Bridging socioeconomic pathways of CO2 emission and credit risk

Florian Bourgey, Emmanuel Gobet, Ying Jiao

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Abstract

This paper investigates the impact of transition risk on a firm's low-carbon production. As the world is facing global climate changes, the Intergovernmental Panel on Climate Change (IPCC) has set the idealized carbon-neutral scenario around 2050. In the meantime, many carbon reduction scenarios, known as *Shared Socioeconomic Pathways* (SSPs) have been proposed in the literature for different production sectors in more comprehensive socio-economic context. In this paper, we consider, on the one hand, a firm that aims to optimize its emission level under the double objectives of maximizing its production profit and respecting the emission mitigation scenarios. Solving the penalized optimization problem provides the optimal emission according to a given SSP benchmark. On the other hand, such transitions affect the firm's credit risk. We model the default time by using the structural default approach. We are particularly concerned with how the adopted strategies by following different SSPs scenarios may influence the firm's default probability.

Key words: Climate risk, transition risk, credit risk, Shared Socioeconomic Pathways, carbon emission reduction, optimal production profit, structural credit model.

MSC Classification (2020): 91B38, 91G40

JEL Classification: Q54, G32, C61, G38.

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1 Introduction

The context of CO_2 emission scenario. The continuous increase in anthropogenic greenhouse gas (GHG) emissions over the two last centuries is leading to a climate change, that is characterized in particular by an increase in the average surface temperature of the land and oceans. According to the Intergovernmental Panel on Climate Change [21], the global average temperature has already increased by 1^oC since the pre-industrial era. Without proactive measures to limit these CO_2 and other GHG emissions, one can expect a global warming by 3 or 4^oC, and maybe more, by 2100. This current global warming is and will continue to profoundly disrupt environmental, geographical and economic balances, if no mitigation and adaptation measures are taken. The Paris Agreement at the twenty-first session of the Conference of the Parties (COP 21) is an important milestone in international climate policy as it establishes a global mitigation framework towards 2030 and sets the ground for a global warming with stabilization around 1.5^oC only. This idealized scenario is based on a carbon neutrality around 2050, with some variations depending on the countries according to their Nationally Determined Contributions (NDC). Actually there are many other scenarios, depending on the ecological transition trajectory that countries, economic actors and populations will follow. In the most recent scientific literature, these scenarios are known as *Shared Socioeconomic Pathways* (SSPs), see [32] and references therein for an overview. See Figure 1 for the global CO_2 emission in different sectors, in the Organization for Economic Co-operation and Development (OECD), according to the scenarios chosen, [32, 33, 16]: the data are available on the SSP Public Database https://tntcat.iiasa.ac.at/SspDb. While the impacts of climate change are already perceptible, these scenarios are essential tools to help us to understand and anticipate mid-term and long-term consequences of near-term decisions. Our work aims to quantify how one firm's transition effort and strategy facing the climate change, in one of these scenarios, will propagate to the firm's credit risk. Credit risk is the possibility of loss resulting from the default of a borrower (typically a company) to the lender (typically a bank): see [25, 34] and references therein for possible modeling approaches to credit risk (without climate concerns).

Climate risks in finance. Climate change generates new sources of risk (so-called climate risks), in particular physical and transition risks as described by the solemn resounding speech by Mark Carney, the former Bank of England Governor [8]. Very quickly, regulators and financial institutions took up the issue, and gathered under the aegis of the Network for the Greening of the Financial System (NGFS) [28]. The Basel Committee on Banking Supervision published in April 2021 a report [3] exploring how climate-related risk drivers, including physical risks and transition risks, can arise and affect both banks and the banking system via micro and macro-economic transmission channels. In this work, we mainly focus on the transition risks and explore how to link projection scenarios such as those described in the SSPs by Phase 6 of Coupled Model Intercomparison Project (CMIP 6) into credit risk projections for firms. More specifically, we provide a quantitative model where the inputs are some desired paths of CO_2 emission $(e_t)_{t\geq 0}$, the production characteristics of a company, its sensitivity to CO_2 emissions, its climate-free credit spread and the outputs are the stochastic evolution of credit spread in an uncertain commercial demand. We consider the firm who aims to

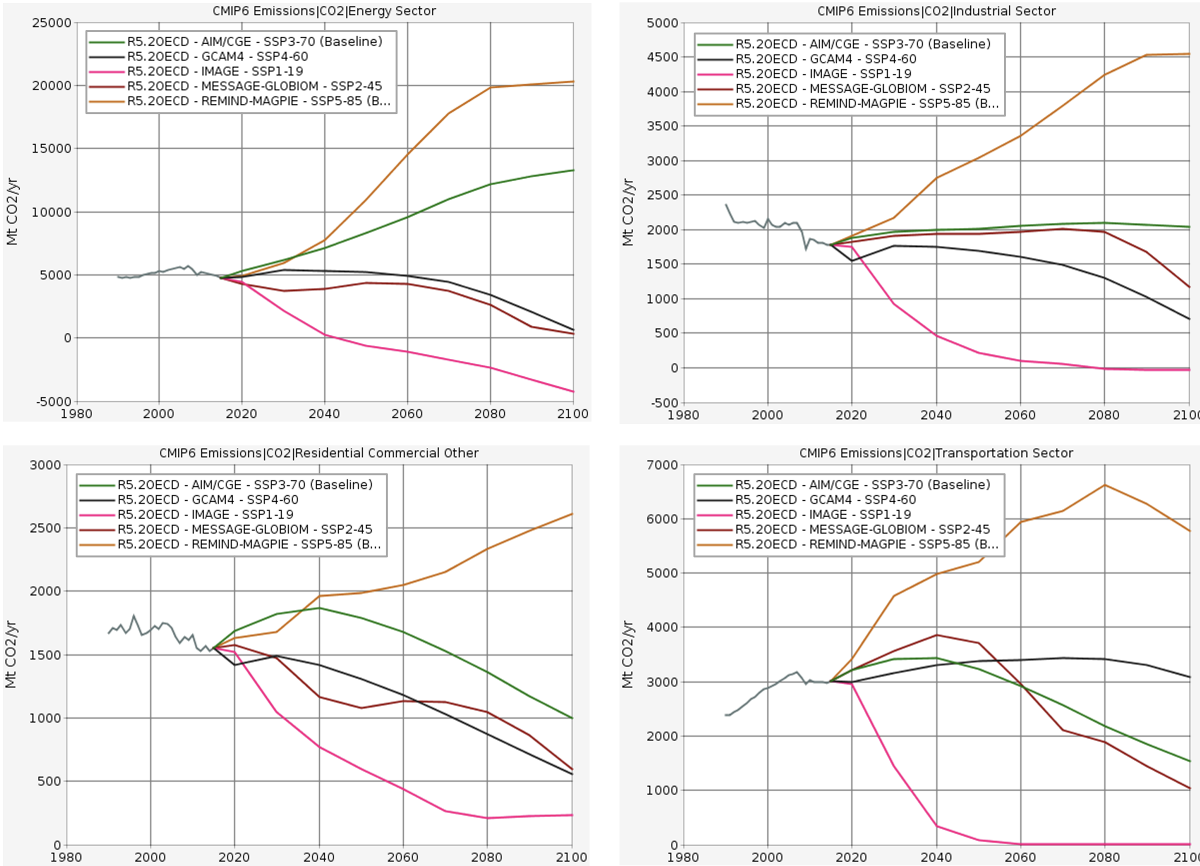

Figure 1: Historical and scenario-based CO_2 emission, from 1980 to 2100, in Mt/yr in the OECD, according to the activity sectors: Energy (top left), Industry (top right), Residential Commercial (bottom left), Transportation (bottom right). Source: Project CMIP 6, https://tntcat.iiasa.ac.at/SspDb.

maximize its production profit and at the same time takes into consideration the CO_2 reduction plan described by SSPs. Over-emission compared to the target may induce penalty. From the point of view of the firm, the objective is to determine the optimal strategy of its effective emission by solving a penalized optimization problem. The credit quality of the firm can be impacted by such carbon emission transition via its cash flow. In the classic structural credit approach such as Merton [26] or Black-Cox [5] model, a default event occurs when the firm's value is inferior to its debt level. We describe the firm's value process as the discounted value of all its future cash flow according to the optimal emission and production profit and study the corresponding default probability.

The optimal production problem in order to maximize expected profit of a company is well studied, see for example Guo and Pham [18]. We suppose that the firm's production depends on its energy consumption and in particular on the carbon emission level. We consider in addition the over-emission penalty under certain probabilistic risk measure constraints in the spirit of Föllmer and Leukert [13] where a loss function is concerned. The optimal emission strategy is obtained by adopting the Pontryagin's maximum principle approach [30] which is a standard method in optimal control theory especially for convex-type features. We then follow the classic structural models in credit risk to compute the climate-related default probability of the firm and show by numerical examples the impact of different relevant parameters and SSPs scenarios on the default probabilities and intensities. In this paper, we concentrate on the impact of transition risk and ignore at this stage the possible physical risk and related losses which may be caused by extreme weathers. In other words, the joint influence of both transition and physical risks is not in the scope of this paper and will be explored in future works. Suggestions about how to combine both are given in Section 3.

State of the art. Quantitative modeling of the impact of climate change on banking is in its infancy, see [7] for a review of the challenges. Several works have performed a thorough qualitative analysis, see for example [2], [27], [9]; we differ from them by providing a more quantitative description of the transmission mechanism of transition risk to credit risk. The author of [15] has proposed an extended (top-down) approach to the usual credit models incorporating new factors for physical and transition risks, with proposals for estimating the characteristics of these extra factors. On the other hand, bottom-up approaches attempt to start from financial statements and balance sheet (to be impacted by shocks or climate scenarios), to derive the firm value and then deduce its default probability and credit risk. Very recently, a statistical approach has been conducted by the European Central Bank [1] on 4 million companies: the authors have calibrated a multivariate econometric model. Although intuitively meaningful, it leads to disappointing results on the default probability (the R squared is about 11.9%, see [1, Table 4, p.85]). This bottum-up approach has been developed by [4] in the first reference article on climate stress tests. See also [31] for the Dutsch bankink system. The authors of [6] have studied the sensitivity of the credit risk of nearly 800 companies to the price of carbon, looking closely at the impact on the balance sheet. Such an analysis implicitly assumes a lack of adaptation of the firm to carbon prices. See also the recent literature review summarized in [6, Figure 3]. This paper is presumably the closest to the current work, but still, our contribution is different. Also dealing with a bottom-up approach, we also aim at modeling the firm's response (by optimizing its production accordingly) to a target CO2 trajectory and an announced climate policy: the possible response (see Proposition 1) of the firm may depend on its sector of activity and its own parameters. As often pointed out in this field, to enable climate risk management some actors need a simple but meaningful model (which we provide in this work) with a small number of features and parameters, as the data is often deficient or provided at an excessively low granularity.

