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Pricing formulae for derivatives in insurance using the Malliavin calculus*

Caroline Hillairet[†] Ying Jiao[‡] Anthony Réveillac[§]

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Abstract

In this paper we provide a valuation formula for different classes of actuarial and financial contracts which depend on a general loss process, by using the Malliavin calculus. In analogy with the celebrated Black-Scholes formula, we aim at expressing the expected cash flow in terms of a building block. The former is related to the loss process which is a cumulated sum indexed by a doubly stochastic Poisson process of claims allowed to be dependent on the intensity and the jump times of the counting process. For example, in the context of Stop-Loss contracts the building block is given by the distribution function of the terminal cumulated loss, taken at the Value at Risk when computing the Expected Shortfall risk measure.

1 Introduction

Risk analysis in the context of insurance or reinsurance is often based on the study of properties of a so-called *cumulative loss process* $L := (L_t)_{t\in[0,T]}$ over a period of time $[0,T]$ where $T>0$ denotes the maturity of a contract. Usually, L takes the form

$$L_t := \sum_{i=1}^{N_t} X_i, \quad t \in [0,T],$$

where $N := (N_t)_{t\in[0,T]}$ is a counting process, and the random variables $(X_i)_{i\in\mathbb{N}^*}$ represent the amount of the claims. A typical contract in reinsurance is the *Stop-Loss contract* that offers protection against an increase in either (or both) severity and frequency of a company's loss experience. More precisely, Stop-loss contracts provide to its buyer (another insurance company) the protection against losses which are larger than a given level K and its payoff function is given by a "call" function. In some cases, there is also an upper limit given by some

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real number M, which specifies the maximal reimbursement amount. Thus the payoff of such a contract is given by

$$\Phi(L_T) = \begin{cases} 0, & \text{if } L_T < K; \\ L_T - K, & \text{if } K \leq L_T < M; \\ M - K, & \text{if } L_T \geq M. \end{cases} \tag{1.1}$$

In full generality the risk carried out by the claims is neither hedgeable nor related to a financial market, hence the premium of the Stop-Loss is equal to $\mathbb{E}[\Phi(L_T)]$ which immediately re-writes as

$$\mathbb{E}[\Phi(L_T)] = \mathbb{E}\left[L_T \mathbf{1}_{\{L_T \in [K,M]\}}\right] - K\mathbb{P}\left[L_T \in [K, M]\right] + (M - K)\mathbb{P}\left[L_T \geq M\right]. \tag{1.2}$$

There is a large number of papers describing how to approximate the compound distribution function of the cumulated loss L_T, and to compute the Stop-Loss premium. The aggregate claims distribution function can in some cases be calculated recursively, using, for example, the Panjer recursion formula, see Panjer [12] and Gerber [11]. Various approximations of Stop-Loss reinsurance premiums are described in the literature, some of them assuming a specific dependence structure.

In analogy with the celebrated Black-Scholes formula, we aim in this paper to express the first term of the right-hand side of (1.2) in terms of a building block which represents the distribution function of the terminal loss L_T. This feature is hidden in the Black-Scholes model since the terminal value of the stock has an explicit lognormal distribution. More specifically, we aim in computing $\mathbb{E}\left[L_T \mathbf{1}_{\{L_T \in [K,M]\}}\right]$ by using the building block $x \mapsto \mathbb{P}\left[L_T \in [K - x, M - x]\right]$. Note that, on the credit derivative market, the payoff function (1.1) can also be related to Collateralized Debt Obligations (CDOs) where there are several tranches, and so several K and M levels, which are expressed in proportion of the underlying which is the loss of a given asset portfolio.

Stop-Loss contracts are the paradigm of reinsurance contracts, but we aim in dealing with more general payoffs whose valuation involves the computation of the quantity

$$\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right], \tag{1.3}$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borelian map and where $\hat{L}$ is of the form $\hat{L}_T := \sum_{i=1}^{N_T} \hat{X}_i$, involving claims $\hat{X}_i$ which are related to the ones X_i of the original loss L_T. To be more precise, $\hat{L}_T$ will be the effective loss covered by the reinsurance company whereas L_T is the loss quantity that activates the contract. Typical examples will be given in Section 2.1. Once again, this is similar to the valuation of CDOs tranches where the recovery rate is often supposed to be a random variable of beta distribution with mean 40% whereas the realized rate, often revealed only after the formal bankruptcy, does not necessarily match with this value.

In this paper we provide an exact formula for (1.3) in terms of the building block $x \mapsto \mathbb{E}\left[h(L_T + x)\right]$ (or of a related quantity for the more general situation (1.3), see (3.4) for a precise statement). This goal will be achieved by using one of the Malliavin calculus available for jump processes. Before turning to the exposition of the model, we emphasize that this methodology goes beyond the analysis of pricing and finds for instance application in the computation of the Expected Shortfall of contingent claims in the realm of risk measures. Indeed, the expected shortfall is a useful risk measure, that takes into account the size of the expected loss above the value at risk. Formally it is defined as

$$ES_\alpha(-L_T) = \mathbb{E}\left[-L_T | -L_T > V@R_\alpha(-L_T)\right], \quad \alpha \in (0,1).$$

As it is well-known, the expected shortfall coincides with Average Value at Risk (AV@R), that is

$$ES_\alpha(-L_T) = AV@R(-L_T) := \frac{1}{1-\alpha}\int_\alpha^1 V@R_s(-L_T)ds,$$

if and only if $\mathbb{P}[-L_T \leq q^+_{-L_T}(t)] = t$, $t \in (0,1)$, where $q^+_{-L_T}(t)$ denotes the quantile of level t of $-L_T$ (see Section 2.2.2 for a precise definition). However, already in the trivial example where the size claims X_i are constant equal to 1, this property fails as $L_T = N_T$ is a Poisson random variable which exhibits a discontinuous distribution function. However, our approach gives an alternative explicit computation of $\mathbb{E}[L_T \mathbf{1}_{\{L_T<\beta\}}]$ and thus of $ES_\alpha(-L_T)$ as

$$ES_\alpha(-L_T) = \frac{-\mathbb{E}[L_T \mathbf{1}_{\{L_T<\beta\}}]}{\mathbb{P}(L_T < \beta)}, \quad \beta := -V@R_\alpha(-L_T).$$

We conclude this section with some comments about the modeling of the claims X_i and $\hat{X}_i$. In the classic Cramer-Lundberg model, the claims are independent and identically distributed (i.i.d.) and in addition independent of the counting process N which happens to be an inhomogeneous Poisson process. In this work we consider a doubly stochastic Poisson process N and we allow dependency between the size of the claims, their arrivals and the intensity of N. In particular we do not assume a Markovian setting. The impact of certain dependence structure on the Stop-Loss premium is studied in the reinsurance literature, such as in Albers [1], Denuit et al. [9] or De Lourdes Centeno [8], but those works usually assume dependency between the successive claim sizes and the arrival intervals. Nevertheless, in the ruin theory literature, some contributions already propose explicit dependencies among inter-arrival times and the claim sizes, such as Albrecher and Boxma [2], Boudreault, Cossette, Landriault and Marceau [7] and related works. A general framework of dependencies is proposed by Albrecher, Constantinescu, and Loisel [4] in which the dependence arises via mixing through a so-called frailty parameter. Recently, Albrecher et al. [3] extend duality results that relate survival and ruin probabilities in the insurance risk model to waiting time distributions in the 'corresponding' queueing model. The risk processes have a counterpart in workload models of queueing theory, and a similar mixing dependencies structure is considered in a queueing context. Besides, our framework extends the mixing approach of [4] and [3] by allowing non-exchangeable family of random variables for the claims amounts. In a similar way, in the credit risk modeling we can also suppose that the recovery rate depend on the underlying default intensity such as in Bakshi, Madan and Zhang [5].

