration $\mathbb{F}$. Therefore, the expression (1) writes, for $t<T$,

$$\mathbb{P}_{\boldsymbol{x}}(\tau_i>T|\tau_{(1)}>t)=e^{-\int_t^T\mu_{\boldsymbol{x}}^i(t,s)ds},\quad i\in\{1,2\}, \tag{9}$$

where $\mu_{\boldsymbol{x}}^i(t,s)$ is a bivariate deterministic function with respect to t and s, the age vector $\boldsymbol{x}=(x_1,x_2)$ regarded as a fixed parameter. The assumptions in Subsection 3.1 and the obtained results therein remain to be the same provided that the dependency on the filtration $\mathbb{F}$ is removed. Among others, we write the joint probability $\mathbb{P}_{\boldsymbol{x}}(\tau_1>t_1,\tau_2>t_2)$ similar to (2) as the product of two marginal probabilities and the Sibuya's function $\rho_{\boldsymbol{x}}(t_1,t_2)$ which represents the dependency between the spouses. More precisely $\mathbb{P}_{\boldsymbol{x}}(\tau_1>t_1,\tau_2>t_2)=\mathbb{P}_{\boldsymbol{x}}(\tau_1>t_1)\mathbb{P}_{\boldsymbol{x}}(\tau_2>t_2)\rho_{\boldsymbol{x}}(t_1,t_2)$. Then we get the joint probability under (9).

Proposition 4. *If $\rho_{\boldsymbol{x}}(t_1,t_2)$ is of class $C^{1,1}$, then the joint probability is given by*

$$\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2) = \exp\left(-\int_0^{t_1} \mu^1_{\boldsymbol{x}}(s\wedge t_2, s)ds - \int_0^{t_2} \mu^2_{\boldsymbol{x}}(s\wedge t_1, s)ds\right). \tag{10}$$

Suppose in addition $\mu^i_{\boldsymbol{x}}(t,s)$, $i=1,2$, is of class $C^{1,1}$, and let $\mu^i_{\boldsymbol{x}}(t,s) = \mu^i_{\boldsymbol{x}}(0,s) - \int_0^t \varphi^i_{\boldsymbol{x}}(u,s)du$. Then the Sibuya's function is given by

$$\rho_{\boldsymbol{x}}(t_1,t_2) = \exp\left\{\int_0^{t_1}\int_0^{t_1\wedge t_2} \varphi^1_{\boldsymbol{x}}(u,s)duds + \int_0^{t_2}\int_0^{t_1\wedge t_2} \varphi^2_{\boldsymbol{x}}(u,s)duds\right\}. \tag{11}$$

Moreover, it holds $\varphi^1_{\boldsymbol{x}}(t,t) = \varphi^2_{\boldsymbol{x}}(t,t)$.

Proof. The equality (10) is a direct consequence of Proposition 1. By replacing μ^i in (10) with the integral form and taking integration by part, we get

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2) = \exp\Bigg(&-\int_0^{t_1} \mu^1_{\boldsymbol{x}}(0,s)ds - \int_0^{t_2} \mu^2_{\boldsymbol{x}}(0,s)ds \\ &+ \int_0^{t_1}\int_0^{s\wedge t_2} \varphi^1_{\boldsymbol{x}}(u,s)duds + \int_0^{t_2}\int_0^{s\wedge t_1} \varphi^2_{\boldsymbol{x}}(u,s)duds\Bigg).\end{aligned} \tag{12}$$

and then (11). When $t_1 \geq t_2$, $\partial^2_{1,2}\ln\rho_{\boldsymbol{x}}(t_1,t_2) = -\partial_2\mu^1_{\boldsymbol{x}}(t_2,t_1)$, and when $t_1 \leq t_2$, $\partial^2_{1,2}\ln\rho_{\boldsymbol{x}}(t_1,t_2) = -\partial_1\mu^2_{\boldsymbol{x}}(t_1,t_2)$. Then

$$\partial^2_{1,2}\ln\rho_{\boldsymbol{x}}(t,t) = -\partial_2\mu^1_{\boldsymbol{x}}(t,t) = -\partial_1\mu^2_{\boldsymbol{x}}(t,t),$$

which implies $\varphi^1_{\boldsymbol{x}}(t,t) = \varphi^2_{\boldsymbol{x}}(t,t)$.

□

Corollary 4.1. *If $\rho_{\boldsymbol{x}}(t_1,t_2) \in C^{\{1,1\}}$, and so are $\mu^1_{\boldsymbol{x}}(t,s)$ and $\mu^2_{\boldsymbol{x}}(t,s)$, then*

$$\mu^i_{\boldsymbol{x}}(t,s) = \mu^i_{\boldsymbol{x}}(0,s) - \int_0^t \varphi_{\boldsymbol{x}}(u,s)du, \tag{13}$$

where $\varphi_{\boldsymbol{x}}(t,t) = \partial^2_{1,2}|_{t_1=t_2=t}\ln\rho_{\boldsymbol{x}}(t_1,t_2)$. In addition, we obtain

$$\begin{aligned}\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2) = \exp\Bigg(&-\int_0^{t_1} \mu^1_{\boldsymbol{x}}(0,s)ds - \int_0^{t_2} \mu^2_{\boldsymbol{x}}(0,s)ds \\ &+ \int_0^{t_1}\int_0^{s\wedge t_2} \varphi_{\boldsymbol{x}}(u,s)duds + \int_0^{t_2}\int_0^{s\wedge t_1} \varphi_{\boldsymbol{x}}(u,s)duds\Bigg).\end{aligned} \tag{14}$$