Organization of the paper. We present models and main results in Section 2: the production model and the optimization problem are introduced in Section 2.1; the optimal emission strategies are presented in Section 2.2, first in a special case where closed-form formula can be obtained and then in the general case; Section 2.3 focuses on the credit risk and default probability. Finally numerical applications are given in Section 3 to illustrate the main results.

2 Model and results

2.1 Production and carbon emission constraint

Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual assumptions. We consider a firm whose production $P = (P_t, t \geq 0)$ depends on the energy consumption and in particular on its effective CO_2-emission volume $\gamma = (\gamma_t, t \geq 0)$ and solves the following stochastic differential equation (SDE),

$$\mathrm{d}P_t = P_t \left(\mu\left(t, P_t, \gamma_t\right) \mathrm{d}t + \sigma \mathrm{d}W_t\right), \quad P_0 > 0, \tag{1}$$

where γ_t is the instantaneous emission rate at time t, $W = (W_t, t \geq 0)$ is an $\mathbb{F}$-Brownian motion which represents the uncertainty in demand and supply for the production and σ is a positive constant volatility parameter. The emission rate is in general positive. The function $\mu : (t, x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ characterizes the production rate and satisfies the local Lipschitz condition on x, i.e., for all $t \geq 0$,

$$\forall y \in \mathbb{R}, \forall x, x' \in \mathbb{R}_+, \qquad \left|\mu(t, x, y) - \mu(t, x', y)\right| \leq K\left|x - x'\right|, \tag{2}$$

for an independent positive constant K. We suppose in addition that for any $t \in \mathbb{R}_+$, $\mu(t, \cdot, \cdot)$ is of class C^1. Empirical studies show that overproduction can lead to a decrease of production rate for example due to excess of supply, whereas the effect of emissions on production growth is positive (see, e.g., [22]). We thus suppose that μ is decreasing with respect to the production P and increasing with respect to the emission γ.

In response to the greenhouse gas emission reduction target, the firm has an objective emission plan described by $e = (e_t, t \geq 0)$, which represents the emission benchmark suggested by a SSP projection or accredited to the firm by the European Commission. This quantity could be deterministic or stochastic meaning respectively the allowances fixed or recommended with certain tolerance by regulation. The firm must keep watching on its emission evolution trajectory γ_t compared to the objective benchmark e_t at any time $t \geq 0$. In this paper, we consider the effective emission which is in general supposed to be positive. However, in other context such as carbon sequestration, or the emission allowance compensation within the European Emission Trading System (ETS), the SSPs may take negative values (see e.g. Figure 1 Energy sector). In other words, the objective emission e_t may be negative, meaning that it has to be considered as net emission, once that all processes related to compensation and carbon capture have been taken into account. We suppose in addition that e is bounded: this technical condition is anything but a restriction in practice. Exceeding the allocated benchmark can induce penalty losses to the firm as it may need to pay for the carbon tax or to purchase an extra quantity of emission allocation through ETS. In the meantime, if the effective emission is lower than the reference value, then the firm can obtain a certain form of award for its effort from regulators to stimulate additional future actions[1]. These regulation rules suggest measuring the impact of effective emission transition trajectory on the firm's financial plan by using a loss function related to risk measures. In literature, we often focus on the downside risk by using loss functions such as expected shortfall at a terminal date, see for example [13, 10]. In our case, we may also take into account the possibility of an upside award so the loss function can take both positive and negative values, as in [14, Section 4.9]. More precisely, concerning the regulation risk constraints, the evolving emission trajectory is traced continuously and the penalty is described by a loss function $\ell : \mathbb{R} \to \mathbb{R}$ which is an increasing and convex function with standardization condition $\ell(0) = 0$. We suppose in addition the following technical assumption that the loss function ℓ is of quadratic growth at most, i.e., $\ell(x) = \mathcal{O}(|x|^2)$ as $|x| \to +\infty$.

The firm's goal is to maximize its production profit and, at the same time, manage the effective emission level by taking into account the advertised constraints. We let the instantaneous profit of the firm be described by a function $\pi : \mathbb{R}_+ \to \mathbb{R}$ on the production P. We suppose that it is increasing and concave, and belongs to C^1, the class of all differentiable functions whose derivative is continuous, as in [18]. In addition, π satisfies the Inada conditions, i.e., $\lim_{x\to 0+} \pi'(x) = +\infty$ and $\lim_{x\to+\infty} \pi'(x) = 0$, which is a standard assumption on production functions in economic growth theory, initially proposed by Inada [20].

We also consider the firm's production cost function $\mathcal{C} : \mathbb{R}_+ \to \mathbb{R}_+$ on the effective emission which represents the firm's energy-related costs such as supply and use, and the technical reform cost to reduce the emission during its production process. It is natural that the cost function $\mathcal{C}$ is increasing with respect to emission. We suppose that $\mathcal{C}$ is convex meaning that higher emission can imply over-cost to the firm such as accelerating equipment's deterioration and increasing repair and replacement expenses. At this stage, we do not consider other types of costs such as the labor cost of the firm.

Given a benchmark emission projection, the firm chooses its optimal effective emission to maximize the expected profit by controlling the related production, cost, and emission constraints. The optimization problem is presented as below. We consider the profit maximization over all future time under the pathwise emission constraint and define the objective optimization function as

$$J(\gamma) := \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\pi(P_t) - \mathcal{C}(\gamma_t) - \ell(\gamma_t - e_t)\right)\mathrm{d}t\right]. \tag{3}$$

where $r > 0$ is a constant[2] discount rate. We aim to solve

$$\widehat{J} = \sup_{\gamma\in\mathcal{A}} J(\gamma) \tag{4}$$

[1] see https://ec.europa.eu/clima/policies/effort/regulation_en.

[2] This rate could be a deterministic function of time without significantly changing the following results.

where $\mathcal{A}$ is the admissible strategy set for the positive progressively measurable processes γ such that for some $\eta \in (0, r)$,

$$\mathbb{E}\left[\int_0^\infty e^{-\eta t}\gamma_t^2 \mathrm{d}t\right] < +\infty.$$

Note that bounded emission γ – as that for Proposition 1 – automatically fulfills the above integrability condition. We may also consider a fixed horizon time such as 2050, the year by which the COP wishes the world achieve the carbon-neutrality. In this case, besides the pathwise constraint, an extra penalty function could be included for the total cumulative emission at the terminal time.

2.2 Optimal emission strategy

In this section, we solve the optimization problems and characterize the optimal effective emission. We begin by presenting an explicit model where the optimal emission strategy is obtained in closed-form formula. We then study the general case by using the Pontryagin's maximum principle approach [30] which provides a general set of necessary conditions for the optimal strategy using the method of Lagrange multipliers applied to constrained optimization problems.

2.2.1 Profit maximization in an explicit model

We consider a logarithmic profit function $\pi(x) = \log x$ and let the cost and penalty functions be given respectively as

$$\mathcal{C}(x) = \frac{x^2}{2} \qquad \text{and} \qquad \ell(x) = \omega\frac{(x_+)^2}{2}, \tag{5}$$

where $\omega \geq 0$ is a constant coefficient characterizing the penalty force of the CO_2 emission constraint and the function x_+ denotes $\max(x, 0)$. The choice of a quadratic penalty function is to accentuate higher quantities of over-emission compared to the benchmark. The objective function (3) rewrites as

$$J(\gamma) = \mathbb{E}\left[\int_0^\infty e^{-rt}\Big(\log P_t - \frac{\gamma_t^2}{2} - \omega\frac{\big(\gamma_t - e_t\big)_+^2}{2}\Big)\mathrm{d}t\right]. \tag{6}$$

Introduce a variable transform and define the log-production $p_t := \log P_t$ which by (1) solves the SDE

$$\mathrm{d}p_t = \overline{\mu}(t, p_t, \gamma_t)\mathrm{d}t + \sigma \mathrm{d}W_t, \tag{7}$$

where the drift coefficient satisfies $\overline{\mu}(t, x, y) := \mu(t, e^x, y) - \frac{1}{2}\sigma^2$ for every $t \in \mathbb{R}_+, x \in \mathbb{R}, y \in \mathbb{R}_+$. We then set the following explicit model and suppose that the drift function of log-production has an affine form

$$\overline{\mu}(t, x, y) = a + bx + cy, \quad t \in \mathbb{R}_+, x \in \mathbb{R}, y \in \mathbb{R}_+, \tag{8}$$

where the coefficients $a \geq 0$ corresponds to an average production level, $b \leq 0$ is a mean-reverting parameter with the negative sign meaning that over-production may deteriorate the production ability and $c \geq 0$ represents the dependence of the firm's production upon CO_2 emission.