We proceed as follows. We first make clear in Section 2 our model for the loss process and present the insurance contracts for which we will propose a pricing formula. The latter will be stated and proved as Theorem 3.5 in Section 3. Particular cases of this result to several types of contracts in insurance are also given in this section. Finally, explicit examples are presented in Section 4.

2 Model Setup

In this section, we describe the loss process and the associated reinsurance contracts we will study. Throughout this paper, T will denote a positive finite real number which represents the final horizon time.

2.1 The Loss process

We begin by introducing the loss process $L := (L_t)_{t\in[0,T]}$ where the size of claims and their arrival times are correlated. Let $(N_t)_{t\in[0,T]}$ be a Cox process (also called doubly stochastic Poisson process) with random intensity $(\lambda_t)_{t\in[0,T]}$, whose jump times, denoted by $(\tau_i)_{i\in\mathbb{N}^*}$, model the arrival times of the claims. We suppose that the claim size X_i depends on both the cumulated intensity defined by $\Lambda_t := \int_0^t \lambda_s ds$ and the claim arrival time τ_i. Moreover, it will also depend on some random variable ε_i where we suppose that $(\varepsilon_i)_{i\in\mathbb{N}^*}$ is a sequence of positive i.i.d. random variables independent of the Cox process N. More precisely, the loss is given by

$$L_t := \sum_{i=1}^{N_t} X_i \, e^{-\kappa(t-\tau_i)}, \qquad \text{with } X_i := f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i), \quad t \in [0,T], \tag{2.1}$$

where κ is the discounting factor and $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ is a bounded deterministic function. We provide several examples as below.

Example 2.1.

1. In the classic ruin theory, the claim size is often supposed to be independent of the arrival and the intensity process. In this case, we have $f(t,\ell,x) = x$.

2. In the second example, we suppose that the dependence of f on the exogenous factor ε is linear and the linear coefficient is a function of the cumulated intensity Λ rescaled by time, *i.e.*, $\frac{\Lambda_t}{t}$, which stands for some mean level of the intensity. For instance, let

$$f(t,\ell,x) = \sqrt{\frac{\ell}{t}}x.$$

   In this example, if ε_i follows an exponential distribution with parameter 1, then $X_i = f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)$ follows an exponential distribution with parameter $\sqrt{\frac{\tau_i}{\Lambda_{\tau_i}}}$ conditionally to the vector $(\tau_i, \Lambda_{\tau_i})$.

2.1.1 Generalized loss process

We can also consider a more general case where the realized claim sizes $(X_i)_{i\in\mathbb{N}^*}$ are not exactly the ones that are computed to activate the reinsurance contract. More precisely, assume that in addition to the factors $(\varepsilon_i)_{i\in\mathbb{N}^*}$, there exists a family of i.i.d. positive random variables $(\vartheta_i)_{i\in\mathbb{N}^*}$ which may depend on the random variables ε_i's. Let $g : \mathbb{R}_+^4 \to \mathbb{R}_+$ be a deterministic bounded function. We can define a modified cumulative loss process as

$$\hat{L}_t := \sum_{i=1}^{N_t} g(\tau_i, \Lambda_{\tau_i}, \varepsilon_i, \vartheta_i) e^{-\kappa(t-\tau_i)}, \quad t \in [0,T]. \tag{2.2}$$

More precisely, although the insurance contract is triggered by the loss process L, the compensation amount can depend on some other exogenous factors $(\vartheta_i)_{i\in\mathbb{N}^*}$. This would mean for instance that the amounts ϑ_i's are much lower than the ε_i's. A typical example is given by the housing insurance market on the American East Coast. Indeed, this region is seasonally exposed to hurricanes of different magnitudes. Most of the damages impacts the houses of the insured who may as well buy contracts on other belongings such as cars which are much less valuable. After a hurricane episode, the re-insurance Stop-Loss contract will be activated on the basis of the total damages L_T on the houses (which are represented by the claims ε_i) whereas the effective damages $\hat{L}_T$ will also include all other insured belongings (which would be modeled by the ϑ_i). In the special case where the function g does not depend on the fourth variable, the general loss $\hat{L}_t$ reduces to the standard loss defined in (2.1). We give below some examples of the joint distribution $(\varepsilon_i, \vartheta_i)$.

Example 2.2.

1. The first natural case is that ε_i and ϑ_i are independent random variables. For example, each of them can follow an exponential distribution (or Erlang distribution) with different positive parameters θ_1 and θ_2.
2. We can introduce dependence between ε_i and ϑ_i by using the mixing method in [4]. Let ε_i and ϑ_i follow Pareto marginal distributions respectively and a dependence structure according to a Clayton copula (according to Example 2.3 in [4], this can be achieved by mixing the two Pareto marginal distributions where the mixing parameter follows a Gamma distribution).
3. Case of explicit dependence : let ε_i follow a Pareto distribution and ϑ_i follow a Weibull distribution with form or scaling parameter depending of ε_i.

2.2 Reinsurance contracts and related quantities

2.2.1 Generalized Stop-loss Contrats

We have seen in the introduction the Stop-Loss contract whose payoff is given by $\Phi(L_T)$ where Φ has been defined in (1.1) and corresponds to a call spread, that is, the difference of two call functions. Our approach allows us to go beyond the case of the Stop-Loss contract. Consider now a contract where the reinsurance company pays

$$\widetilde{\Phi}(L_T, \hat{L}_T) = \begin{cases} 0, & \text{if } L_T \leq K \\ \hat{L}_T - K, & \text{if } K \leq L_T \leq M \\ M - K, & \text{if } L_T \geq M \end{cases}, \tag{2.3}$$

with $\hat{L}_T$ defined in (2.2) if the *a priori* loss L_T excesses some amount K or belongs to some interval $[K, M]$. Then the price of such a contract is :

$$\mathbb{E}\left[\hat{L}_T \mathbf{1}_{\{L_T > K\}}\right] - K\mathbb{P}\left[L_T \in [K, M]\right] + (M - K)\mathbb{P}\left[L_T \geq M\right]. \tag{2.4}$$