Proof. Notice that from Proposition 4 when $t_1 \geq t_2$, $\partial^2_{t_1,t_2}\ln\rho_{\boldsymbol{x}}(t_1,t_2) = -\frac{\partial}{\partial t_2}\mu^1_{\boldsymbol{x}}(t_2,t_1)$, and when $t_1 \leq t_2$, $\partial^2_{t_1,t_2}\ln\rho_{\boldsymbol{x}}(t_1,t_2) = -\frac{\partial}{\partial t_1}\mu^2_{\boldsymbol{x}}(t_1,t_2)$, Then we have

$$\partial^2_{1,2}|_{t_1=t_2=t}\ln\rho_{\boldsymbol{x}}(t_1,t_2) = -\frac{\partial}{\partial t_2}\mu^1_{\boldsymbol{x}}(t_2,t_1) = -\frac{\partial}{\partial t_1}\mu^2_{\boldsymbol{x}}(t_1,t_2).$$

By the definition of φ, we have $-\frac{\partial}{\partial t_2}\mu^1_{\boldsymbol{x}}(t_2,t_1) = \varphi_{\boldsymbol{x}}(t_2,t_1)$, and $-\frac{\partial}{\partial t_1}\mu^2_{\boldsymbol{x}}(t_1,t_2) = \varphi_{\boldsymbol{x}}(t_1,t_2)$, which implies immediately (13). By replacing μ^i in Proposition 4 with the integral form (13) and taking integration by part, we get (14). ■

Remark 1. *We obtain, as a direct consequence of (14), the explicit form of $\rho_{\boldsymbol{x}}(t_1,t_2)$ given by*

$$\rho_{\boldsymbol{x}}(t_1,t_2) = \exp\left\{\int_0^{t_1}\int_0^{t_1\wedge t_2}\varphi_{\boldsymbol{x}}(u,s)duds + \int_0^{t_2}\int_0^{t_1\wedge t_2}\varphi_{\boldsymbol{x}}(u,s)duds\right\}.$$

Therefore, the function φ plays an important role in determining the correlation structure of default times.

1. *The survival probability should be decreasing with respect to time, which implies*

$$\frac{\partial}{\partial t_i}\ln\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2) \leq 0, \quad \forall t_1, t_2 \in [0,T].$$

2. *The probability density function should be positive, which implies*

$$\frac{\partial^2}{\partial t_1\partial t_2}\mathbb{P}_{\boldsymbol{x}}(\tau_1 > t_1, \tau_2 > t_2) \geq 0.$$

Therefore, a model consistent $\mu^i_{\boldsymbol{x}}(0,t)$ and $\varphi_{\boldsymbol{x}}(t_1,t_2)$ should verify for any $(t_1,t_2) \in [0,+\infty)^2$ the following conditions

$$\begin{cases} \int_0^{t_2}\varphi_{\boldsymbol{x}}(t_1\wedge s_2, s_2\vee t_1)ds_2 - \mu^1_{\boldsymbol{x}}(0,t_1) \leq 0, \\ \int_0^{t_1}\varphi_{\boldsymbol{x}}(s_1\wedge t_2, t_2\vee s_2)ds_1 - \mu^2_{\boldsymbol{x}}(0,t_2) \leq 0, \\ \left(\int_0^{t_2}\varphi_{\boldsymbol{x}}(t_1\wedge s_2, s_2\vee t_1)ds_2 - \mu^1_{\boldsymbol{x}}(0,t_1)\right)\left(\int_0^{t_1}\varphi_{\boldsymbol{x}}(s_1\wedge t_2, t_2\vee s_2)ds_1 - \mu^2_{\boldsymbol{x}}(0,t_2)\right) \\ \qquad\qquad + \varphi_{\boldsymbol{x}}(t_1\wedge t_2, t_2\vee t_1) \geqslant 0 \end{cases} \tag{15}$$

Remark 2. *In Equation (9), we considered a deterministic intensity which is motivated by operational uses. In fact, the simplest specification in real world application consists in considering an exponentially distributed remaining life times with a constant parameter. However, this is very restrictive and not consistent with the phenomenon of interest in the sense that a constant intensity does not have an aging property, i.e., the age of the considered individual has no effect on his or her residual lifetime. Therefore, there is various generalisations of the exponential model, making it possible to obtain increasing hazard functions with regards to the age. For instance, we can refer the well-celebrated parametric forms of Weibull, Gompertz and Makeham [22].*

We now focus on the first death and its impact on the surviving spouse. When the first bereavement occurs, combining Proposition 3 and (14), we can derive the conditional survival probability.

Proposition 5. *For any $0 \le t \le T$, the conditional survival probability of the individual i of the couple $(i = 1, 2)$ is given by*

$$\mathbb{P}(\tau_i > T|\mathcal{D}_t) = \mathbb{1}_{\{\tau_{(1)}>t\}} \exp\left(-\int_t^T \mu^i_{\boldsymbol{x}}(t,s)ds\right) + \mathbb{1}_{\{\tau_i>t,\tau_j\le t\}} \exp\left(-\int_t^T \mu^i_{\boldsymbol{x}}(\tau_{(1)},s)ds\right)\left(1 - \frac{\int_t^T \varphi^i_{\boldsymbol{x}}(\tau_{(1)},s)ds}{\mu^j_{\boldsymbol{x}}(\tau_{(1)},\tau_{(1)}) - \int_{\tau_{(1)}}^t \varphi^i_{\boldsymbol{x}}(\tau_{(1)},s)ds}\right).$$