Proposition 1. *Suppose $r > b$. The optimal emission strategy $\widehat{\gamma}$ is given by*

$$\widehat{\gamma}_t = \min\left\{\frac{c}{r-b},\ \frac{1}{1+\omega}\big(\omega\, e_t + \frac{c}{r-b}\big)\right\}. \tag{9}$$

Proof. Let $\gamma \in \mathcal{A}$ be any admissible strategy to the problem (6). Then

$$J(\widehat{\gamma}) - J(\gamma) = \mathbb{E}\left[\int_0^\infty e^{-rt}\Big((\widehat{p}_t - p_t) - \frac{1}{2}(\widehat{\gamma}_t^2 - \gamma_t^2) - \frac{\omega}{2}\big((\widehat{\gamma}_t - e_t)_+^2 - (\gamma_t - e_t)_+^2\big)\Big)\mathrm{d}t\right] \tag{10}$$

where $\widehat{p}_t$ is given by

$$\mathrm{d}\widehat{p}_t = (a + b\widehat{p}_t + c\widehat{\gamma}_t)\mathrm{d}t + \sigma \mathrm{d}W_t.$$

By Itô's formula, we have

$$\mathrm{d}\big(e^{-rt}(\widehat{p}_t - p_t)\big) = e^{-rt}\Big((b-r)(\widehat{p}_t - p_t) + c(\widehat{\gamma}_t - \gamma_t)\Big)\mathrm{d}t. \tag{11}$$

Since $\gamma \in \mathcal{A}$ and

$$\mathbb{E}\left[\Big(\int_0^\infty e^{-\eta t}|\gamma_t|\mathrm{d}t\Big)^2\right] \le \Big(\int_0^\infty (e^{-\eta t/2})^2\mathrm{d}t\Big)\mathbb{E}\left[\int_0^\infty (e^{-\eta t/2}|\gamma_t|)^2\mathrm{d}t\right] < +\infty \tag{12}$$

for some $\eta \in (0, r)$, the integral $\int_0^\infty e^{-\eta t}|\gamma_t|\mathrm{d}t$ is finite almost surely, the same holds obviously for $\hat{\gamma}$ since the benchmark e is bounded. By solving the linear equation (11), we get

$$\begin{aligned}
e^{-rt}(\widehat{p}_t - p_t) &= e^{-rt}\int_0^t e^{b(t-s)}c(\widehat{\gamma}_s - \gamma_s)\mathrm{d}s,\\
|e^{-rt}(\widehat{p}_t - p_t)| &\le ce^{-rt}\int_0^t |\widehat{\gamma}_s - \gamma_s|\mathrm{d}s \le ce^{(\eta-r)t}\int_0^{+\infty} e^{-\eta s}|\widehat{\gamma}_s - \gamma_s|\mathrm{d}s
\end{aligned}$$

using $b \le 0$. Hence, since $r > \eta$,

$$\lim_{t\to+\infty} e^{-rt}(\widehat{p}_t - p_t) = 0,\ \ a.s.$$

which leads to

$$\int_0^\infty e^{-rt}\Big((b-r)(\widehat{p}_t - p_t) + c(\widehat{\gamma}_t - \gamma_t)\Big)\mathrm{d}t = 0,\ \ a.s. \tag{13}$$

owing to (11). So (10) can be rewritten as

$$\begin{aligned}
J(\widehat{\gamma}) - J(\gamma) &= \mathbb{E}\left[\int_0^\infty e^{-rt}\Big(\frac{c}{r-b}(\widehat{\gamma}_t - \gamma_t) - \frac{1}{2}(\widehat{\gamma}_t^2 - \gamma_t^2) - \frac{\omega}{2}\big((\widehat{\gamma}_t - e_t)_+^2 - (\gamma_t - e_t)_+^2\big)\Big)\mathrm{d}t\right]\\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}\Big(f(\widehat{\gamma}_t, e_t) - f(\gamma_t, e_t)\Big)\mathrm{d}t\right]
\end{aligned}$$

where

$$f(x,e) := \frac{c}{r-b}x - \frac{x^2}{2} - \frac{\omega}{2}(x-e)_+^2. \tag{14}$$

For given e, the maximal value in x of $f(x,e)$ is attained at

$$x = \min\left\{\frac{c}{r-b}, \frac{1}{1+\omega}\left(\omega\, e + \frac{c}{r-b}\right)\right\},$$

which concludes the proof. □

In the above proposition, we note that the constant value in (9)

$$\overline{\gamma} := \frac{c}{r-b} \tag{15}$$

corresponds to a critical level for the optimal emission strategy for the firm and corresponds to its desired emission level without carbon penalty, i.e., when $\omega = 0$.

- If the benchmark e_t is superior to $\overline{\gamma}$, then the optimal strategy is to remain at the constant level $\overline{\gamma}$ which means that no effort needs to be provided by the company. In particular, when $c = 0$, i.e., the firm's production does not depend on emission at all, then the optimal emission is expected to be zero.

- On the contrary, when e_t is inferior to $\overline{\gamma}$, meaning that the regulation requires a stricter emission reduction plan, then the optimal strategy is to proceed as an affine function of the benchmark.

The penalty weight ω also plays an essential role. A larger value of ω implies that the firm will adopt a stronger mitigation strategy. In particular, when ω tends to infinity, then the optimal emission converges to the benchmark.

Remark 1. The positive function ℓ defined in (5) focuses on the downside penalty corresponding to the out-performance part of the effective emission compared to the SSP benchmark. It is possible to describe the compensation or award when the firm's emission is below the benchmark. In such a case, one may consider instead a more general function ℓ that can take both positive and negative values such as

$$\ell(x) = \omega_1 \frac{(x_+)^2}{2} - \omega_2 \frac{(x_-)^2}{2}$$

where ω_1 and ω_2 are two positive real numbers and $x_- = \max(-x, 0)$. Similar as in the proof of Proposition 1, we can obtain in this case the explicit optimal effective emission with the function (14) replaced by

$$f(x,e) := \frac{c}{r-b}x - \frac{x^2}{2} - \frac{\omega_1}{2}(x-e)_+^2 + \frac{\omega_2}{2}(x-e)_-^2.$$

When $0 \le \omega_2 < 1$,

$$\widehat{\gamma}_t = \begin{cases} \frac{1}{1+\omega_1}(\omega_1 e_t + \overline{\gamma}), & \text{if } e_t \le \overline{\gamma}; \\ \frac{1}{1-\omega_2}(\overline{\gamma} - \omega_2 e_t), & \text{otherwise.} \end{cases} \tag{16}$$

When $\omega_2 \ge 1$,

$$\widehat{\gamma}_t = \begin{cases} \frac{1}{1+\omega_1}(\omega_1 e_t + \overline{\gamma}), & \text{if } e_t \le \overline{\gamma} \text{ and } \overline{b}(e_t) \le 0; \\ 0, & \text{otherwise;} \end{cases} \tag{17}$$

where $\overline{b}(\cdot)$ is a boundary function given by

$$\overline{b}(e) = \frac{\omega_1\omega_2 + \omega_1 + \omega_2}{2(1+\omega_1)} e^2 - \frac{\omega_1 \overline{\gamma}}{1+\omega_1} e - \frac{\overline{\gamma}^2}{2(1+\omega_1)}.$$

In particular, when e_t is superior to $\overline{\gamma}$, we note that the optimal emission $\widehat{\gamma}$ is mitigated with respect to $\overline{\gamma}$ (see Proposition 1) when the reward force parameter ω_2 becomes strictly positive, i.e., $\omega_2 \in (0,1)$, by noticing that

$$\frac{1}{1-\omega_2}(\overline{\gamma} - \omega_2 e_t) < \overline{\gamma} < e_t.$$

When the reward force is large enough, i.e. $\omega_2 > 1$, the optimal emission becomes zero , i.e. $\widehat{\gamma}_t = 0$. The comparison with Proposition 1 shows that the fact of adding the reward mechanism when the firm achieves better mitigation results (besides the penalty upon over-emission) can stimulate more efficiently the firms to attain the carbon neutral objective.

2.2.2 General case by stochastic Pontryagin's maximum principle

We now consider the general problem (3). We still use the log-production p_t given by (7) with a general drift term μ (under the assumptions of Subsection 2.1), and define accordingly, for every $x \in \mathbb{R}$, the auxiliary cost function $\overline{\pi}(x) := \pi(e^x)$. Then

$$\pi(P_t) = \overline{\pi}(p_t). \tag{18}$$

Besides, we deal with more general cost and penalty functions $\mathcal{C}$ and ℓ than those previously chosen in (5); still, $\mathcal{C}$ and ℓ are assumed to be convex and C^1. We characterize the optimal effective emission as below. In the statement below, we are willingly a bit vague about the growth assumptions on the functions ℓ, $\mathcal{C}$, π and μ (and their derivatives), these assumptions and some monotone conditions play an important role to justify the next computations of Pontryagin maximum principle and ensure the existence of control and adjoint process Y (like those in Proposition 2) in infinite horizon in suitable weighted L_2 space, see for instance the book of [29] for a broad account on the subject. The take-home message from the following derivation is rather to get the optimality condition equation characterizing the optimal emission in a fairly eneral situation, the verification of technical conditions would be done in a second step, according to the cost and dynamics coefficients.

Proposition 2. *For any $\gamma \in \mathcal{A}$, let $Y_\cdot(\gamma)$ denote the conditional expectation*

$$Y_t(\gamma) = \mathbb{E}\left[\int_t^\infty e^{-ru+\int_t^u \partial_x \overline{\mu}(t,p_s,\gamma_s)\mathrm{d}s}\,\overline{\pi}'(p_u)\mathrm{d}u \middle| \mathcal{F}_t\right], \quad t \geq 0$$

which is an $\mathbb{F}$-adapted cÃădlÃăg process and supposed to satisfy $\mathbb{E}[\int_0^\infty Y_t(\gamma)^2\mathrm{d}t] < +\infty$. If $\widehat{\gamma}$ is an optimal control

$$J(\widehat{\gamma}) = \sup_{\gamma\in\mathcal{A}} J(\gamma),$$

then it must solve the following equation

$$\mathcal{C}'(\widehat{\gamma}_t) + \ell'\left(\widehat{\gamma}_t - e_t\right) = e^{rt}\partial_y \overline{\mu}\left(t, \widehat{p}_t, \widehat{\gamma}_t\right) Y_t(\widehat{\gamma}), \quad \mathrm{d}t \otimes \mathrm{d}\mathbb{P}\ a.e., \tag{19}$$

where $\mathrm{d}\widehat{p}_t = \overline{\mu}\left(t, \widehat{p}_t, \widehat{\gamma}_t\right)\mathrm{d}t + \sigma \mathrm{d}W_t$.