2.2.2 Expected Shortfall

The expected shortfall is a useful risk measure which takes into account the size of the expected loss above the value at risk. We recall the Expected Shortfall with level α as

$$ES_\alpha(-L_T) = \mathbb{E}\left[-L_T | -L_T > V@R_\alpha(-L_T)\right], \quad \alpha \in (0, 1).$$

where the definition of $V@R$ is

$$V@R_\alpha(X) = -q_X^+(\alpha) = q_{-X}^-(1-\alpha)$$

with

$$q_X^+(t) = \inf\{x|\,\mathbb{P}[X \le x] > t\} = \sup\{x|\,\mathbb{P}[X < x] \le t\}$$
$$q_X^-(t) = \sup\{x|\,\mathbb{P}[X < x] < t\} = \inf\{x|\,\mathbb{P}[X \le x] \ge t\}.$$

It is well known that $ES_\alpha(X)$ is equal to $AV@R(X) := \frac{1}{1-\alpha}\int_\alpha^1 V@R_s(X)ds$ if and only if $\mathbb{P}[X \le q_X^+(t)] = t$, $t \in (0,1)$, which is in particular satisfied if the distribution function of X is continuous (see *e.g.* [10, Relation (4.38)]). However, the latter property fails already in the case where the size claims X_i are constant. Thus one can not rely on the above relation and has to compute directly the conditional expectation $ES_\alpha(-L_T)$.

We will provide an alternative expression for the expected shortfall. We denote by $\beta := -V@R_\alpha(-L_T)$, then

$$ES_\alpha(-L_T) = \frac{-\mathbb{E}\left[L_T \mathbf{1}_{\{L_T<\beta\}}\right]}{\mathbb{P}[L_T < \beta]}$$

where

$$\beta = q_{-L_T}^+(\alpha) = \inf\{x|\,\mathbb{P}[L_T > -x] > \alpha.\}$$

So once again the key term to compute turns out to be the expectation $\mathbb{E}\left[L_T \mathbf{1}_{\{L_T<\beta\}}\right]$.

2.3 General payoffs

More generally, we are interested in computing quantities of the form

$$\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right],$$

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borelian map with $\mathbb{E}[h(L_T)] < \infty$. Since in our model, the counting process is given by a Cox process with stochastic intensity, the building block becomes the following mapping by using the conditional expectation

$$x \mapsto \mathbb{E}\left[h(L_T + x)|(\lambda_t)_{t\in[0,T]}\right].$$

Note that the examples of Section 2.2.1 (respectively of Section 2.2.2) are contained in this setting by choosing $h := \mathbf{1}_{[K,M]}$ for some $-\infty \le K < M \le +\infty$ (respectively $h := \mathbf{1}_{[-\infty,\beta]}$ and $\hat{L}_T = L_T$).

Our approach calls for some stochastic analysis material that we present in the next section.

3 The pricing formulae using the Malliavin calculus

In this section, we establish our main pricing formulae by using the Malliavin calculus. To this end, we first make precise the Poisson space associated to the loss process. Then we provide basic tools for the Malliavin calculus.

3.1 Construction of the Poisson space

3.1.1 The counting process and intensity process

We recall that the loss process involves the Cox process $(N_t)_{t\in[0,T]}$ with its intensity and jump times, and the family of random variables $(\varepsilon_i)_{i\in\mathbb{N}^*}$. We begin by introducing a general counting process which will be useful for the construction of $(N_t)_{t\in[0,T]}$ on a suitable space. Let Ω_1 be the set of (finite or infinite) strictly increasing sequences in $]0,+\infty[$. We define a continuous-time stochastic process $\mathcal{C}$ on the set Ω_1 as

$$\forall\,(t,\omega_1)\in[0,+\infty[\times\Omega_1,\quad \mathcal{C}_t(\omega_1):=\mathrm{card}([0,t]\cap\omega_1).$$

Let $\mathbb{F}^{\mathcal{C}}=(\mathcal{F}_t^{\mathcal{C}})$ be the filtration generated by the process $\mathcal{C}$, namely $\mathcal{F}_t^{\mathcal{C}}:=\sigma(\mathcal{C}_s,\,s\leq t)$. It is known that there exists a unique probability measure $\mathbb{P}_1$ on $(\Omega_1,\mathcal{F}_\infty^{\mathcal{C}})$ under which the process $\mathcal{C}$ is a Poisson process of intensity 1, that is, for every $(s,t)\in[0,+\infty)^2$, with $s<t$, the random variable $\mathcal{C}_t-\mathcal{C}_s$ is independent of $\mathcal{F}_s^{\mathcal{C}}$ and Poisson distributed with parameter $t-s$.

We then consider a probability space $(\Omega_2,\mathcal{A},\mathbb{P}_2)$ on which is defined :

(i) a positive stochastic process $(\lambda_t)_{t\in[0,T]}$ such that

$$\int_0^T\lambda_s ds<+\infty,\quad \mathbb{P}_2\text{ - a.s..}$$

(ii) a collection of i.i.d. $\mathbb{R}_+^2$-valued bounded random variables $(\varepsilon_i,\vartheta_i)_{i\in\mathbb{N}^*}$ and a $\mathbb{R}_+^2$-random variable $(\overline{\varepsilon},\overline{\vartheta})$ independent from $(\varepsilon_i,\vartheta_i)_{i\in\mathbb{N}^*}$, with $(\overline{\varepsilon},\overline{\vartheta})\overset{\mathcal{L}}{=}(\varepsilon_1,\vartheta_1)$ (where $\overset{\mathcal{L}}{=}$ stands for the equality of probability distributions). We set μ the law of the pair $(\overline{\varepsilon},\overline{\vartheta})$.

Assumption 3.1. *We assume that λ is independent of $(\varepsilon_i,\vartheta_i)_{i\in\mathbb{N}^*}$, and of $(\overline{\varepsilon},\overline{\vartheta})$.*

$\mathbb{F}^\lambda=(\mathcal{F}_t^\lambda)_{t\in[0,T]}$ be the right-continuous complete filtration generated by the stochastic process λ. Moreover, we set

$$\Lambda_t:=\int_0^t\lambda_s ds,\quad t\in[0,T]. \tag{3.1}$$

Let $\mathcal{F}^{\varepsilon,\vartheta}$ be the σ-algebra generated by $(\varepsilon_i)_{i\in\mathbb{N}^*}$ and $(\vartheta_i)_{i\in\mathbb{N}^*}$. Note that only $(\varepsilon_i)_{i\in\mathbb{N}^*}$ and $(\vartheta_i)_{i\in\mathbb{N}^*}$ will be involved in the loss process and $\overline{\varepsilon}$ and $\overline{\vartheta}$ are just independent copies which play an auxiliary role. We denote by μ the probability law of the couple $(\varepsilon_i,\vartheta_i)$.

Assumption 3.2. *Throughout this paper, we assume that : $\Lambda_T<+\infty$, $\mathbb{P}_2-a.s.$.*

3.1.2 The doubly stochastic Poisson process

We now consider the product space $(\Omega:=\Omega_1\times\Omega_2,\mathcal{F}:=\mathcal{F}_\infty^{\mathcal{C}}\otimes\mathcal{A},\mathbb{P}:=\mathbb{P}_1\otimes\mathbb{P}_2)$. By abuse of notation, any random variable Y on Ω_1 can be considered as a random variable on Ω which sends $\omega=(\omega_1,\omega_2)$ to $Y(\omega_1)$. Similarly, any random variable Z on Ω_2 can be considered as a random variable on Ω which sends $\omega=(\omega_1,\omega_2)$ to $Z(\omega_2)$.