Proof. We have $\mathbb{P}(\tau_i > T|\mathcal{D}_t) = \mathbb{1}_{\{\tau_{(1)}>t\}}\mathbb{E}[\mathbb{1}_{\{\tau_i>T\}}|\mathcal{D}_t] + \mathbb{1}_{\{\tau_i>t,\tau_j=\tau_{(1)}\le t\}}\mathbb{E}[\mathbb{1}_{\{\tau_i>T\}}|\mathcal{D}_t]$. Thus, by using the joint survival probability function given by Equation (14) and supposing that $\mathbb{F}$ is a trivial filtration in Proposition 3, we obtain the results by direct computations. □

Hence, in order to characterize the broken-heart syndrome, we focus on the jump in the mortality of one spouse given the bereavement of the other. Let us consider the implied hazard rate, denoted $b^i(t,T)$, of the surviving spouse, which is defined as

$$\begin{aligned} b^i_{\boldsymbol{x}}(t,T) &= -\partial_T \ln \mathbb{P}(\tau_i > T|\mathcal{D}_t), \\ &= \mu^i_{\boldsymbol{x}}(0,T) - \int_0^{t\wedge\tau_{(1)}} \varphi_{\boldsymbol{x}}(u,T)du + \mathbb{1}_{\{\tau_{(1)}\le t\}}\left(\frac{\varphi_{\boldsymbol{x}}(\tau_{(1)},T)}{\mu^j_{\boldsymbol{x}}(0,\tau_{(1)}) - \int_0^t \varphi_{\boldsymbol{x}}(u\wedge\tau_{(1)},\tau_{(1)}\vee u)du}\right) \end{aligned} \tag{16}$$

where the second equality is deduced from Proposition 5. Before the first death, it equals $\mu^i_{\boldsymbol{x}}(t,T)$, and after the first default occurs, we observe that if $\varphi > 0$ (resp. $\varphi < 0$), there exists a positive (resp. negative) jump at the first death time $t = \tau_{(1)}$. The magnitude of this jump is then given by

$$\Delta b^i_{\boldsymbol{x}}(t,T) = \mathbb{1}_{\{\tau_{(1)}\le t\}}\frac{\varphi_{\boldsymbol{x}}(\tau_{(1)},T)}{\mu^j_{\boldsymbol{x}}(0,\tau_{(1)}) - \int_0^t \varphi_{\boldsymbol{x}}(u\wedge\tau_{(1)},\tau_{(1)}\vee u)du}. \tag{17}$$

This gives an explicit measure of the hazard rate change depending on the age $\boldsymbol{x}$ of the couple, which is appropriate to capture the impact of the first death. Indeed, as noted by Dufresne *et al.* [8] not only the initial ages but also the age difference are key arguments characterizing the broken-heart syndrome [31]. Their results show that a model accounting for this aspect captures some additional association between lifetime of the spouses that would not be reflected in a model without age difference. In Equation (17), this can be accommodated by specifying the density φ as a function of the ages, the differences between the ages and other relevant combinations. However, unlike the common interpretation of the broken-heart syndrome, the magnitude $\Delta b^1_{\boldsymbol{x}}(t,T)$ depends on a *maturity* date T. Therefore, when dealing with such a quantity one should interpret (17) as the impact over a period $[t,T]$. In other words, the jump can be interpreted as the survivor mortality shift over the period $[t,T]$ due to the death of a spouse. Its behavior will be investigated numerically in the following section.

5 Numerical illustrations

In this section, we present some numerical examples for illustration. To begin with, we consider the case where mortality intensity of the spouses is deterministic for the sake of readability, see Section 4. Formally, based on Remark 2, we will focus on an example widely in use actuarial and demographic applications and assume that the intensity of each individual is described using a Gompertz parametric form [30]. Subsequently, we study some explicit examples of the above introduced framework, which will illustrate the choice of the function $\varphi_{\boldsymbol{x}}(t,s)$ defined in Theorem 4.1. Recall that this will characterize the dependence structure as well as the impact of the first death on the intensity of survivor. In order to inspect the impact of the dependence structure $\rho_{\boldsymbol{x}}(t,s)$ which is characterized by a parameter α in the following examples, we rather look at the linear correlation between the individuals remaining lifetimes. This will ease the understanding of the dependence making it possible to compare the outputs with the literature. Recall that the function characterizing the linear correlation between two individuals can be defined as the corresponding correlation between $\mathbb{1}_{\{\tau_1>T\}}$ and $\mathbb{1}_{\{\tau_2>T\}}$, i.e.

$$\rho = \frac{\text{Cov}[\mathbb{1}_{\{\tau_1>T\}}, \mathbb{1}_{\{\tau_2>T\}}]}{\sqrt{\text{Var}[\mathbb{1}_{\{\tau_1>T\}}]}\sqrt{\text{Var}[\mathbb{1}_{\{\tau_2>T\}}]}}. \tag{18}$$

We use marginal distributions that are of Gompertz form [3]: $\mu^1_{\boldsymbol{x}}(0,t) = e^{h_1(x_1+t)}$ and $\mu^2_{\boldsymbol{x}}(0,t) = e^{h_2(x_2+t)}$, where h_i for $i = 1,2$ are constants. Unlike the common use of such distributions in bivariate mortality modelling, we rely on the framework introduced above to assess the impact of the dependence in the spouses mortality. In fact, in many applications, the Gompertz law is used for the marginal before fitting a copula to link the two residual lifetimes. In our case, as noted in the previous sections, the dependence structure defined in Theorem 4.1 acts on a multiplicative manner, which is specified also in item 1, allowing for a more tractable specification of the dependency. Lastly, we will consider the affine framework introduced in Subsection 3.2 to analyse the effect of the dependence on prices at the contract issuing as well as on the premium updating during the life of contract.