Proof. We first introduce, for two given controls $\gamma, \tilde{\gamma} \in \mathcal{A}$, the directional derivatives for the log-production

$$\dot{p}_t := \partial_\varepsilon p_t^{\gamma+\varepsilon\tilde{\gamma}}\Big|_{\varepsilon=0}.$$

From (7), it holds that

$$\mathrm{d}\dot{p}_t = \left(\partial_x \overline{\mu}\left(t, p_t, \gamma_t\right)\dot{p}_t + \partial_y \overline{\mu}\left(t, p_t, \gamma_t\right)\tilde{\gamma}_t\right)\mathrm{d}t, \quad \dot{p}_0 = 0. \tag{20}$$

Similarly, we define for the objective value function

$$\dot{J}\left(\gamma, \tilde{\gamma}\right) := \partial_\varepsilon J\left(\gamma + \varepsilon\tilde{\gamma}\right)\big|_{\varepsilon=0},$$

which from (3) equals

$$\dot{J}\left(\gamma, \tilde{\gamma}\right) = \mathbb{E}\Big[\int_0^\infty e^{-rt}\left(\overline{\pi}'\left(p_t\right)\dot{p}_t - \mathcal{C}'(\gamma_t)\tilde{\gamma}_t - \ell'\left(\gamma_t - e_t\right)\tilde{\gamma}_t\right)\mathrm{d}t\Big]. \tag{21}$$

By the definition of $Y_\cdot(\gamma)$, the process

$$M_t = Y_t(\gamma) + \int_0^t e^{-ru+\int_t^u \partial_x \overline{\mu}(t,p_s,\gamma_s)\mathrm{d}s}\,\overline{\pi}'(p_u)\mathrm{d}u, \quad t \geq 0$$

is an $\mathbb{F}$-martingale and

$$\lim_{t\to+\infty} Y_t(\gamma) = 0.$$

Observing by (20) that $Y_0(\gamma)\dot{p}_0 = 0$ and that $\dot{p}$ is a finite-variation process, we apply Ito's formula on $Y_t(\gamma)\dot{p}_t$,

$$0 = \mathbb{E}\left[\int_0^\infty \left(Y_t(\gamma)\mathrm{d}\dot{p}_t + \dot{p}_t \mathrm{d}Y_t(\gamma)\right)\right]$$

$$= \mathbb{E}\Big[\int_0^\infty Y_t(\gamma)\big(\partial_x \overline{\mu}\,(t,p_t,\gamma_t)\,\dot{p}_t + \partial_y \overline{\mu}\,(t,p_t,\gamma_t)\,\tilde{\gamma}_t\big)\mathrm{d}t - \int_0^\infty \dot{p}_t\big(e^{-rt}\overline{\pi}'\,(p_t) + \partial_x \overline{\mu}\,(t,p_t,\gamma_t)\,Y_t(\gamma)\big)\mathrm{d}t\Big]$$
$$= \mathbb{E}\Big[\int_0^\infty Y_t(\gamma)\partial_y \overline{\mu}\,(t,p_t,\gamma_t)\,\tilde{\gamma}_t\mathrm{d}t - \int_0^\infty e^{-rt}\overline{\pi}'\,(p_t)\,\dot{p}_t\mathrm{d}t\Big],$$

using that $\mathbb{E}\left[\int_0^\infty \dot{p}_t \mathrm{d}M_t\right] = 0$, which implies that

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\overline{\pi}'\,(p_t)\,\dot{p}_t\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty Y_t(\gamma)\partial_y \overline{\mu}\,(t,p_t,\gamma_t)\,\tilde{\gamma}_t\mathrm{d}t\right].$$

We can then rewrite (21) as

$$\dot{J}\,(\gamma,\tilde{\gamma}) = \mathbb{E}\Big[\int_0^\infty \big(\partial_y \overline{\mu}\,(t,p_t,\gamma_t)\,Y_t(\gamma) - e^{-rt}\mathcal{C}'(\gamma_t) - e^{-rt}\ell'\,(\gamma_t - e_t)\,\big)\tilde{\gamma}_t\mathrm{d}t\Big],$$

which holds for any control process $\tilde{\gamma} \in \mathcal{A}$. Consequently, if $\widehat{\gamma}$ is an optimal control, then necessarily $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$ a.e.

$$\partial_y \overline{\mu}\,(t,\widehat{p}_t,\widehat{\gamma}_t)\,Y_t(\widehat{\gamma}) = e^{-rt}\left(\mathcal{C}'(\widehat{\gamma}_t) + \ell'\,(\widehat{\gamma}_t - e_t)\right)$$

where $Y_t(\widehat{\gamma})$ is written as

$$Y_t(\widehat{\gamma}) = \mathbb{E}\left[\int_t^\infty e^{-ru+\int_t^u \partial_x \overline{\mu}(t,\widehat{p}_s,\widehat{\gamma}_s)\mathrm{d}s}\,\overline{\pi}'(\widehat{p}_u)\mathrm{d}u\right|\mathcal{F}_t\Bigg].$$

In other words, $\widehat{\gamma}$ is the solution to Equation (19), which ends the proof. □

Remark 2. We can consider an alternative problem with finite horizon $T > 0$ and introduce the effective cumulative emission process up to time t as $\Gamma_t := \int_0^t \gamma_s \mathrm{d}s$. It is then possible to incorporate a final regulation for the cumulative emission compared to the target $E_t = \int_0^t e_s \mathrm{d}s$ at the horizon time. The objective function is then defined as

$$J_T(\gamma) := \mathbb{E}\left[\int_0^T e^{-rt}\left(\pi(P_t) - \mathcal{C}(\gamma_t) - \ell_1(\gamma_t - e_t)\right)\mathrm{d}t - e^{-rT}\ell_2\left(\Gamma_T - E_T\right)\right], \tag{22}$$

where ℓ_1 and ℓ_2 are two loss functions and the optimization problem becomes

$$\widehat{J}_T = \sup_{\gamma\in\mathcal{A}_T} J_T(\gamma), \tag{23}$$

where $\mathcal{A}_T$ is the admissible strategy set such that for any $T > 0$, $\mathbb{E}[\Gamma_T^2] < +\infty$, and that for any $x \geq 0$, $\int_0^T |\mu\,(t,x,\gamma_t)|^2\,\mathrm{d}t < +\infty$ almost surely. Similar as Proposition 2, the solution of the finite time horizon can be characterized as below. The difference lies in the extra terminal constraint. More precisely, let

$$Y_t^1(\gamma) = \mathbb{E}\left[\int_t^T e^{-ru+\int_t^u \partial_x \overline{\mu}(t,p_s,\gamma_s)\mathrm{d}s}\,\overline{\pi}'(p_u)\mathrm{d}u\right|\mathcal{F}_t\Bigg].$$

If $\widehat{\gamma}$ is an optimal solution to (23) then it must satisfy

$$e^{-rt}\left(\mathcal{C}'(\widehat{\gamma}_t) + \ell_1'\big(\widehat{\gamma}_t - e_t\big)\right) + \mathbb{E}\big[e^{-rT}\ell_2'\big(\widehat{\Gamma}_T - E_T\big)|\mathcal{F}_t\big] = \partial_y \overline{\mu}\,(t,\widehat{p}_t,\widehat{\gamma}_t)\,Y_t^1(\widehat{\gamma}). \tag{24}$$

2.3 Credit risk under emission transition

In this section, we study the credit risk of the firm induced by the transition towards the low-carbon emission and production pattern. We use the effective production obtained in the previous optimization problem to deduce the firm's value process and then compute the default probability in a structural modelling approach.

2.3.1 Firm's value process

In the classical structural credit models such as Merton [26] and Black-Cox [5] models, a default event is triggered when the firm's value no longer allows to cover the reimbursement of the debt. In the following, we describe the value process of the firm by using the discounted cash flow approach which traces back to Keynesian economics. More precisely, let the firm's value $V^\gamma = (V_t^\gamma, t \geq 0)$ be defined as the conditional discounted value of all future cash flows including the global production income deduced by the cost and penalty depending on the effective emission γ, that is

$$V_t^\gamma = \mathbb{E}\left[\int_t^\infty e^{-r(u-t)} \left(\pi(P_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)\right) \mathrm{d}u | \mathcal{F}_t\right]. \tag{25}$$

At the initial time when $t = 0$, we have $V_0^\gamma = J(\gamma)$. The firm would produce according to the optimal production quantity associated to the emission strategy $\widehat{\gamma}$ which are obtained from the maximization procedure in Propositions 1 or 2.

For any $\nu \in \mathcal{A}$ and any $t \geq 0$, the firm's dynamic optimal value (viewed at time t) is given by

$$\widehat{V}_t(\gamma) = \operatorname*{ess\,sup}_{\gamma \in \mathcal{A}(t,\nu)} V_t^\gamma = \operatorname*{ess\,sup}_{\gamma \in \mathcal{A}(t,\nu)} \mathbb{E}\left[\int_t^\infty e^{-r(u-t)} \left(\overline{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)\right) \mathrm{d}u | \mathcal{F}_t\right] \tag{26}$$

where the set $\mathcal{A}(t, \nu) := \{\gamma \in \mathcal{A} \text{ such that } \gamma_{\cdot \wedge t} = \nu_{\cdot \wedge t}\}$ represents all controls which coincide with ν up to time t. By the dynamic programming principle in El Karoui and Quenez [12], for any $\gamma \in \mathcal{A}$, the process $\left(\widehat{V}_t(\gamma) + \int_0^t e^{-r(u-t)} \left(\overline{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)\right) \mathrm{d}u,\, t \geq 0\right)$ is a supermartingale. In particular, for an optimal $\widehat{\gamma}$, $\left(\widehat{V}_t(\widehat{\gamma}) + \int_0^t e^{-r(u-t)} \big(\overline{\pi}(\widehat{p}_u) - \mathcal{C}(\widehat{\gamma}_u) - \ell(\widehat{\gamma}_u - e_u)\big) \mathrm{d}u,\, t \geq 0\right)$ is a martingale. The value process associated to $\widehat{\gamma}$ by Propositions 1 or 2 corresponds to the firm's optimal value.