We define a counting process $N:=(N_t)_{t\in[0,T]}$ on Ω by using a time change as

$$N_t(\omega_1,\omega_2):=\mathcal{C}_{\Lambda_t(\omega_2)}(\omega_1)=\mathcal{C}_{\int_0^t\lambda_s(\omega_2)ds}(\omega_1),\quad t\in[0,T],\ (\omega_1,\omega_2)\in\Omega.$$

Note that for any t, N_t is $\mathcal{F}^{\mathcal{C}}_\infty \otimes \mathcal{F}^\lambda_T$-measurable random variable. Moreover, for any fixed ω_2 in Ω_2, $N_t(\cdot,\omega_2)$ is an inhomogeneous Poisson process on Ω_1 with intensity $t \mapsto \lambda_t(\omega_2)$ with respect to the filtration $(\mathcal{F}^{\mathcal{C}}_{\Lambda_t(\omega_2)})_{t\in[0,T]}$ which reads as[1]

$$\mathbb{E}\left[e^{iu(N_t-N_s)}\Big|\mathcal{F}^\lambda_s\right] = \mathbb{E}\left[\exp\left((e^{iu}-1)\int_s^t \lambda_r dr\right)\Big|\mathcal{F}^\lambda_s\right], \quad 0 \leq s < t \leq T,$$

where $\mathbb{E}$ denotes the expectation with respect to the measure $\mathbb{P}$. For a process $(u_t)_{t\in[0,T]}$ such that :

$$\begin{cases} u_t \text{ is } \mathcal{F}\text{-measurable}, \quad t \in [0,T], \\ \text{for a.e. } \omega_2 \in \Omega_2, (u_t(\cdot,\omega_2))_{t\in[0,T]} \text{ is } (\mathcal{F}^{\mathcal{C}}_{\Lambda_t(\omega_2)})_{t\in[0,T]}\text{-predictable}, \\ \mathbb{E}\left[\int_0^T |u_t|dt\right] < +\infty, \end{cases} \tag{3.2}$$

we denote by $\left(\int_0^T u_s dN_s\right)(\omega_1,\omega_2)$ the Lebesgue-Stieltjes integral of $u(\omega_1,\omega_2)$ against the measure $N(\omega_1,\omega_2)$.

For any $i \in \mathbb{N}$, we let τ_i be the i-th jump time of the process N, namely

$$\forall\, \omega = (\omega_1,\omega_2) \in \Omega, \quad \tau_i(\omega) := \inf\{t > 0,\ N_t = \mathcal{C}_{\Lambda_t(\omega_2)}(\omega_1) \geq i\},$$

with the convention $\tau_0 = 0$.

3.2 The Malliavin integration by parts formula

We can now state the Malliavin integration by parts formula on the product space. For any $t \in [0,T]$, and $\omega_1 \in \Omega_1$ which is of finite length or has a limit greater than t, we define $\omega_1 \cup \{t\}$ in Ω_1 as the increasing sequence whose underlying set is the union of ω_1 and t. The effect of this operator is to add a jump at time t to the Poisson process N. Finally, for $\omega := (\omega_1,\omega_2) \in \Omega$, and $t \in [0,T]$, we set

$$\omega \cup \{t\} := (\omega_1 \cup \{t\}, \omega_2),$$

provided that $\omega_1 \cup \{t\}$ is well defined. The following lemma is a direct extension of the one presented for example in [13, Corollaire 5] or [14] (see also [15]).

Lemma 3.3. *Let $u : \Omega \times [0,T] \to \mathbb{R}$ be a stochastic process which enjoys (3.2), and $F : \Omega \to \mathbb{R}$ be a bounded $\mathcal{F}$-measurable random variable. Then the stochastic process $(\omega,t) \mapsto F(\omega \cup \{t\})$ is well-defined $\mathbb{P} \otimes dt$-a.e. and*

$$\mathbb{E}\left[F\int_0^T u_s dN_s\Big|\mathcal{F}^\lambda_T \vee \mathcal{F}^{\varepsilon,\vartheta}\right] = \mathbb{E}\left[\int_0^T u_t\, F(\cdot \cup \{t\})\lambda_t dt\Big|\mathcal{F}^\lambda_T \vee \mathcal{F}^{\varepsilon,\vartheta}\right]. \tag{3.3}$$

3.3 The main result

In this section we present our main result concerning the computation of the quantity

$$\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right],$$

[1]By a slight abuse of notation, $\mathbb{E}\left[\cdot|\mathcal{F}^\lambda_T\right] := \mathbb{E}\left[\cdot|\mathcal{F}^{\mathcal{C}}_0 \otimes \mathcal{F}^\lambda_T\right]$ and $\mathbb{E}\left[\cdot|\mathcal{F}^\lambda_T \vee \mathcal{F}^{\varepsilon,\vartheta}\right] := \mathbb{E}\left[\cdot|\mathcal{F}^{\mathcal{C}}_0 \otimes (\mathcal{F}^\lambda_T \vee \mathcal{F}^{\varepsilon,\vartheta})\right]$.

where $h : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borelian map with $\mathbb{E}[h(L_T)] < \infty$ and where L_T and $\hat{L}_T$ are respectively defined in (2.1) and (2.2). We set

$$\varphi_\lambda^h(x) := \mathbb{E}\left[h(L_T + x)|\mathcal{F}_T^\lambda\right], \quad x \in \mathbb{R}_+. \tag{3.4}$$

It might be surprising at first glance to consider the conditional expectation given λ in the building block. In fact, as the intensity λ of N is random, it can be compared to a Black-Scholes model with independent stochastic volatility. In that context the Black-Scholes formula would be written in terms of the conditional law of the terminal value of the stock given the volatility (which would simply be a lognormal distribution with variance given by the volatility). Recall that for the insurance contract presented in Section 2.2.1, $h := \mathbf{1}_{[K,M]}$ and thus φ_λ^h coincides with the conditional distribution function of L_T.

Before turning to the statement and the proof of the main result, note that

$$\hat{L}_T = \int_0^T \hat{Z}_s dN_s, \tag{3.5}$$

with

$$\hat{Z}_s := \sum_{i=1}^{+\infty} g(s, \Lambda_s, \varepsilon_i, \vartheta_i) e^{-\kappa(T-s)} \mathbf{1}_{(\tau_{i-1}, \tau_i]}(s), \quad s \in [0, T]. \tag{3.6}$$

Moreover on the set $\{\Delta_s N = 0\}$, one has

$$\hat{Z}_s = g(s, \Lambda_s, \varepsilon_{1+N_s}, \vartheta_{1+N_s}) e^{-\kappa(T-s)}. \tag{3.7}$$

As Λ is a continuous process, $\hat{Z}$ satisfies Relation (3.2), provided that $\mathbb{E}\left[\int_0^T |\hat{Z}_t| dt\right] < +\infty$.

We start our analysis with the following lemma.