5.1 Dependence Structure and Impact on the Intensities

Example 1. *Let $\varphi_{\boldsymbol{x}}(t,s) = \alpha$ where α is a constant. Recall that $\varphi_{\boldsymbol{x}}(t,s) = \partial^2_{1,2}|_{t_1=t_2=t} \ln \rho_{\boldsymbol{x}}(t_1,t_2)$, see Theorem 4.1 and Remark 1, which allows using Equation (14), to write the joint survival probability as follows*

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \exp\left(- \int_0^{t_1} \mu^1_{\boldsymbol{x}}(0,s)ds - \int_0^{t_2} \mu^2_{\boldsymbol{x}}(0,s)ds + \alpha t_1 t_2 \right).$$

Note that the dependence between the individuals is increasing exponentially over time. Moreover, in such a case, the constraints (15) satisfied by the parameter α can be rewritten as

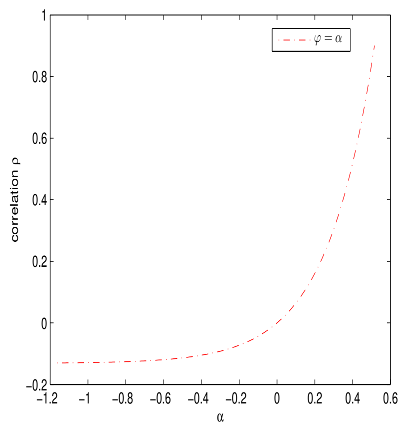
$$-\mu^1_{\boldsymbol{x}}(0,T)\mu^2_{\boldsymbol{x}}(0,T) \leq \alpha \leq \frac{\min\left(-\mu^1_{\boldsymbol{x}}(0,T), -\mu^2_{\boldsymbol{x}}(0,T)\right)}{T}.$$

Hence, using (16), the implied hazard rate is given by

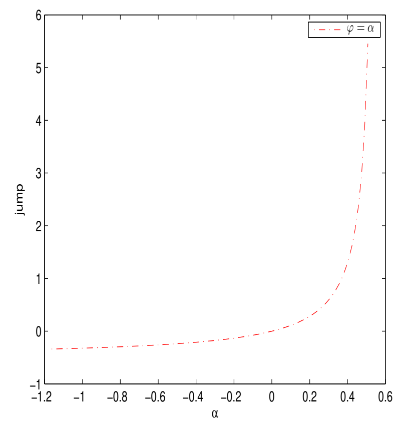
$$b^i(t,T) = \mu^i_{\boldsymbol{x}}(0,T) - \alpha(t \wedge \tau_{(1)}) + \mathbb{1}_{\{\tau_{(1)} \leq t\}} \frac{\alpha}{\mu^j_{\boldsymbol{x}}(0,\tau_{(1)}) - \alpha T},$$

and the jump defined in Equation (17) at the death of the first individual $t = \tau_{(1)}$ can be written as follows

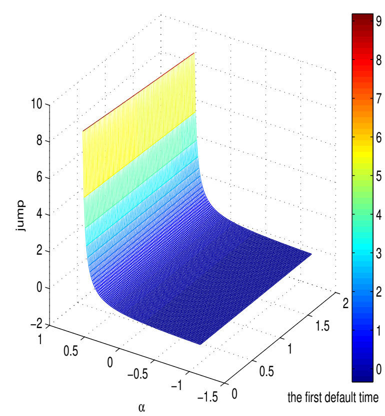
$$\Delta b^i(t,T) = \frac{\alpha}{e^{h_j(x_j+\tau_{(1)})} - \alpha T}.$$



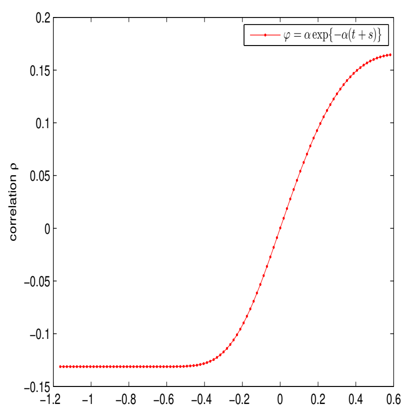
(a) ρ vs α



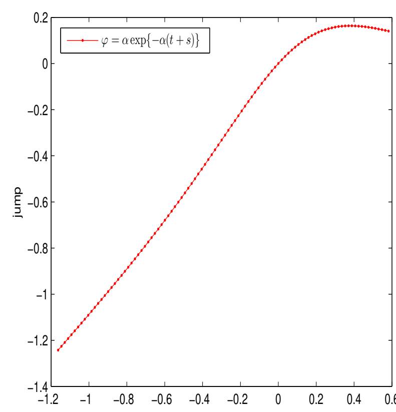
(b) $\Delta b^1(t,T)$ vs α



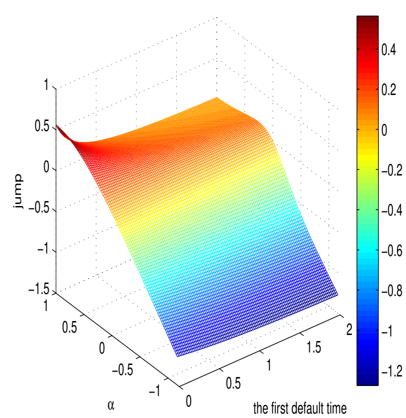
(c) $\Delta b^1(t,T)$ vs $\tau_{(1)}$ and α



(d) ρ vs α



(e) $\Delta b^1(t,T)$ vs α



(f) $\Delta b^1(t,T)$ vs $\tau_{(1)}$ and α

Figure 1: (Gompertz Law) Numerical illustration of Example 1 (top) and Example 2 (bottom) showing the relationship between the correlation ρ in Equation (18), the jump size $\Delta b^1(t,T)$, and the dependence function $\varphi(\boldsymbol{x},t,s)$ as well as the time of occurrence of the first death $\tau^{(1)}$.