Proposition 3. *Let $\widehat{\gamma}$ be an optimal emission strategy which solves (3). The following equality holds for any $t \geq 0$:*

$$\mathbb{E}[\widehat{V}_t(\widehat{\gamma})] = \mathbb{E}[V_t^{\widehat{\gamma}}]. \tag{27}$$

Proof. By definition of the essential supremum in (26), we have $\mathbb{E}[\widehat{V}_t(\widehat{\gamma})] \geq \mathbb{E}[V_t^{\widehat{\gamma}}]$ for any $t \geq 0$. To prove the converse inequality, fix $t \geq 0$ and for any $\gamma \in \mathcal{A}(t, \widehat{\gamma})$,

$$
\begin{aligned}
&\mathbb{E}\left[V_t^{\gamma} + \int_0^t e^{-r(u-t)} \left(\overline{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)\right) \mathrm{d}u\right] \\
&= \mathbb{E}\left[\int_0^{\infty} e^{-r(u-t)} \left(\overline{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)\right) \mathrm{d}u\right] \\
&\leq \mathbb{E}\left[\int_0^{\infty} e^{-r(u-t)} \left(\overline{\pi}(\widehat{p}_u) - \mathcal{C}(\widehat{\gamma}_u) - \ell(\widehat{\gamma}_u - e_u)\right) \mathrm{d}u\right]
\end{aligned}
$$

where the last inequality stems from the optimality of $\widehat{\gamma}$ in the optimization problem (3). By the definition of $A(t, \widehat{\gamma})$, it holds $\widehat{\gamma}_s = \gamma_s$ for any $s \leq t$, we then obtain from the above inequality

$$
\mathbb{E}[V_t^{\gamma}] \leq \mathbb{E}\left[\int_t^{\infty} e^{-r(u-t)} \left(\overline{\pi}(\widehat{p}_u) - \mathcal{C}(\widehat{\gamma}_u) - \ell(\widehat{\gamma}_u - e_u)\right) \mathrm{d}u\right] = \mathbb{E}[V_t^{\widehat{\gamma}}], \tag{28}
$$

since

$$
\mathbb{E}\left[\int_0^t e^{-r(u-t)} \left(\overline{\pi}(p_u) - \mathcal{C}(\gamma_u) - \ell(\gamma_u - e_u)\right) \mathrm{d}u\right] = \mathbb{E}\left[\int_0^t e^{-r(u-t)} \left(\overline{\pi}(\widehat{p}_u) - \mathcal{C}(\widehat{\gamma}_u) - \ell(\widehat{\gamma}_u - e_u)\right) \mathrm{d}u\right].
$$

Note by (26) that $\mathbb{E}[\widehat{V}_t(\widehat{\gamma})] = \mathbb{E}[\operatorname{ess\,sup}_{\gamma \in \mathcal{A}(t,\widehat{\gamma})} V_t^{\gamma}]$. By using a measurable selection argument (see e.g. [35]), for any $\epsilon > 0$, there exists $\gamma_{\epsilon} \in \mathcal{A}(t, \widehat{\gamma})$ such that

$$
\operatorname*{ess\,sup}_{\gamma \in \mathcal{A}(t,\widehat{\gamma})} V_t^{\gamma} \leq V_t^{\gamma_{\epsilon}} + \epsilon.
$$

We then have from (28) that

$$
\mathbb{E}\Big[\operatorname*{ess\,sup}_{\gamma \in \mathcal{A}(t,\widehat{\gamma})} V_t^{\gamma}\Big] \leq \mathbb{E}\left[V_t^{\gamma_{\epsilon}}\right] + \epsilon \leq \mathbb{E}[V_t^{\widehat{\gamma}}] + \epsilon.
$$

From the arbitrariness of $\epsilon > 0$, we obtain the following inequality

$$
\mathbb{E}\left[\widehat{V}_t(\widehat{\gamma})\right] = \mathbb{E}\Big[\operatorname*{ess\,sup}_{\gamma \in \mathcal{A}(t,\widehat{\gamma})} V_t^{\gamma}\Big] \leq \mathbb{E}[V_t^{\widehat{\gamma}}]
$$

and hence the required equality (27). □

2.3.2 Structural credit model and default probability

In a structural credit model, the firm is considered to default when its value process gets below a default threshold. We let the default barrier be described by a deterministic function $L(t)$ which depends on time and represents the minimal level of the firm's liability payment such as the debt reimbursement together with labor and other functioning costs (as the operational and capital expenditure) at time t. When the firm's value is higher than the threshold, then it is in a financially sustainable situation and can function normally. In the contrary case, the firm encounters fiscal difficulty and a default event may be triggered. We are interested in the default probability (PD in short) at a certain date t, which is given by

$$\mathrm{PD}_t = \mathbb{P}\left(V_t^{\widehat{\gamma}} \leq L(t)\right). \tag{29}$$

This corresponds to the classic Merton model where the default-or-not state of the firm is determined by the instantaneous value of the firm's value and of the default threshold. From the regulation point of view, financial and insurance institutions are required by the Solvency II Directive to evaluate their solvency risk and associated default probability at certain given time. From the computation viewpoint, the default probability depends on the distribution of the firm's value process $V^{\widehat{\gamma}}$ at the time t. In the next section, we give quasi-explicit formulas for PD_t in the model of Proposition 1, in particular as a function of the SSP describing the scenario of target emission $(e_t)_t$.

In other cases such as the Black-Cox model, the default event depends on the trajectory of the value process. At a given time t, the firm is considered in default if its value has crossed the threshold during the period $[0, t]$, that is,

$$\mathrm{PD}_t = \mathbb{P}\left(\exists s \in [0, t], \text{such that } V_s^{\widehat{\gamma}} \leq L(s)\right) \tag{30}$$

This corresponds to a random default time τ which is a first-hitting-time given as

$$\tau := \inf\{t \geq 0, V_t^{\widehat{\gamma}} \leq L(t)\}, \tag{31}$$

with convention $\inf \emptyset = \infty$. Then τ is a stopping-time with respect to the filtration $\mathbb{F}$. In this case, we are concerned with a passage time to the curved boundary which is a deterministic function on time. Such problems are studied in probability theory and applications with different approaches in literature (see for example the book of [23]). Since explicit expressions for probability density of the passage time can be obtained only in very particular cases, computational approach is often adopted either to approximate the density function, e.g., [11], or to simulate directly the first passage time, e.g. [17, 19].

3 Numerical illustrations

3.1 SSPs scenarios and optimal emission

We consider the following CO_2 emission scenarios which correspond to different socioeconomic reference pathways provided by CMIP6: SSP1-2.6, SSP2-4.5, SSP3-LowNTCF, SSP4-6.0, and SSP5-3.4-OS. These scenarios are illustrative pathways adopted by the IPCC in the sixth Assessment Report indicating the CO_2 concentrations in atmosphere from lowest (SSP1) to highest (SSP5). In other words, SSP1 and SSP5 describe respectively economic growth pattern via sustainable and fossil-fuel pathways. We choose two sectors: Transportation (Figure 2) and Industrial (Figure 3) sectors for which the year 2015 is our starting point. For each sector, we consider the above five SSPs including two baseline scenarios (Tier 1, c.f. [24]): SSP1-2.6 which is the most mitigated scenario corresponding approximately to the previous scenario generation Representative Concentration Pathway (RCP) 2.6, and SSP2-4.5 with is a moderate scenario similar to RCP-4.5. We also consider three (Tier 2) supplementary scenarios: SSP3-LowNTCF (Near-Term Climate Forcing) which provides a comparison scenarios with high NTCF emissions (notably SOx and methane), SSP4-6.0 focusing on a socio-economic context of inequality, and SSP5-34-OS (OverShoot) which allows for large overshoot by mid-century followed by substantive policy tools in the latter half of the century.

The numerical computations are based on the explicit model of Proposition 1. We start with the transportation sector. The different emission benchmarks (left) $t \mapsto e_t$ are normalized so that $e_0 = \widehat{\gamma}_0 = \frac{c}{r-b}(= \overline{\gamma})$. The associated optimal emission (right) are provided by (9) which is equal to $\overline{\gamma}$ if the benchmark is above and is given as an affine function of e_t in the contrary case. As one can observe, for both sectors, the scenario SSP1-2.6 is the hardest benchmark which imposes immediate reduction from the beginning 2015. For the transportation sector, the scenario SSP4-6.0 imposes no emission constraints that the optimal emission remains constant at level $\overline{\gamma}$. The situation is similar for the scenario SSP3-LowNTCF in the industrial sector. Other scenarios correspond to relatively soft emission constraints with overshoot where the reduction may begin from a later date. Figure 4 shows the impact of the parameter ω. More precisely, the optimal emission is decreasing with respect to the penalty force. In other words, a stronger penalty policy will induce larger emission reduction from the firm.

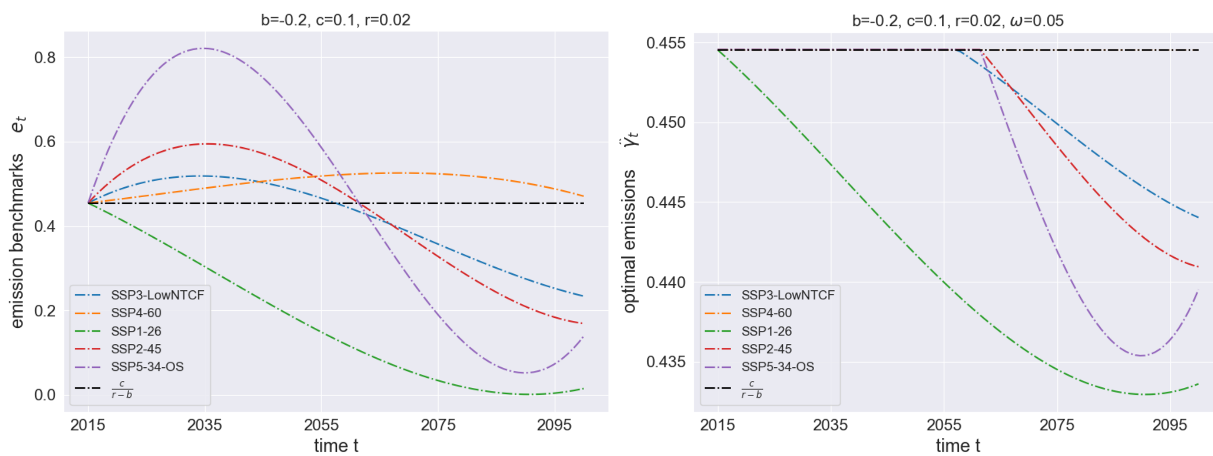

Figure 2: Transportation sector: SSPs emission scenarios v.s. associated optimal emission.