Lemma 3.4. *Under Assumptions 3.1 and 3.2, for any $t \in [0, T]$, it holds that*

$$\left(g(t, \Lambda_t, \varepsilon_{1+N_t}, \vartheta_{1+N_t}) e^{-\kappa(T-t)}, L_T(\cdot \cup \{t\}), \lambda_t\right) \overset{\mathcal{L}}{=} \left(g(t, \Lambda_t, \overline{\varepsilon}, \overline{\vartheta}) e^{-\kappa(T-t)}, L_T + f(t, \Lambda_t, \overline{\varepsilon}) e^{-\kappa(T-t)}, \lambda_t\right).$$

Proof. We set

$$L_t := \sum_{i=1}^{N_t} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i) e^{-\kappa(T-\tau_i)}, \quad L_t^+ := \sum_{i=1}^{N_t} f(\tau_i, \Lambda_{\tau_i}, \varepsilon_{i+1}) e^{-\kappa(T-\tau_i)}, \quad t \in [0, T].$$

We first precise the value of $L_T(\omega \cup \{t\})$ for a fixed element $t \in (0, T)$ and for $\omega := (\omega_1, \omega_2)$ in Ω such that $t \notin \omega_1$ and $\omega_1 \cup \{t\}$ is well defined (the set of such ω has probability 1). By definition, we have that

$$L_T(\omega \cup \{t\}) = \sum_{i=1}^{N_T(\omega \cup \{t\})} f(\tau_i(\omega \cup \{t\}), \Lambda_{\tau_i(\omega \cup \{t\})}(\omega_2), \varepsilon_i(\omega_2)) e^{-\kappa(T - \tau_i(\omega \cup \{t\}))}$$

Note that one has

$$\forall\, i \in \mathbb{N}, \quad \tau_i(\omega \cup \{t\}) = \begin{cases} \tau_i(\omega), & \text{if } i \leq N_t(\omega), \\ t, & \text{if } i = N_t(\omega) + 1 \\ \tau_{i-1}(\omega), & \text{if } i > N_t(\omega) + 1. \end{cases}$$

Therefore we can write $L_T(\omega \cup \{t\})$ as the sum of three terms as follows

$$\begin{aligned} L_T(\omega \cup \{t\}) = &\sum_{i=1}^{N_t(\omega)} f(\tau_i(\omega_1), \Lambda_{\tau_i(\omega)}(\omega_2), \varepsilon_i(\omega_2))e^{-\kappa(T-\tau_i(\omega_1))} \\ &+ f(t, \Lambda_t(\omega_2), \varepsilon_{1+N_t(\omega)}(\omega_2))e^{-\kappa(T-t)} \\ &+ \sum_{i=N_t(\omega)+2}^{N_T(\omega)+1} f(\tau_{i-1}(\omega_1), \Lambda_{\tau_{i-1}(\omega)}(\omega_2), \varepsilon_i(\omega_2))e^{-\kappa(T-\tau_{i-1}(\omega_1))}. \end{aligned} \tag{3.8}$$

By definition, the first term in the sum is just $L_t(\omega)$. Moreover, by a change of index we can write the third term as

$$\sum_{i=N_t(\omega)+1}^{N_T(\omega)} f(\tau_i(\omega), \Lambda_{\tau_i(\omega)}(\omega_2), \varepsilon_{i+1}(\omega_2))\mathrm{e}^{-\kappa(T-\tau_i(\omega))} = L_T^+(\omega) - L_t^+(\omega). \tag{3.9}$$

Therefore by (3.8) the following equality holds almost surely

$$f(t, \Lambda_t, \varepsilon_{1+N_t})e^{-\kappa(T-t)} = (L_T(\cdot \cup \{t\}) - L_t) - (L_T^+ - L_t^+). \tag{3.10}$$

Moreover, from the decomposition formula (3.8) we also observe that ε_{1+N_t} is independent of $L_t + L_T^+ - L_t^+$ given $\mathcal{F}_\infty^{\mathcal{C}} \otimes \mathcal{F}_T^\lambda$. In addition, by Assumption 3.1 the conditional law of ε_{1+N_t} given $\mathcal{F}_\infty^{\mathcal{C}} \otimes \mathcal{F}_T^\lambda$ identifies with the law of $\overline{\varepsilon}$ since $\mathcal{F}^\varepsilon$ is independent of $\mathcal{F}_T^\lambda$.

We now compute the characteristic functions of the two random vectors of interest. Let χ be the characteristic function of the random vector

$$\Big(g(t, \Lambda_t, \varepsilon_{1+N_t}, \vartheta_{1+N_t})e^{-\kappa(T-t)}, L_T(\cdot \cup \{t\}), \lambda_t\Big).$$

Let $(u_1, u_2, u_3) \in \mathbb{R}^3$. One has

$$\begin{aligned} \chi(u_1, u_2, u_3) &:= \mathbb{E}\left[e^{iu_1 g(t,\Lambda_t,\varepsilon_{1+N_t},\vartheta_{1+N_t})e^{-\kappa(T-t)} + iu_2 L_T(\cdot\cup\{t\}) + iu_3\lambda_t}\right] \\ &= \mathbb{E}\left[e^{iu_3\lambda_t} e^{iu_1 g(t,\Lambda_t,\varepsilon_{1+N_t},\vartheta_{1+N_t})e^{-\kappa(T-t)} + iu_2\left(L_t + e^{-\kappa(T-t)}(f(t,\Lambda_t,\varepsilon_{1+N_t}) + L_T^+ - L_t^+\right)}\right] \\ &= \mathbb{E}\left[e^{iu_3\lambda_t} e^{iu_2(L_t + L_T^+ - L_t^+)} e^{iu_2 e^{-\kappa(T-t)} f(t,\Lambda_t,\varepsilon_{1+N_t})} e^{iu_1 e^{-\kappa(T-t)} g(t,\Lambda_t,\varepsilon_{1+N_t},\vartheta_{1+N_t})}\right]. \end{aligned}$$

Since ε_{1+N_t} and ϑ_{1+N_t} are independent of $L_t + L_T^+ - L_t^+$ given $\mathcal{F}_\infty^{\mathcal{C}} \otimes \mathcal{F}_T^\lambda$, we obtain that

$$\chi(u_1, u_2, u_3) = \mathbb{E}\left[e^{iu_3\lambda_t} e^{iu_2 e^{-\kappa(T-t)} f(t,\Lambda_t,\overline{\varepsilon})} e^{iu_1 e^{-\kappa(T-t)} g(t,\Lambda_t,\overline{\varepsilon},\overline{\vartheta})} \mathbb{E}\left[e^{i\mu_2(L_t + L_T^+ - L_t^+)} \,\Big|\, \mathcal{F}_\infty^{\mathcal{C}} \otimes \mathcal{F}_T^\lambda\right]\right],$$