Example 2. *Let $\varphi_{\boldsymbol{x}}(t,s) = \alpha \exp\left(-\alpha(t+s)\right)$ where α is a constant parameter. When $\alpha \neq 0$, using (14), we have*

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \exp\left(-\int_0^{t_1} e^{h_1(x_1+s)}ds - \int_0^{t_2} e^{h_2(x_2+s)}ds + \frac{1}{\alpha}e^{-\alpha(t_1+t_2)} - \frac{1}{\alpha}e^{-\alpha t_1} - \frac{1}{\alpha}e^{-\alpha t_2} + \frac{1}{\alpha}\right).$$

When $\alpha \to 0$, taking the limit in the above expression gives rise to

$$\lim_{\alpha \to 0} \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \exp\left(-\int_0^{t_1} e^{h_1(x_1+s)}ds - \int_0^{t_2} e^{h_2(x_2+s)}ds\right).$$

Then the implied hazard rate is given by

$$b^i(t,T) = e^{h_i(x_i+T)} - e^{-\alpha T}\left(e^{-\alpha(t\wedge\tau_{(1)})} - 1\right) + \mathbb{1}_{\{\tau_{(1)}\leq t\}} \frac{\alpha e^{-\alpha\left(T+\tau_{(1)}\right)}}{e^{h_j(x_j+\tau_{(1)})} - e^{-\alpha\tau_{(1)}}\left(e^{-\alpha T} - 1\right)},$$

and the jump in the intensity of the survivor at the first death can be written as follows

$$\Delta b^i(t,T) = \frac{\alpha e^{-\alpha(T+\tau_{(1)})}}{e^{h_j(x_j+\tau_{(1)})} - e^{-\alpha\tau_{(1)}}\left(e^{-\alpha T} - 1\right)}.$$

In Examples 1 and 2, we depict the impact of the above dependence assumptions on different quantities of interest. For illustration, we let $h_1 = 0.01, h_2 = 0.012, x_1 = 45$, and $x_2 = 55$. First of all, for Example 1, in Figure 1(a) and Figure 1(b), we depict the evolution of the linear correlation and the jumps size as a function of the parameter α. For Example 2, the corresponding behaviours are presented in Figure 1(d) and Figure 1(e). We notice that for these examples, the correlation and the jump size are increasing according the parameter α. This is, by the definition of the correlation function, an intuitive behaviour as discussed earlier. Indeed, it does increase with respect to α in Figure 1(a). It is also clear form Figure 1(a), which depicts the evolution of the linear correlation ρ in (18), for fixed α, that for a fixed correlation ρ, the jump size is increasing with respect to the first default time. Regarding the first example, we can notice for high levels of α, that we are approaching a perfect collinearity between the two life statuses. However, this should not be coherent regarding some empirical analyses available in the literature. In fact, as noted by the empirical study of Frees *et al.* [11], the time of death of the paired lives are highly correlated but would not be greater than 41%. Although this is based on a specific real world dataset, it should be taken into account when establishing the optimal values for the parameter. More specifically, one should consider a fitting procedure that can be used to estimate α. However, this is not in the scope of this paper and it will be explored on future works. Moreover, we can observe reactions and correlation in different directions. In fact, as pointed out by Gourieroux and Lu [12], this arises when an individual is devastated by the death of his spouse, with an increase of his mortality intensity, whereas, in other cases, the death of the spouse may provide more freedom to the surviving spouse and possibly a decrease of his mortality rate. Regarding the jump in the intensity, Figure 1(b) and 1(e) depict the evolution of this effect for different values of α. We remark that the intensity reacts also in different directions depending on the values of the parameter α, which can replicate the observed phenomenon discussed earlier [12]. We should note that Example 2 produces more coherent values which tends to be in line with the empirically estimated values in some real-life portfolios as in Lu [18].

On the other hand, Figures 1(c) and 1(f) depict the combined impact on the mortality of the survivor in terms of the dependence parameter α and the time of occurrence of the first death. We pay a particular attention to the impact of the timing of the first death. In fact, in these figures, we see that the parametrization of the dependence has not the same outputs. In these figures, we reported the direct impact on the intensity. However, one should consider the residual impact on the intensity at the considered age. In other words, the quantity of interest is $\Delta b^i_x(t,T)/\mu^i_{\boldsymbol{x}}(t,T)$, which indicates the impact of the first death in the intensity of the survivor, which depends on the attained age. Therefore, the impact of the timing of the first death should take into account the age of the survivor at the occurrence. This is accommodated for in the above examples as the jump $\Delta b^i_x(t,T)$ does incorporate the term $x_j + \tau_{(1)}$. However, to better understand it, we should recall that the jump is the accumulated effect over the period. Of course, the constant dependence in Example 1 does take into account this effect as one can confirm from its expression in Example 1 and Figure 1(c). The intensity's jump in this case is slightly decreasing as the spouses live longer together. In Figure 2, we reported the relative impact $\Delta b^1_x(t,T)/\mu^i_{\boldsymbol{x}}(t,T)$ in terms of the timing of the first death. We can see that the impact of the first death on the intensity increases with the timing of the first death. This is a desired effect as long as late deaths infer on a long common life. Thus, the individuals living longer together are more likely to develop and share common lifestyles.