3.2 Default probability for different sectors

We now present the emission-related default probability in the explicit case and analyze the impact of carbon emission reduction given different SSPs benchmarks. We still consider the explicit model in Proposition 1 and use a slightly different definition for the value process associated to the optimal

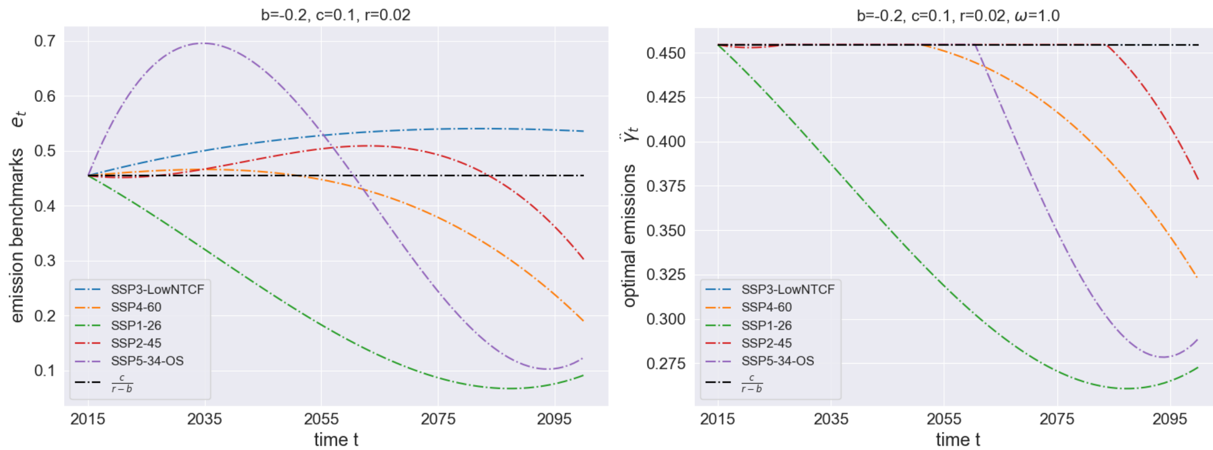

Figure 3: Industrial sector: SSPs emission scenarios v.s. associated optimal emission .

emission $V^{\widehat{\gamma}}$. More precisely, instead of choosing a logarithmic function, we let the profit of the firm be given directly as the average price $N > 0$ multiplied by the total production and write the firm's value process $\widehat{V}$ as

$$\widehat{V}_t = \mathbb{E}\left[\int_t^\infty e^{-r(u-t)}\left(N\widehat{P}_u - \mathcal{C}(\widehat{\gamma}_u) - \ell(\widehat{\gamma}_u - e_u)\right)\mathrm{d}u\middle|\mathcal{F}_t\right]. \tag{32}$$

The Default Probability is given by (29). Suppose that $\widehat{V}_0 > L(0)$. We aim to compute the firm's value (32) by specifying a benchmark e_t. Note that if the benchmark is given as a deterministic function such as a SSP scenario, then it is also the case for the optimal emission $\widehat{\gamma}$ from (9). We then have

$$\widehat{V}_t = N\int_t^\infty e^{-r(u-t)}\mathbb{E}\big[\widehat{P}_u|\mathcal{F}_t\big]\mathrm{d}u - \int_t^\infty e^{-r(u-t)}(\mathcal{C}(\widehat{\gamma}_u) + \ell(\widehat{\gamma}_u - e_u))\mathrm{d}u. \tag{33}$$

From (7) and (8), the optimal log-production rewrites as

$$\widehat{p}_u = e^{b(u-t)}\widehat{p}_t + \frac{a}{b}(e^{b(u-t)} - 1) + c\int_t^u e^{b(u-v)}\widehat{\gamma}_v\mathrm{d}v + \sigma\int_t^u e^{b(u-v)}\mathrm{d}W_v.$$

Hence for every $u \geq t \geq 0$, conditionally on $\mathcal{F}_t$, the optimal log-production $\widehat{p}_u$ is Gaussian with mean

$$e^{b(u-t)}\widehat{p}_t + \frac{a}{b}(e^{b(u-t)} - 1) + c\int_t^u e^{b(u-v)}\widehat{\gamma}_v\mathrm{d}v =: e^{b(u-t)}\widehat{p}_t + m_{u,t},$$

and variance

$$\sigma_{u,t}^2 := \frac{\sigma^2}{2b}\big(e^{2b(u-t)} - 1\big).$$

Figure 4: Industrial sector: different optimal emissions w.r.t. ω for scenarios SSP1-26 and SSP4-60.

Therefore, concerning the optimal production $\widehat{P}$, we have

$$\mathbb{E}[\widehat{P}_u|\mathcal{F}_t] = \exp\left(e^{b(u-t)}\widehat{p}_t + m_{u,t} + \frac{\sigma^2_{u,t}}{2}\right).$$

From (33), the firm's value with an optimal emission is thus given by

$$\widehat{V}_t = N\int_t^\infty e^{-r(u-t)}\exp\left(e^{b(u-t)}\widehat{p}_t + m_{u,t} + \frac{\sigma^2_{u,t}}{2}\right)\mathrm{d}u - \int_t^\infty \frac{e^{-r(u-t)}}{2}\left(\widehat{\gamma}_u^2 + \omega\left((\widehat{\gamma}_u - e_u)_+\right)^2\right)\mathrm{d}u$$
$$=: h(t,\widehat{p}_t). \tag{34}$$

We can then compute the emission-related default probabilities with the explicit form of the firm's value $\widehat{V}$ according to Merton model. Here, the default probability rewrites as

$$\mathrm{PD}_t = \mathbb{P}\big(\widehat{V}_t \leq L(t)\big) = \mathbb{P}\left(\widehat{p}_t \leq (h(t,\cdot))^{-1}(L(t))\right) = \Phi\Big(\frac{(h(t,\cdot))^{-1}(L(t)) - e^{bt}p_0 - m_{t,0}}{\sigma_{t,0}}\Big), \tag{35}$$

where Φ is the cumulative distribution function of a standard normal random variable.

For our experiments, the default boundary $L(t)$ is determined as follows. We consider the firm's default probability without climate concern (that is, $\omega = 0$) as a baseline where the default intensity λ_{ref} is supposed to be a fixed reference value in our numerical tests and is set at 3%. More precisely, we let

$$\mathbb{P}\big(\widehat{V}_t^{\mathrm{ref}} \leq L(t)\big) = 1 - e^{-\lambda_{\mathrm{ref}}t},$$

where $\widehat{V}_t^{\mathrm{ref}}$ corresponds to the optimal value without any emission constraint, i.e., $\omega = 0$, so that the value of $L(t)$ is obtained. We have by (34) that

$$L(t) = h\left(t, \Phi^{-1}(1 - e^{-\lambda_{\mathrm{ref}}t})\sigma_{t,0} + \frac{e^{bt}-1}{b}\big(a + \frac{c^2}{r-b}\big) + e^{bt}p_0\right),$$

using that $m_{t,0} = \frac{e^{bt}-1}{b}\big(a + \frac{c^2}{r-b}\big)$ when $\omega = 0$ and $\sigma_{t,0}$ is independent of e_t.

In Figures 5 and 6, we plot for the two sectors Transportation and Industrial the default probability PD_t given by (35) and the respective intensity $t \mapsto \lambda_t$ for the the different scenarios. The intensity λ_t is computed as

$$\mathbb{P}(\widehat{V}_t \leq L(t)) = 1 - \exp\left(-\int_0^t \lambda_s \mathrm{d}s\right).$$

The integral $\int_0^t \lambda_s \mathrm{d}s$ is approximated with a left-point rectangle scheme, i.e., between two times $0 \leq s_{i-1} < s_i \leq t$, we have

$$\lambda_{s_{i-1}} = \frac{1}{s_i - s_{i-1}} \log\left(\frac{\mathbb{P}(\widehat{V}_{s_{i-1}} > L(s_{i-1}))}{\mathbb{P}(\widehat{V}_{s_i} > L(s_i))}\right).$$

As expected the initial value of the intensity coincides with the prefixed value $\lambda_0 = 3\%$. When time evolves, the more constrained scenarios are associated to larger default probabilities and higher intensities. The SSP1-2.6 scenario is the most impacted one, which is quite natural given its immediate and hard reduction strategy. The scenario which follows is SSP5-3.4-OS: although this benchmark allows for a large overshoot up to 2060, the relatively strict mitigation during the latter period makes the default probability and intensity increase significantly. Observe that SSP4-6.0 corresponds to a fixed intensity of $\lambda_0 = 3\%$ in the Transportation sector and the same phenomenon appears for the SSP3-LowNTCF scenario in the Industrial sector, as the optimal emission is unconstrained in these two cases. We note that in this study we only investigate the transition risk related to the firm's mitigation strategy and ignore the possible physical risks under each scenario for example the more frequent damage and natural catastrophes under scenarios with higher temperature increase such as a SSP5 scenario (see for example [1] for more discussions). Although physical climate risk is not the main modeling concern of this work, it could easily incorporated consistently with our approach, by adding a suitable physical climate damage contribution in the firm's value (32)-(34), in order to broadly account for climate risks.