where we also use the fact that the probability law of $(\varepsilon_{1+N_t}, \vartheta_{1+N_t})$ given $\mathcal{F}_\infty^{\mathcal{C}} \otimes \mathcal{F}_T^\lambda$ coincides with μ (which, we recall, is the probability law of $(\overline{\varepsilon}, \overline{\vartheta})$). Moreover, from (3.9) we observe that $L_t + L_T^+ - L_t^+$ has the same law as L_T conditioned on $\mathcal{F}_\infty^{\mathcal{C}} \otimes \mathcal{F}_T^\lambda$. Therefore, we obtain

$$\begin{aligned} \chi(u_1, u_2, u_3) &= \mathbb{E}[e^{iu_3\lambda_t} e^{iu_2 e^{-\kappa(T-t)} f(t,\Lambda_t,\overline{\varepsilon})} e^{iu_1 e^{-\kappa(T-t)} g(t,\Lambda_t,\overline{\varepsilon},\overline{\vartheta})} e^{iu_2 L_T}] \\ &= \mathbb{E}[e^{iu_2 e^{-\kappa(T-t)} f(t,\Lambda_t,\overline{\varepsilon})} e^{iu_1(e^{-\kappa(T-t)} g(t,\Lambda_t,\overline{\varepsilon},\overline{\vartheta}) + L_T)} e^{iu_3\lambda_t}], \end{aligned}$$

which shows that χ coincides with the characteristic function of the vector

$$\Big(g(t, \Lambda_t, \overline{\varepsilon}, \overline{\vartheta})e^{-\kappa(T-t)}, L_T + f(t, \Lambda_t, \overline{\varepsilon})e^{-\kappa(T-t)}, \lambda_t\Big).$$

The lemma is thus proved. $\square$

We now turn to the statement and the proof of the main result of this paper.

Theorem 3.5. *Recall that $(\varepsilon_i, \vartheta_i)_{i\in\mathbb{N}^*}$ and $(\overline{\varepsilon}, \overline{\vartheta})$ are i.i.d. with common law μ. Under the Assumptions 3.1 and 3.2, it holds that*

$$\begin{aligned}
&\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] \\
&= \int_0^T e^{-\kappa(T-t)}\mathbb{E}\left[g(t,\Lambda_t,\overline{\varepsilon},\overline{\vartheta})\,\lambda_t\,\varphi_\lambda^h\left(f(t,\Lambda_t,\overline{\varepsilon})e^{-\kappa(T-t)}\right)\right]dt \\
&= \int_{\mathbb{R}_+^2}\int_0^T e^{-\kappa(T-t)}\mathbb{E}\left[g(t,\Lambda_t,x,y)\,\lambda_t\,\varphi_\lambda^h\left(f(t,\Lambda_t,x)e^{-\kappa(T-t)}\right)\right]\mu(dx,dy)\,dt,
\end{aligned} \tag{3.11}$$

where $\hat{L}_T$ is defined in (2.2) and the mapping $\varphi_\lambda^h(x) := \mathbb{E}\left[h(L_T+x)|\mathcal{F}_T^\lambda\right]$ is defined in (3.4).

Proof. Assumptions 3.1 and 3.2 are in force. Using the relation (3.5) and the integration by parts formula on the Poisson space (3.3), it holds that

$$\begin{aligned}
\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\middle|\mathcal{F}^{\varepsilon,\vartheta}\vee\mathcal{F}_T^\lambda\right]\right] \\
= \mathbb{E}\left[\mathbb{E}\left[h\left(L_T\right)\int_0^T Z_t dN_t\middle|\mathcal{F}^{\varepsilon,\vartheta}\vee\mathcal{F}_T^\lambda\right]\right] &= \mathbb{E}\left[\int_0^T Z_t h\left(L_T(\cdot\cup\{t\})\right)\lambda_t dt\right]
\end{aligned}$$

By Relation (3.7) and the fact that the set $\{\Delta_t N\neq 0\}$ is negligeable, we obtain

$$\begin{aligned}
\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] &= \mathbb{E}\left[\int_0^T g(t,\Lambda_t,\varepsilon_{1+N_t},\vartheta_{1+N_t})e^{-\kappa(T-t)}h\left(L_T(\cdot\cup\{t\})\right)\lambda_t dt\right] \\
&= \int_0^T \mathbb{E}\left[g(\Lambda_t,\varepsilon_{1+N_t},\vartheta_{1+N_t})e^{-\kappa(T-t)}h\left(L_T(\cdot\cup\{t\})\right)\lambda_t\right]dt.
\end{aligned}$$

Finally, by Lemma 3.4, the above formula leads to

$$\mathbb{E}\left[L_T\mathbf{1}_{\{L_T\in[K,M]\}}\right] = \int_0^T \mathbb{E}\left[g(t,\Lambda_t,\overline{\varepsilon},\overline{\vartheta})e^{-\kappa(T-t)}h\left(L_T+f(t,\Lambda_t,\overline{\varepsilon})e^{-\kappa(T-t)}\right)\lambda_t\right]dt.$$

Since $\overline{\varepsilon}$ is independent of $\mathcal{F}_T^\lambda\vee\mathcal{F}^\varepsilon$, one has

$$\mathbb{E}\left[h\left(L_T+f(\Lambda_t,\overline{\varepsilon})e^{-\kappa(T-t)}\right)\middle|\mathcal{F}_T^\lambda\vee\sigma(\overline{\varepsilon})\right] = \varphi_\lambda^h\left(f(t,\Lambda_t,\overline{\varepsilon})e^{-\kappa(T-t)}\right).$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] &= \int_0^T \mathbb{E}\left[g(t,\Lambda_t,\overline{\varepsilon},\overline{\varepsilon})e^{-\kappa(T-t)}\lambda_t\varphi_\lambda^h\left(f(t,\Lambda_t,\overline{\varepsilon})e^{-\kappa(T-t)},f(t,\Lambda_t,\overline{\varepsilon})e^{-\kappa(T-t)}\right)\right]dt \\
= \int_{\mathbb{R}_+^2}\int_0^T & e^{-\kappa(T-t)}\mathbb{E}\left[g(t,\Lambda_t,x,y)\,\lambda_t\,\varphi_\lambda^h\left(f(t,\Lambda_t,x)e^{-\kappa(T-t)}\right)\right]dt\,\mu(dx,dy),
\end{aligned}$$

as asserted by the theorem. □

Remark 3.6. 1. Note that from Equality (3.11), it is clear that our approach only requires the knowledge of the conditional law of L_T given λ (*via* the mapping φ_λ) and not the one of the pair $(L_T, \hat{L}_T)$. This seems to be particularly useful for the numerical approximation of the aforementioned expectation.

2. The theorem above provides us the relation of the pricing formula with respect to the intensity process $(\lambda_t)_{t\geq 0}$ of the counting process.

Relation (3.11) allows us to give a lower (respectively upper) bound on the price if h is assumed to be convex (respectively concave).

