Optimal Portfolio Choice with ESG Considerations and Asymmetric Information

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Abstract

We study portfolio optimization problems incorporating environmental, social, and governance (ESG) factors for three types of investors, namely *brown* investors who care only about their wealth, *green* investors who prefer assets with high ESG ratings, and *mixed* investors who take a position between the two. Given that some ESG ratings are not publicly available, we distinguish between different levels of ESG information accessibility for these investors. We explore the impact of ESG preferences and ESG information accessibility on investment strategies, addressing issues such as the components of optimal portfolios of different investors and the utility indifference value of nonpublic ESG information.

Keywords: portfolio choice, ESG rating, utility maximization, asymmetric and partial information, filtering.

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1 Introduction

Traditionally, investors seek to optimize their investments primarily based on financial factors. Environmental, social, and governance (ESG) investing, which incorporates ESG factors into the investment decision-making process, has been on the rise in recent years. Due to the prevalence of ESG investing, there is a growing literature focusing on how to integrate ESG factors into portfolio choice. Pedersen et al. [19] extend the classical mean-variance framework by adding an ESG preference function and characterize the solution to the investor's portfolio problem by an ESG-efficient frontier. Pástor et al. [18] study a utility maximization problem, in which an agent derives non-pecuniary benefits from holding assets determined by their ESG characteristics. They show that ESG-driven investing makes firms greener and shifts real investment toward green firms, producing positive social impact. Later, Avramov et al. [2] extend their framework by considering ESG rating uncertainty and analyze the implications of the uncertainty for portfolio choice and asset pricing in equilibrium. These works are carried out under a one-period framework, in which ESG scores or ratings are either constants or random variables and hence do not change during the time period under consideration.

In this paper, we study portfolio optimization problems in the presence of ESG factors under a multi-period continuous-time model. The aim is to capture the impact of ESG investing over a medium- to long-term investment horizon. To this end, we assume that the ESG information flow is updated over time. To account for ESG information, we focus on ESG ratings, which, according to Boffo and Patalano [5], represent *one of the key ways in which investors and other market participants make use of ESG information.* ESG ratings are provided by evaluation agencies such as MSCI and are typically updated at discrete time points. For instance, MSCI ESG ratings are updated monthly, as shown in Figure 1. To capture this feature, we model ESG ratings using a finite-state discrete-time Markov chain, inspired by the literature on credit rating modeling (see, e.g., Duffie and Singleton [7]).

Recent empirical studies show that ESG factors are related to both financial performance and stock performance of firms. For example, high ESG-rated firms are associated with higher stock returns and lower stock volatility during times of crises; see Engelhardt et al. [8] and references therein. See also Yang et al. [23] who study how a firm's ESG performance can affect its downside