Figure 7 illustrates the default intensity for Industrial sector under the impact of the parameter c representing the dependence of the firm on emission and ω representing the penalty force. We consider the hardest scenario SSP1-2.6 and a moderate one SSP4-6.0. For both scenarios, the increase of one of the two parameters implies a higher default intensity and this phenomenon is particularly accentuated when the firm's production is highly dependent on the CO2 emission (when c is large). For such a firm, a strong penalty policy together with a hard mitigation scenario such as SSP1-2.6 (left) could have a significant impact on the firm's default probability.

We finally present in Figure 8 the default intensity obtained in the path-dependent Black-Cox model (30) in comparison with the Merton model (29): since in the Black-Cox approach the default event is monitored throughout a given period and not only at the end, this naturally gives rise to a more important measure of credit risk. For our experiments, the default event is considered at annual intermediate dates, and the default probability is then computed as a discrete approximation

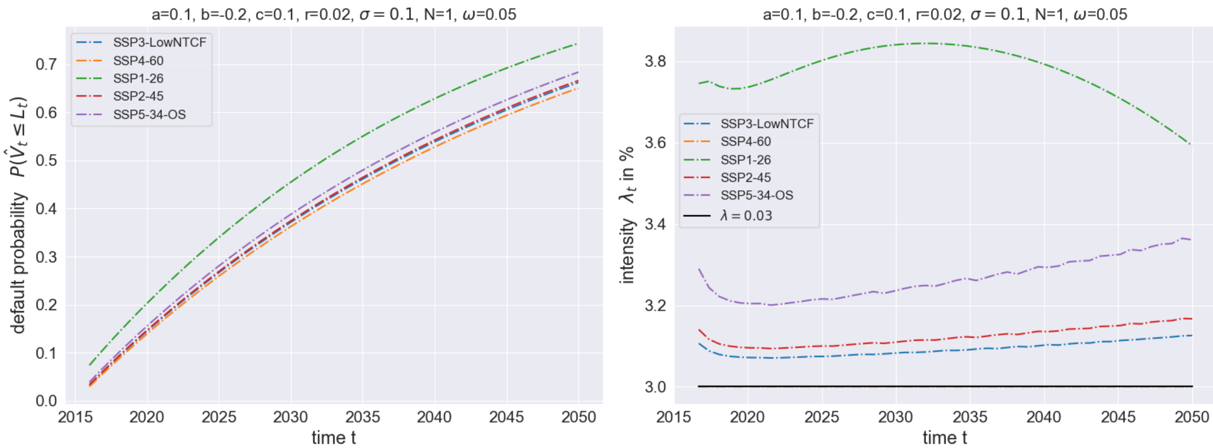

Figure 5: Transportation sector: default probability and associated intensity.

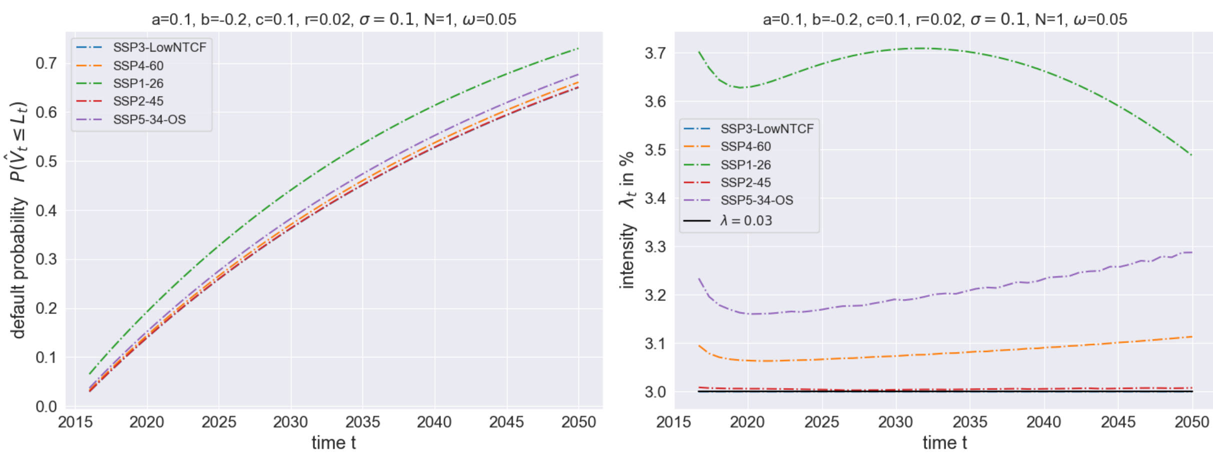

Figure 6: Industrial sector: default probability and associated intensity.

of (30) given by the following definition of default probability:

$$t \mapsto \mathbb{P}\left(\exists s = t_0, t_1, ..., t_n \in [0, t] : \widehat{V}_s \leq L(s)\right) = \mathrm{PD}_t.$$

The Black-Cox default intensity is then obtained as

$$1 - \mathrm{PD}_{t_{k+1}} = (1 - \mathrm{PD}_{t_k}) \exp\left(-\lambda_{t_k}(t_{k+1} - t_k)\right).$$

The two scenarios in Figure 8 are SSP1-2.6 and SSP5-3.4-OS for Industrial sector which are the most impacted scenarios by mitigation strategy (as shown in Figure 6). As expected, we observe that

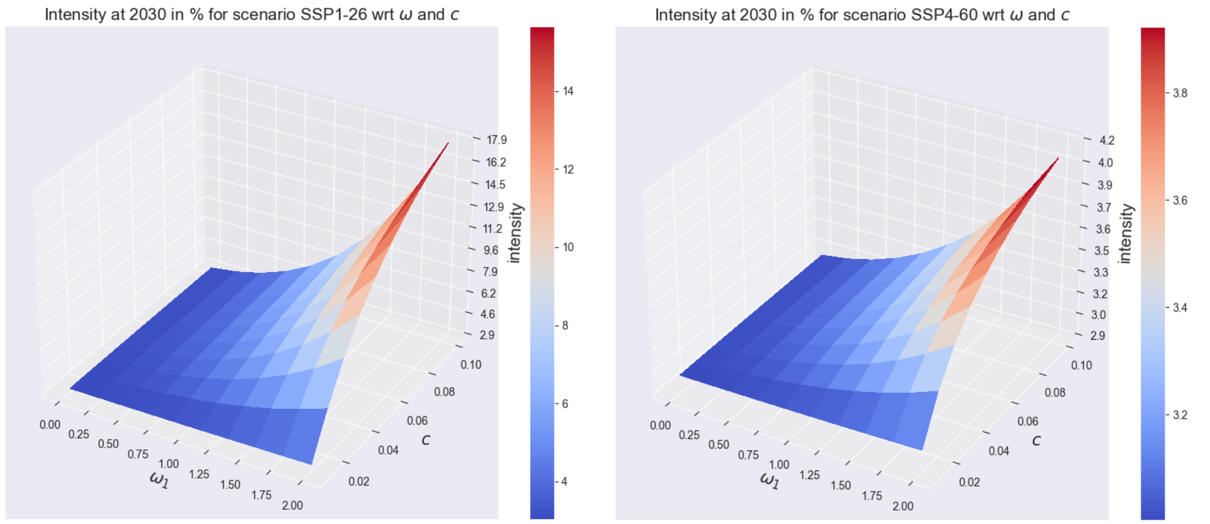

Figure 7: Industrial sector: intensity surface at 2030 for the scenarios SSP1-26 and SSP4-60

the possibility to include intermediate default will induce a gap – for both scenarios – between the default probabilities and intensities obtained in the two credit models. Anyway, choosing a Merton or Black-Cox approach is up to the end-user of credit risk; in our "SSP emission-to-firm production" modeling both are possible and allow in all cases to bridge a projected transition scenario and the credit risk of a company in a given sector.

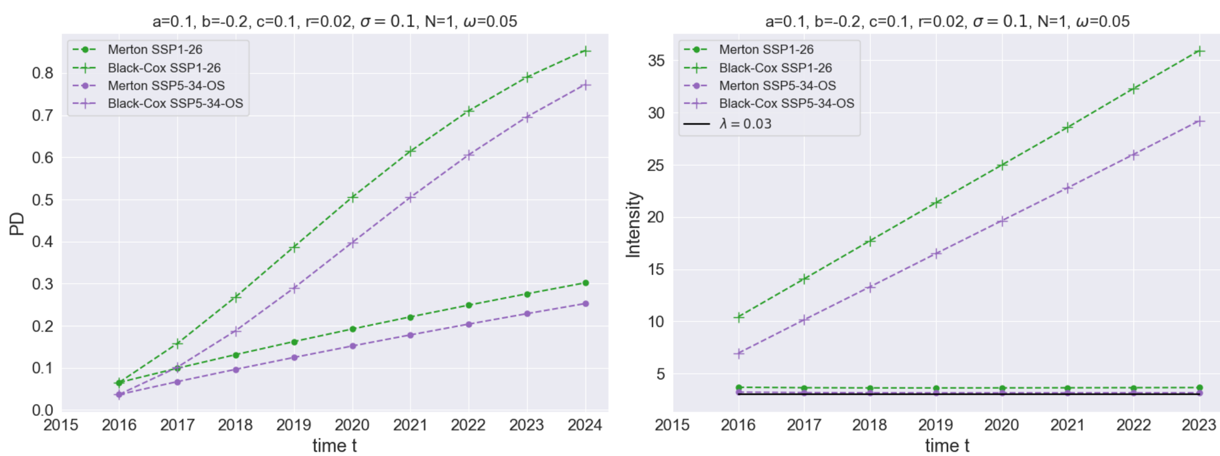

Figure 8: Industrial sector: Black–Cox default probability and associated intensity for the industrial sector.

4 Conclusion

This paper focuses on a firm's transition risk towards low-carbon production and shows how to quantify the credit risk embedded in the CO2 emission mitigation strategy. We consider a profit maximization problem under some loss constraints (related to CO2 emission) and then use the obtained optimal emission to construct the firm's value process and investigate default probabilities and intensities under different SSPs scenarios. An explicit model of production is proposed to derive closed-form computation for related quantities: this is a valuable building block in the climate risk-credit risk nexus.