Corollary 3.7. *Under the assumptions of Theorem 3.5, it holds that :*

(i) if h is convex, then

$$\begin{aligned}
&\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] \\
&\geq \int_{\mathbb{R}_+^2}\int_0^T e^{-\kappa(T-t)}\mathbb{E}\left[g(t,\Lambda_t,x,y)\,\lambda_t\, h\left(\mathbb{E}[L_T\middle|\mathcal{F}_T^\lambda] + f(t,\Lambda_t,x)e^{-\kappa(T-t)}\right)\right]\mu(dx,dy)\,dt.
\end{aligned}$$

(i) if h is concave, then

$$\begin{aligned}
&\mathbb{E}\left[\hat{L}_T h\left(L_T\right)\right] \\
&\leq \int_{\mathbb{R}_+^2}\int_0^T e^{-\kappa(T-t)}\mathbb{E}\left[g(t,\Lambda_t,x,y)\,\lambda_t\, h\left(\mathbb{E}[L_T\middle|\mathcal{F}_T^\lambda] + f(t,\Lambda_t,x)e^{-\kappa(T-t)}\right)\right]\mu(dx,dy)\,dt.
\end{aligned}$$

Proof. We prove (i) as statement (ii) follows the same line. As h is assumed to be convex, Jensen's inequality implies that

$$\varphi_\lambda^h(x) \geq h\left(\mathbb{E}[L_T\middle|\mathcal{F}_T^\lambda] + x\right), \quad x \in \mathbb{R}_+.$$

The result is then obtained by plugging this estimate in Relation (3.11). □

4 Applications and examples

In this section, we provide some application examples of our main result, in particular for the (generalized) stop-loss contract. Such explicit computations will also be useful for the CDO tranches and expected shortfall risk measure.

4.1 Computation of the building block

We first focus on the building block φ_λ^h (defined in (3.4)) when $h := \mathbf{1}_{\{[K,M]\}}$:

$$\varphi_\lambda(x) := \varphi_\lambda^h(x) = \mathbb{P}\left[L_T \in [K-x, M-x] | \mathcal{F}_T^\lambda\right], \quad x \in \mathbb{R}_+$$

which corresponds to the payoff of a stop-loss contract or a CDO tranche. Let $\mathcal{F}^\varepsilon := \sigma(\varepsilon_i,\ i \in \mathbb{N}^*)$. For any $i \in \mathbb{N}^*$, we set $X_i := f(\tau_i, \Lambda_{\tau_i}, \varepsilon_i)$ and x in $\mathbb{R}_+$, we have

$$\begin{aligned}
&\mathbb{P}\left[L_T \in [K-x, M-x]\middle|\mathcal{F}_T^\lambda \vee \mathcal{F}^\varepsilon\right] \\
&= \mathbb{P}\left[\sum_{i=1}^{N_T} X_i e^{\kappa\tau_i} \in [(K-x)e^{\kappa T}, (M-x)e^{\kappa T}]\middle|\mathcal{F}_T^\lambda \vee \mathcal{F}^\varepsilon\right]
\end{aligned}$$

$$= \sum_{k=1}^{+\infty} \mathbb{E}\left[\sum_{i=1}^{k} X_i e^{\kappa \tau_i} \in [(K-x)e^{\kappa T}, (M-x)e^{\kappa T}] \middle| N_T = k, \mathcal{F}_T^{\lambda} \vee \mathcal{F}^{\varepsilon} \right] \mathbb{P}[N_T = k | \mathcal{F}_T^{\lambda}]$$

$$= \sum_{k=1}^{+\infty} e^{-\int_0^T \lambda_s ds} \int_{\mathcal{S}_k} \mathbb{P}\left[\sum_{i=1}^{k} X_i e^{\kappa t_i} \in [(K-x)e^{\kappa T}, (M-x)e^{\kappa T}] \middle| \mathcal{F}_T^{\lambda} \vee \mathcal{F}^{\varepsilon} \right] \lambda_{t_1} dt_1 \cdots \lambda_{t_k} dt_k$$

$$= \sum_{k=1}^{+\infty} e^{-\int_0^T \lambda_s ds} \int_{\mathcal{S}_k} \int_{\mathbb{R}_+^k} \mathbf{1}_{\{\sum_{i=1}^k x_i e^{\kappa t_i} \in [(K-x)e^{\kappa T}, (M-x)e^{\kappa T}]\}} \mathcal{L}^{|\lambda}_{X_{(1:k)}}(dx_1, \ldots, dx_k) \lambda_{t_1} dt_1 \cdots \lambda_{t_k} dt_k, \tag{4.1}$$

where $\mathcal{S}_k := \{0 < t_1 < \cdots < t_k \leq T\}$, $X_{(1:k)} := (X_1, \ldots, X_k)$ and

$$\mathcal{L}^{|\lambda}_{X_{(1:k)}}(dx_1, \ldots, dx_k) := \mathbb{P}\left[X_{(1:k)} \in (dx_1, \ldots, dx_k) \middle| \mathcal{F}_T^{\lambda} \right].$$

It just remains to compute the joint distribution of the claims $X_{(1:k)}$ in different situations. In particular, we provide below an explicit example.

Model on ε_i **:** We assume that $(\varepsilon_i)_{i\in\mathbb{N}^*}$ are i.i.d. random variables with Pareto distributions $\mathcal{P}(\alpha_\varepsilon, \beta_\varepsilon)$ with $(\alpha_\varepsilon, \beta_\varepsilon) \in (\mathbb{R}_+^*)^2$ whose density ψ_ε is defined as

$$\psi_\varepsilon(z) = \left(\beta_\varepsilon \frac{\alpha_\varepsilon^{\beta_\varepsilon}}{z^{\beta_\varepsilon + 1}} \right) \mathbf{1}_{\{z \geq \alpha_\varepsilon\}} dz.$$

Choosing $f(t, \ell, x) := \sqrt{\frac{\ell}{t}} x$, the conditional distribution $\mathcal{L}^{|\lambda}_{X_{(1:k)}}(dx_1, \ldots, dx_k)$ in Relation (4.1) becomes

$$\begin{aligned}
&\mathcal{L}^{|\lambda}_{X_{(1:k)}}(dx_1, \ldots, dx_k) \\
&= \mathbb{P}\left[\left(\sqrt{\frac{\Lambda_{t_1}}{t_1}} \varepsilon_1, \cdots, \sqrt{\frac{\Lambda_{t_k}}{t_k}} \varepsilon_k \right) \in (dx_1, \ldots, dx_k) \middle| \mathcal{F}_T^{\lambda} \right] \\
&= \prod_{i=1}^{k} \mathbb{P}\left[\sqrt{\frac{\Lambda_{t_i}}{t_i}} \varepsilon_i \in dx_i \middle| \mathcal{F}_T^{\lambda} \right] \\
&= \prod_{i=1}^{k} \left(\beta_\varepsilon \frac{\left(\sqrt{\frac{t_i}{\Lambda_{t_i}}} \alpha_\varepsilon \right)^{\beta_\varepsilon}}{z_i^{\beta_\varepsilon + 1}} \right) \mathbf{1}_{\left\{ z_i \geq \sqrt{\frac{t_i}{\Lambda_{t_i}}} \alpha_\varepsilon \right\}} dz_i.
\end{aligned}$$

The next step to compute the right-hand side of Relation (3.11) is to specify the joint law of $(\varepsilon_1, \vartheta_1)$.