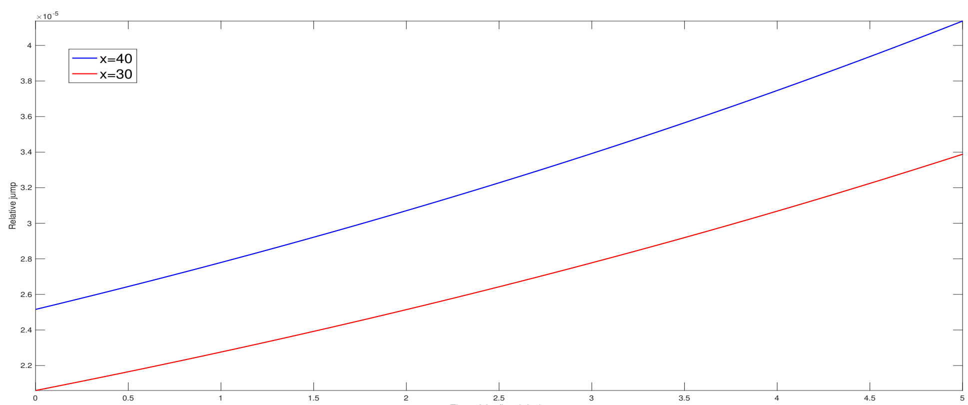
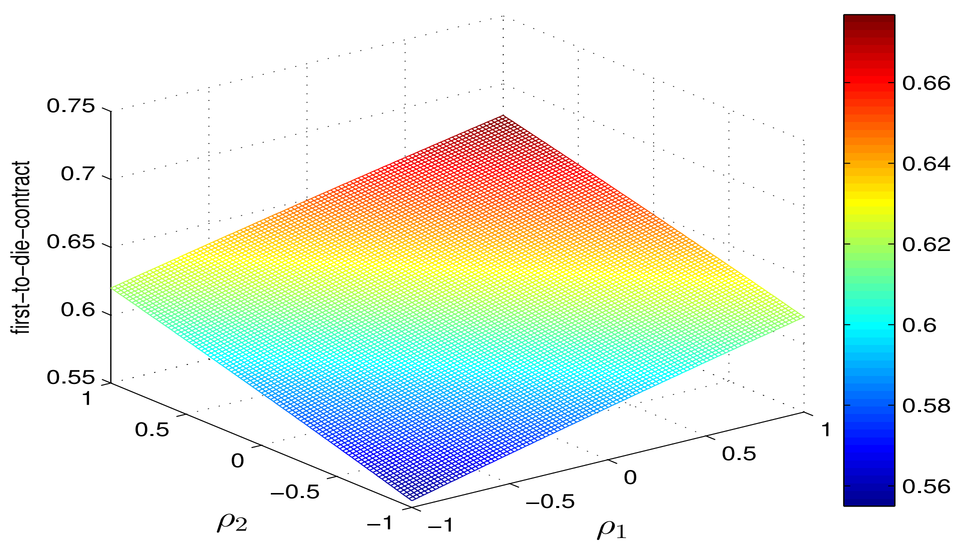


Figure 2: (Gompertz Law) Numerical illustration of the relative impact of the time of the first on the intensity of the survivor for different ages $x = 30$ and 40 using the illustrative Example 2.

5.2 Dependence Structure and Impact on the Joint-Life Contracts Pricing

In this section, we will focus on the pricing formulas for joint lives policies derived in Subsection 3.3. In particular, we consider the first-to-die and last-to-die (last survivor) contracts, see Subsection 2.2. For simplicity of illustration, we suppose that the interest rate is zero. Here, we consider dynamic intensities using the affine framework introduced in Subsection 3.2. Formally, we assume that Y^i_t, $(i = 0, 1, 2)$ are CIR processes as shown in Subsection A.2. We recall that each intensity is given by $\mu^i_{\boldsymbol{x},T^*}(t,s) = \mu^i_{\boldsymbol{x}}(t,s) + Z^i_{T^*}$,$i = 1, 2$, where $\mu^i_{\boldsymbol{x}}(t,s) = e^{h_i(x_i+s)} - \alpha t$, and Z^i_t is the stochastic factor driving the intensity and it is given in terms of the CIR processes Y^i_t as $Z^i_t = \rho_i Y^0_t + Y^i_t$ where $\rho_i \in [-1, 1]$. Henceforth, we analyse the effect of the dependency on prices at the contract inception time as well as on the premium updating during the life of the contract. To this end, for the first-to-die policies, we will consider the input parameters $x_1 = 45, x_2 = 50$, $h_1 = -0.025, h_2 = -0.015$, $y_0 = 0.08$, $a_0 = 0.4$, $b_0 = 0.08$, $\sigma_0 = 0.06$, $y_1 = 0.08$, $a_1 = 0.5$, $b_1 = 0.1$, $\sigma_1 = 0.07$, $y_2 = 0.09$, $a_2 = 0.6$, $b_2 = 0.12$, $\sigma_2 = 0.08$, $\alpha = -5 \times 10^{-4}$ and $T^* = 50$ and a policy with maturity $T = 2$. We further assume that the first death occurs after t and look at the impact of the correlation.