We conclude that loss penalty imposed on over-emission compared to benchmark may induce the firm to decrease its effective CO2 emission. The adopted strategy depends on the relevant SSP scenario and also on the production characteristics of the firm. A suitable choice of loss function such as incorporating award upon under-emission appears to stimulate more efficiently the firm to achieve the carbon-neutral objective. Concerning credit risk, the more constrained mitigation scenarios are associated with higher default probability and intensity, especially for firms whose production is closely related to emission and when the penalty force is strong. A higher value of penalty force enforces the firm to perform larger emission reduction but increases its default risk.

In this work, we do not investigate the physical risk related to different SSPs scenarios and its impact on the credit quality and default probability: however, the model we have proposed could be adapted to analyze the presence of joint impact by both physical and transition risks, see remarks in Section 3. Moreover, in the numerical examples we have presented, we consider the reference intensity to be constant for illustration, it could be extended in a straightforward manner to account for credit intensity curve. A more comprehensive default model including stochastic intensity could also be more realistic. These points are left to future works.

References

[1] S. Alogoskoufis, N. Dunz, T. Emambakhsh, T. Hennig, M. Kaijser, C. Kouratzoglou, M. A. Munoz, L. Parisi, and C. Salleo. ECB's economy-wide climate stress test, methodology and results. *European Central Bank Occasional Paper,* https://www.ecb.europa.eu/pub/pdf/scpops/ecb.op281~05a7735b1c.en.pdf, (281), September 2021.

[2] M. Auffhammer. Quantifying economic damages from climate change. *Journal of Economic Perspectives*, 32(4):33–52, 2018.

[3] Basel Committee on Banking Supervision. Climate-related risk drivers and their transmission channels. *Bank for International Settlements,* https://www.bis.org/bcbs/publ/d517.htm, d517, 2021.

[4] S. Battiston, A. Mandel, I. Monasterolo, F. Schütze, and G. Visentin. A climate stress-test of the financial system. *Nature Climate Change*, 7(4):283–288, 2017.

[5] F. Black and J. C. Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, 31(2):351–367, 1976.

[6] V. Bouchet and T. Le Guenedal. Credit risk sensitivity to carbon price. *Available at SSRN 3574486*, 2020.

[7] E. Campiglio, Y. Dafermos, P. Monnin, J. Ryan-Collins, G. Schotten, and M. Tanaka. Climate change challenges for central banks and financial regulators. *Nature Climate Change*, 8(6):462–468, 2018.

[8] M. Carney. Speech: "breaking the tragedy of the horizon". *Bank of England*, *https://www.bankofengland.co.uk/speech/2015/breaking-the-tragedy-of-the-horizon-climate-change-and-financial-stability*, September, 29th 2015.

[9] J. Colas, I. Khaykin, A. Pyanet, and J. Westheim. Extending our horizons. assessing credit risk and opportunity in a changing climate: outputs of a working group of 16 banks piloting the TCFD recommendations. *UNEP Finance Initiative-Oliver Wyman* *https://www.unepfi.org/wordpress/wp-content/uploads/2018/04/EXTENDING-OUR-HORIZONS.pdf*, 2018.

[10] J. Detemple and M. Rindisbacher. Dynamic asset liability management with tolerance for limited shortfalls. *Insurance: Mathematics and Economics*, 43(3):281–294, 2008.

[11] J. Durbin. The first-passage density of the Brownian motion process to a curved boundary. *Journal of Applied Probability*, 29(2):291–304, 1992.

[12] N. El Karoui and M.-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization*, 33(1):29–66, 1995.

[13] H. Föllmer and P. Leukert. Efficient hedging: cost vs. shortfall risk. *Finance and Stochastics*, 4(2):117–146, 2000.

[14] H. Föllmer and A. Schied. *Stochastic Finance: an Introduction in Discrete Time*, volume 27 of *De Gruyter Studies in Mathematics*. De Gruyter, 2004.

[15] J. Garnier. The climate extended risk model (cerm). *arXiv preprint arXiv:2103.03275*, 2021.

[16] M. J. Gidden, K. Riahi, S. J. Smith, S. Fujimori, G. Luderer, E. Kriegler, D. P. van Vuuren, M. van den Berg, L. Feng, D. Klein, K. Calvin, J. C. Doelman, S. Frank, O. Fricko, M. Harmsen, T. Hasegawa, P. Havlik, J. Hilaire, R. Hoesly, J. Horing, A. Popp, E. Stehfest, and K. Takahashi. Global emissions pathways under different socioeconomic scenarios for use in cmip6: a dataset of harmonized emissions trajectories through the end of the century. *Geoscientific Model Development*, 12(4):1443–1475, 2019.

[17] E. Gobet and S. Menozzi. Stopped diffusion processes: boundary corrections and overshoot. *Stochastic Processes and their Applications*, 120(2):130–162, 2010.

[18] X. Guo and H. Pham. Optimal partially reversible investment with entry decision and general production function. *Stochastic Processes and their Applications*, 115(5):705–736, 2005.

[19] S. Herrmann and E. Tanré. The first-passage time of the Brownian motion to a curved boundary: an algorithmic approach. *SIAM Journal on Scientific Computing*, 38(1):A196–A215, 2016.

[20] K.-I. Inada. Two-sector model of economic growth: Comments and a generalization. *The Review of Economic Studies*, 30(2):119–127, 1963.

[21] IPCC. Summary for policymakers. In V. Masson-Delmotte, P. Zhai, H.-O. PÃűrtner, D. Roberts, J. Skea, P. Shukla, A. Pirani, W. Moufouma-Okia, C. PÃľan, R. Pidcock, S. Connors, J. Matthews, Y. Chen, X. Zhou, M. Gomis, E. Lonnoy, T. Maycock, M. Tignor, , and T. Waterfield, editors, *Global warming of 1.5 ^oC. An IPCC Special Report on the impacts of global warming of 1.5 ^oC above pre-industrial levels and related global greenhouse gas emission pathways, in the context of strengthening the global response to the threat of climate change, sustainable development, and efforts to eradicate poverty.* World Meteorological Organization, Geneva, Switzerland, 2018.

[22] P. Kalaitzidakis, T. P. Mamuneas, and T. Stengos. Greenhouse emissions and productivity growth. *Journal of Risk and Financial Management*, 11(3):38, Septembre 2018.

[23] H. R. Lerche. *Boundary crossing of Brownian motion*, volume 40 of *Lecture Notes in Statistics*. Springer-Verlag, Berlin, 1986. Its relation to the law of the iterated logarithm and to sequential analysis.

[24] M. Malte, Z. R. J. Nicholls, J. Lewis, M. J. Gidden, E. Vogel, M. Freund, U. Beyerle, C. Gessner, A. Nauels, N. Bauer, J. G. Canadell, J. S. Daniel, A. John, P. B. Krummel, G. Luderer, N. Meinshausen, S. A. Montzka, P. J. Rayner, S. Reimann, S. J. Smith, M. van den Berg, G. J. M. Velders, M. K. Vollmer, and R. H. J. Wang. The shared socio-economic pathway (SSP) greenhouse gas concentrations and their extensions to 2500. *Geoscientific Model Development*, 13:3571–3605, 2020.

[25] A. McNeil, R. Frey, and P. Embrechts. *Quantitative risk management.* Princeton series in finance. Princeton University Press, 2005.

[26] R. C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29(2):449–470, 1974.

[27] P. Monnin. Integrating climate risks into credit risk assessment-current methodologies and the case of central banks corporate bond purchases. *Council on Economic Policies, Discussion Note*, 4, 2018.

[28] NGFS. A call for action climate change as a source of financial risk, April 2019.

[29] E. Pardoux and A. Râscanu. *Stochastic differential equations, Backward SDEs, Partial differential equations*, volume 69. Springer, 2014.

[30] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. *The mathematical theory of optimal processes.* Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962. Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt.

[31] H. J. Reinders, D. Schoenmaker, and M. A. Van Dijk. A finance approach to climate stress testing. *Available at SSRN 3573107*, 2020.

[32] K. Riahi, D. P. van Vuuren, E. Kriegler, J. Edmonds, B. C. O'Neill, S. Fujimori, N. Bauer, K. Calvin, R. Dellink, O. Fricko, W. Lutz, A. Popp, J. C. Cuaresma, S. KC, M. Leimbach, L. Jiang, T. Kram, S. Rao, J. Emmerling, K. Ebi, T. Hasegawa, P. Havlik, F. Humpenöder, L. A. D. Silva, S. Smith, E. Stehfest, V. Bosetti, J. Eom, D. Gernaat, T. Masui, J. Rogelj, J. Strefler, L. Drouet, V. Krey, G. Luderer, M. Harmsen, K. Takahashi, L. Baumstark, J. C. Doelman, M. Kainuma, Z. Klimont, G. Marangoni, H. Lotze-Campen, M. Obersteiner, A. Tabeau, and M. Tavoni. The shared socioeconomic pathways and their energy, land use, and greenhouse gas emissions implications: An overview. *Global Environmental Change*, 42:153–168, jan 2017.

[33] J. Rogelj, A. Popp, K. V. Calvin, G. Luderer, J. Emmerling, D. Gernaat, S. Fujimori, J. Strefler, T. Hasegawa, G. Marangoni, V. Krey, E. Kriegler, K. Riahi, D. P. van Vuuren, J. Doelman, L. Drouet, J. Edmonds, O. Fricko, M. Harmsen, P. Havlík, F. Humpenöder, E. Stehfest, and M. Tavoni. Scenarios towards limiting global mean temperature increase below 1.5 ^{o}c. *Nature Climate Change*, 8(4):325–332, mar 2018.

[34] T. Roncalli. *Handbook of Financial Risk Management.* CRC Press, 2020.

[35] D. H. Wagner. Survey of measurable selection theorems: an update. In *Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979)*, volume 794 of *Lecture Notes in Math.*, pages 176–219. Springer, Berlin-New York, 1980.