Model on $(\varepsilon_i, \vartheta_i)$ **:** We assume that $(\varepsilon_i, \vartheta_i)_{i\in\mathbb{N}^*}$ are i.i.d. random vectors, with marginal distributions following Pareto distributions $\mathcal{P}(\alpha_\varepsilon, \beta_\varepsilon)$ and $\mathcal{P}(\alpha_\vartheta, \beta_\vartheta)$ (for a set of parameters α_ε, β_ε, α_ϑ, $\beta_\vartheta > 0$) respectively. The dependence structure is modeled through a Clayton copula with parameter $\theta > 0$. We recall that the Clayton copula is $C(u, v) := \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}$ and the density c of the Clayton copula is given by

$$c(u, v) := (1 + \theta)(uv)^{-1-\theta}(u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta} - 2}.$$

The joint distribution of $(\varepsilon_1, \vartheta_1)$ is then given by

$$\mu(dx, dy) = c\left(F_\varepsilon(x), F_\vartheta(y)\right)\psi_\varepsilon(x)\psi_\vartheta(y)dxdy,$$

with $F_\varepsilon(z) = \left(1 - \frac{\alpha_\varepsilon^{\beta_\varepsilon}}{z^{\beta_\varepsilon}}\right)$ and $F_\vartheta(z) = \left(1 - \frac{\alpha_\vartheta^{\beta_\vartheta}}{z^{\beta_\vartheta}}\right)$.

Joint law of (λ_t, Λ_t) : The final step in the computation of Relation (3.11) is to make precise the joint law of (λ_t, Λ_t). More precisely, we need to compute

$$\mathbb{E}\left[g(t, \Lambda_t, x, y)\, \lambda_t\, \varphi_\lambda\left(K - f(t, \Lambda_t, x)e^{-\kappa(T-t)}, M - f(t, \Lambda_t, x)e^{-\kappa(T-t)}\right)\right]$$

Assume the intensity process $(\lambda_t)_{t\in[0,T]}$ is given by

$$\lambda_t = \lambda_0 \exp(2\beta W_t)$$

where W is a Brownian motion, and β a constant (non null). Then the cumulative intensity is

$$\Lambda_t = \lambda_0 \int_0^t \exp(2\beta W_s)ds.$$

By Borodin and Salminen [6] (page 169), the joint law of (Λ_t, W_t) is given by

$$\mathbb{P}\left(\Lambda_t \in dv, W_t \in dz\right) = \frac{\lambda_0|\beta|}{2v}\exp\left(-\frac{\lambda_0(1+e^{2\beta z})}{2\beta^2 v}\right) i_{\beta^2 t/2}\left(\frac{\lambda_0 e^{\beta z}}{\beta^2 v}\right)\mathbf{1}_{\{v>0\}}dvdz$$

where the function

$$i_y(z) = \frac{z e^{\frac{\pi^2}{4y}}}{\pi\sqrt{\pi y}}\int_0^\infty \exp\left(-zch(x) - \frac{x^2}{4y}\right) sh(x)\sin(\frac{\pi x}{2y})dx.$$

The expectation term in the right-hand side of equation (3.11) is then

$$\mathbb{E}\left[g(t, \Lambda_t, x, y)\, \lambda_t\, \varphi_\lambda\left(K - f(t, \Lambda_t, x)e^{-\kappa(T-t)}, M - f(t, \Lambda_t, x)e^{-\kappa(T-t)}\right)\right]$$

$$= \int_{\mathbb{R}^2} g(t, v, x, y)\varphi_\lambda\left(K - \sqrt{\frac{v}{t}}xe^{-\kappa(T-t)}, M - \sqrt{\frac{v}{t}}xe^{-\kappa(T-t)}\right)\frac{\lambda_0^2|\beta|}{2v}e^{2\beta z}\exp\left(-\frac{\lambda_0(1+e^{2\beta z})}{2\beta^2 v}\right) i_{\beta^2 t/2}\left(\frac{\lambda_0 e^{\beta z}}{\beta^2 v}\right)\mathbf{1}_{\{v>0\}}dvdz.$$

4.2 A Black-Scholes type formula for generalized Stop-Loss contracts in the Cramer-Lundberg

As an illustration, we conclude our analysis by specifying our result in the classic Cramer-Lundberg model. More precisely, we assume that the Cox process is an homogeneous Poisson process with constant intensity $\lambda_0 > 0$ and set $h := \mathbf{1}_{[K,M]}$, with $K < M$. The building block reduces to the distribution function

$$\varphi_{\lambda_0}(x) := \varphi^h_{\lambda_0}(x) = \mathbb{P}\left[L_T \in [K - x, M - x]\right], \quad x \in \mathbb{R}_+. \tag{4.2}$$

In that case we omit the dependency on Λ for the mappings f and g (as $\Lambda_t = t\lambda_0$).

Corollary 4.1. *Under the assumptions of Theorem 3.5, it holds*

$$\mathbb{E}\left[\hat{L}_T \boldsymbol{1}_{L_T\in[K,M]}\right] = \lambda_0 \int_0^T \int_{\mathbb{R}_+^2} e^{-\kappa(T-t)} g(t,x,y)\, \varphi_{\lambda_0}\left(f(t,x)e^{-\kappa(T-t)}\right) \mu(dx,dy)dt,$$

(recall that $\mu := \mathcal{L}_{(\overline{\varepsilon},\overline{\vartheta})}$).

If we assume furthermore that $f(t,x) = g(t,x,y) = x$ and $\kappa = 0$, then the loss process L_T corresponds to the cumulated loss of the classic Cramer-Lundberg model. In this context, a huge literature deals with the computation of the ruin probability and related quantities, such as the discounted penalty function at ruin (Gerber-Shiu function). Others papers are concerned with the pricing of Stop-Loss contract. The pricing relies on the computation of a term of the form $\int_K^M ydF(y)$ with F being the cumulative distribution function of the loss process L_T, and the discussion in the literature mainly focuses on the derivation of the compound distribution function F (usually calculated recursively, using Panjer recursion formula and numerical methods/approximations) *cf.* [12] and [11]. Our Malliavin approach provides another formula which reads as

$$\mathbb{E}\left[\hat{L}_T \mathbf{1}_{L_T\in[K,M]}\right] = \lambda_0 T \int_{\mathbb{R}_+} x\,(F(M-x) - F(K-x))\mu(dx). \tag{4.3}$$

Note that our result indeed coincides with the one obtained in [11]. Indeed, if one translates in a general setting Formula [11, (6)] (as μ is constrained to have a finite support in $\mathbb{N}$ in [11]) the distribution F satisfies

$$ydF(y) = \lambda_0 T \int_{\mathbb{R}_+} xdF(y-x)\mu(dx),$$

from which one deduces that

$$\begin{aligned}\int_K^M ydF(y) &= \lambda_0 T \int_K^M \int_{\mathbb{R}_+} x\mu(dx)dF(y-x)\\ &= \lambda_0 T \int_{\mathbb{R}_+} x \int_K^M dF(y-x)\mu(dx)\\ &= \lambda_0 T \int_{\mathbb{R}_+} x(F(K-x) - F(M-x))\mu(dx),\end{aligned}$$

which is exactly (4.3).

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