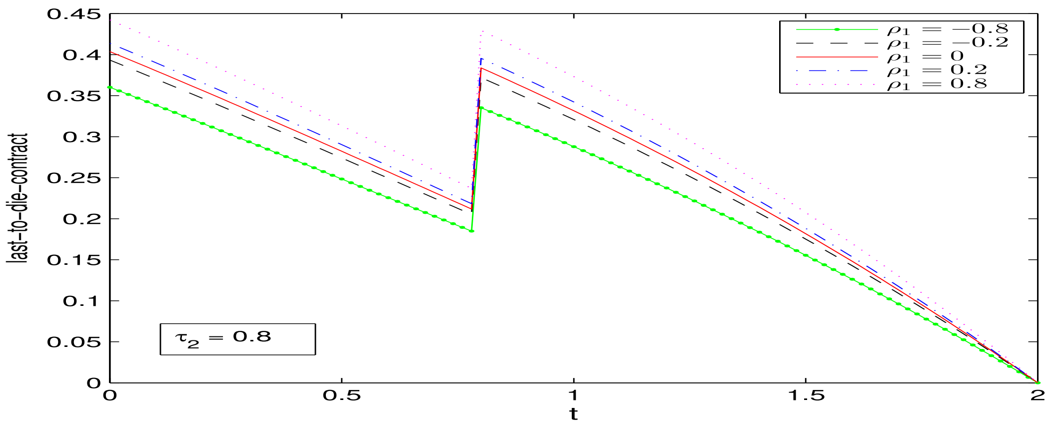
First, we consider the price of the first-to-die contract at time $t = 1$. It is depicted in Figure 3(a), where we show the relationship between the price of such a contract and parameters ρ_1 and ρ_2. In this case, we are considering the sole impact of the correlation on the price as soon as the broken-heart syndrome is not affecting the contract. First, notice that when ρ_1 increases, the conditional expectation of first death conditionally on $\mathcal{G}_t$ also increases. This is reasonable since the environmental noise $Z_t^1 = \rho_1 Y_t^0 + Y_t^1$ where $\rho_1 \in [-1, 1]$ increases with ρ_1, owing to the fact that here Y_t^1 is a CIR process which is always positive, see Appendix A. Intuitively, the greater the environmental risk, the higher the probability of the first death before T. Moreover, we depict in Figure 3(a) the relationship between the price of the first-to-die contract and parameter ρ_1 and ρ_2 before the first death occurs. We can see that the price is decreasing as the ρ_1 and ρ_2 decrease.



(a) price vs ρ_1 and ρ_2

Figure 3: (First-to-Die Contract) Numerical illustration of the price of firs-to-die contracts using the affine framework introduced in Subsection 3.2 and CIR process detailed in Appendix A. Here, we have $\mu^i_{\boldsymbol{x},T^*}(t, s) = \mu^i_{\boldsymbol{x}}(t, s) + Z^i_{T^*}, i = 1, 2$, where $\mu^i_{\boldsymbol{x}}(t, s) = e^{h_i(x_i+s)} - \alpha t$, and Z^i_t is the stochastic factor driving the intensity and it is given in terms of the CIR processes Y^i_t as $Z^i_t = \rho_i Y^0_t + Y^i_t$ where $\rho_i \in [-1, 1]$.

Another example of interest is the last-to-die contract paying out a lump sum at the death of the last survivor. In this case, we are interest both on the impact of correlation on the price as well as the broken-heart syndrome. Figure Figure 4(a) shows the evolution of the price of the contract during the life of the contract. As noted by Gourieroux and Lu [12], this is a key element to take into account to correctly represent the contract on the insurer's balance



(a) price vs t

Figure 4: (Last-to-Die Contract) Numerical illustration of the price of firs-to-die contracts using the affine framework introduced in Subsection 3.2 and CIR process detailed in Appendix A. Here, we have $\mu^i_{T^*}(\boldsymbol{x}, t, s) = \mu^i(\boldsymbol{x}, t, s) + Z^i_{T^*}, i = 1, 2$, where $\mu^i(\boldsymbol{x}, t, s) = e^{h_i(x_i+s)} - \alpha t$, and Z^i_t is the stochastic factor driving the intensity and it is given in terms of the CIR processes Y^i_t as $Z^i_t = \rho_i Y^0_t + Y^i_t$ where $\rho_i \in [-1, 1]$.

sheet, or, if it is securitized, to evaluate the price of the corresponding component. Moreover, the evolution of the price over time is a key element when assessing the so-called solvency capital requirement under the prudential regulation. For illustration, we suppose that the first death occurs at time $\tau_{(1)} = \tau_1 = 0.8$ and assess its impact on the price. Figure 4(a) shows that before the occurrence of the first death, the greater ρ_1, the greater the price of the last-to-die contracts. Hence, after the first death $\tau_{(1)} = \tau_1 = 0.8$ occurs, for different ρ_1, the conditional expectation of the last-to-die contracts on $\mathcal{G}_t$ are following the same pattern with the same impact of the correlation. In fact, as the death time $\tau_{(1)}$ of the first individual the intensity of the survivor jumps with a magnitude that depends on the age and the correlation factor ρ_1, see Proposition 5. The impact of this phenomenon is also observed in the prices as shown in Figure 4(a).

6 Concluding Remarks

This paper introduces a new framework for analyzing the mortality dependence between individuals in a couple. This framework is intended to study the dependence relation that describes the joint mortality of married couples in terms of marginal mortality rates. In fact, It has long been documented that the death of a spouse does impact the mortality of the surviving spouse and this causal effect is known as the "broken-heart syndrome". In addition, the spouses are also impacted by their common lifestyle which induces a spurious correlation. In the actuarial literature, the common methods to handle such a phenomenon are copulas and Markov approaches [12, 19, 29].

The approach developed in this paper aims to reconcile the two above mentioned methods. It is related to the recent advances in credit risk modelling and proposes a framework for assessing and modelling the dynamic of such a joint survival probability. We consider the (conditional) joint survival probability under different scenarios of life statuses of the couple, notably, on the sets before and after the first death respectively. We characterize the dynamics of the involved intensities and quantify the impact of the first death on the intensity of the survivor. We also incorporate some illustrative variants which allow to take into account some main features driving the broken-heart syndrome, such as the gender, the age of the spouses, etc.

Beyond the theoretical establishment of the model, an interesting question is related to the statistical investigation and estimation of the model. In particular, we need to adapt the model to specific challenges encountered in real-world datasets. For instance, among other things, one needs to develop estimation procedures in presence of censored data and ways to estimate the correlation between the married lives. This is, however, beyond the scope of the current paper. Also, the choice of the parametrization remains an open question and will be investigated in future works.

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A Examples of Affine Processes

A.1 Vasicek Process

We suppose that stochastic processes Y_t^i where $i \in \{0,1,2\}$ follow Vasicek model whose dynamics are given by

$$dY_t^i = (b_i + c_i Y_t^i)dt + \sigma_i dW_t^i, \;\; Y^i(0) = y_i,$$

where b_i, c_i and σ_i are some constant parameters. Then, for $i = 1, 2$, the Riccati ODE system (7) in Subsection 3.2 can be rewritten as

$$\begin{cases} \dot{\beta}_i(t) = -c_i \beta_i(t), & \text{with } \beta_i(T^*) = t_i, \\ \dot{\gamma}_i(t) = -\frac{1}{2}\sigma_i^2 \beta_i^2(t) - b_i \beta_i(t), & \text{with } \gamma_i(T^*) = 0. \end{cases}$$

Then, we can easily derive the explicit parameters as follows

$$\beta_i(t) = t_i e^{c_i(T^*-t)}, \tag{19}$$

$$\gamma_i(t) = -\frac{b_i t_i}{c_i}\left(1 - e^{c_i(T^*-t)}\right) - \frac{\sigma_i^2 t_i^2}{4c_i}\left(1 - e^{2c_i(T^*-t)}\right). \tag{20}$$

For $i = 0$, we have the following ODEs

$$\begin{cases} \dot{\beta}_0(t) = -c_0 \beta_0(t), & \text{with } \beta_0(T^*) = \rho_1 t_1 + \rho_2 t_2, \\ \dot{\gamma}_0(t) = -\frac{1}{2}\sigma_0^2 \beta_0^2(t) - b_0 \beta_0(t), & \text{with } \gamma_0(T^*) = 0, \end{cases}$$

which similarly give arise the explicit solutions

$$\beta_0(t) = e^{c_0(T^*-t)}(\rho_1 t_1 + \rho_2 t_2), \tag{21}$$

$$\gamma_0(t) = -\frac{b_0(\rho_1 t_1 + \rho_2 t_2)}{c_0}\left(1 - e^{c_0(T^*-t)}\right) - \frac{\sigma_0^2(\rho_1 t_1 + \rho_2 t_2)^2}{4c_0}\left(1 - e^{2c_0(T^*-t)}\right). \tag{22}$$

Then we easily have the joint (conditional) survival probability (8) and (**??**) in Subsection 3.2. We should, therefore, substitute $\gamma_i(0)$ and $\beta_i(0)$ by the above corresponding forms in Equations (19), (20), (21) and (22).

A.2 Cox-Ingersoll-Ross Process

We suppose that processes Y_t^i where $i \in \{0, 1, 2\}$ follow Cox-Ingersoll-Ross (CIR) model and are described as solution of the following SDE

$$dY_t^i = a_i(b_i - Y_t^i)dt + \sigma_i\sqrt{Y_t^i}dW_t^i, \;\; Y^i(0) = y_i,$$

where b_i, c_i and σ_i are some constant parameters. Then, for $i = 1, 2$, the Riccati ODEs system (7) in Subsection 3.2 are given as follows

$$\begin{cases} \dot{\beta}_i(t) = a_i\beta_i(t) - \frac{1}{2}\sigma_i^2\beta_i(t)^2, & \text{with } \beta_i(T^*) = t_i, \\ \dot{\gamma}_i(t) = -a_i b_i \beta_i(t), & \text{with } \gamma_i(T^*) = 0. \end{cases}$$

Then we can easily have

$$\beta_i(t) = \frac{2a_i}{\sigma_i^2}\frac{1}{1 - \left(1 - \frac{2a_i}{\sigma_i^2 t_i}\right)e^{a_i(T^*-t)}}, \tag{23}$$

$$\gamma_i(t) = -\frac{2a_i b_i}{\sigma_i^2}\log\left(1 - \sigma_i^2 t_i \frac{1 - e^{-a_i(T^*-t)}}{2a_i}\right). \tag{24}$$

For $i = 0$, the Riccati ODE system (7) can be rewritten as

$$\begin{cases} \dot{\beta}_0(t) = a_0\beta_0(t) - \frac{1}{2}\sigma_0^2\beta_0(t)^2, & \text{with } \beta_0(T^*) = \rho_1 t_1 + \rho_2 t_2, \\ \dot{\gamma}_0(t) = -a_0 b_0 \beta_0(t), & \text{with } \gamma_0(T^*) = 0. \end{cases}$$

Then we have

$$\beta_0(t) = \frac{2a_0}{\sigma_0^2} \frac{1}{1 - \left(1 - \frac{2a_0}{\sigma_0^2(\rho_1 t_1 + \rho_2 t_2)}\right) e^{a_0(T^* - t)}}, \tag{25}$$

$$\gamma_0(t) = -\frac{2a_0 b_0}{\sigma_0^2} \log \left(1 - \frac{1 - e^{-a_0(T^* - t)}}{\frac{2a_0}{\sigma_0^2(\rho_1 t_1 + \rho_2 t_2)}}\right). \tag{26}$$

Similarly the Vasicek case, we can also derive the joint (conditional) survival probability (8) and (**??**) using the explicit form of $\gamma_i(0)$ and $\beta_i(0)$ in Equations (23), (24), (25) and (26).