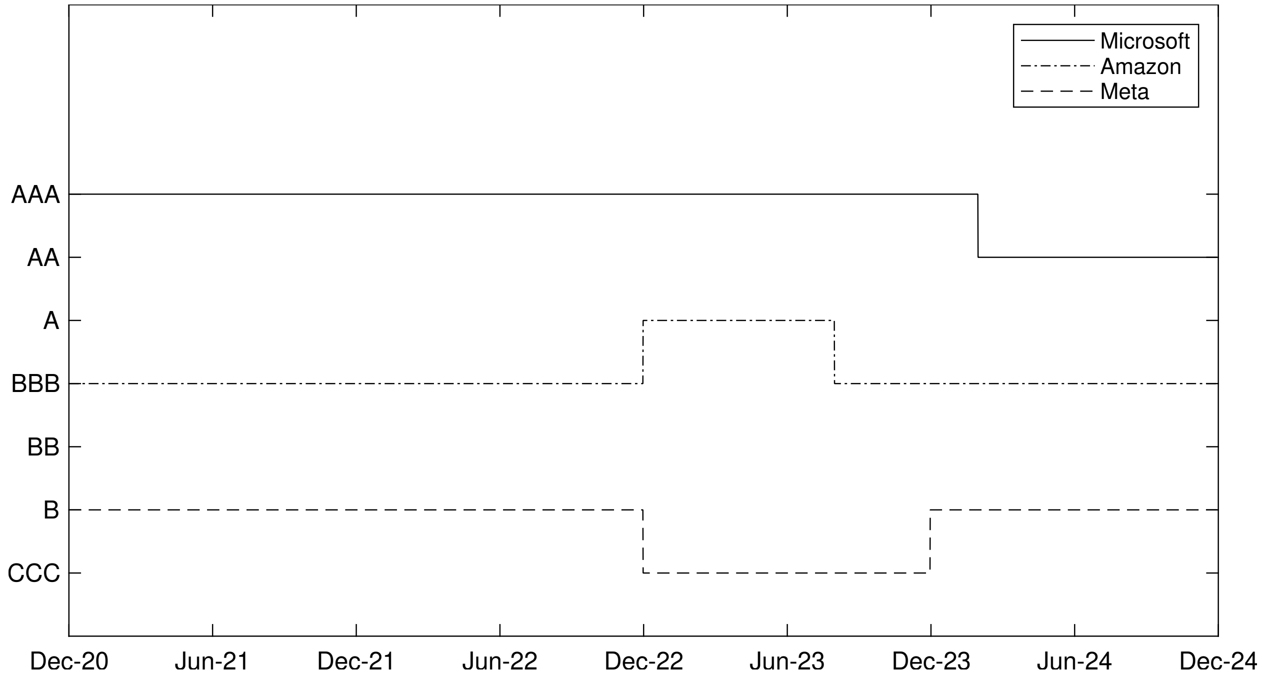

Figure 1. This figure shows the MSCI ESG ratings of Microsoft, Amazon, and Meta from December 2020 to December 2024, which are available at `https://www.msci.com/our-solutions/esg-investing/esg-ratings-climate-search-tool`.

risk. Motivated by these studies, we assume that the evolution of risky asset prices depends on ESG ratings.

We consider exponential-type utility optimization problems for three types of investors who have different ESG preferences and ESG information accessibility, respectively referred to as *brown*, *green*, and *mixed* investors. Specifically, brown investors are only concerned about their wealth, whereas green investors not only care about their wealth but also have preferences for assets with good ESG ratings. In practice, some ESG ratings are not publicly available. Brown investors have knowledge only of publicly available ESG ratings. In contrast, green investors are willing to make an effort to acquire ESG ratings of all assets. Mixed investors are positioned between these two types: they have access to and follow complete ESG rating information but do not necessarily prioritize ESG factors in their decision-making. In other words, the portfolio choice of mixed investors is driven primarily by wealth, with ESG ratings taken as a secondary consideration—resembling a standard utility optimization problem.

For brown investors, nonpublic ESG ratings are treated as hidden factors in their portfolio choice, and the utility maximization problem is studied under partial information. In the literature, optimization with partial information has often been studied in contexts where the coefficients of asset price processes depend on an unobservable process (see, e.g., Honda [13], Bäuerle and Rieder [22], Hata and Iida [12], Rieder and Bäuerle [3], and Björk et al. [4]). In this paper, the hidden ESG ratings are modeled using a discrete-time Markov chain, and we adopt the filtering method to derive the optimal investment strategy for brown investors.

In contrast to brown investors, green and mixed investors have access to all ESG ratings and use the complete rating information to make investment decisions. The asymmetry of information will be described by specifying different filtrations in our model. As the ESG rating of an asset changes over time, investors would adjust their information flow by adding newly available rating information regularly. In the literature, such supplementary information (sometimes specified as insider's information) has been studied using the initial enlargement of filtration setup; see, e.g., Amendinger et al. [1] and Grorud and Pontier [11], in which the privileged information is assumed to be known at the initial time. We will extend this setting by incorporating the rating information that is updated at discrete time points into our model.

Mixed investors have a similar objective function as brown investors, which only depends on terminal wealth. For green investors, we propose an aggregate wealth-weighted ESG rating as a measure of the overall ESG performance of investment strategies, and we incorporate the proposed rating into green investors' utility to reflect their ESG preferences. This treatment is similar to those in Pástor et al. [18] and Avramov et al. [2], where the utility of a green investor depends on both her terminal wealth and the ESG performance of her asset holdings. It is noteworthy that there is a strand of literature on investment problems with multivariate utility; see, e.g., Grant and Satchell [10] and references therein.

We employ a backward induction approach to derive analytical expressions for the value functions and optimal investment strategies of the three types of investors. By examining these results, we find that, for all investors, the optimal portfolios contain the riskless asset and a local mean-variance portfolio derived from the mean-variance problem for the instantaneous increment of wealth. In addition, the optimal portfolio of a brown investor includes a hedging portfolio, which is used to hedge against unfavorable shifts in the filtered probabilities associated with nonpublic ESG ratings. The optimal portfolio of a green investor contains an ESG portfolio determined by ESG ratings. We also find that different types of investors perceive the mean return rates of risky assets differently.

Given different ESG information accessibility and ESG preferences, we are interested in the utility indifference value of the nonpublic ESG information, interpreted as the maximum amount that a brown investor is willing to pay in order to obtain this hidden information. This value is always nonnegative, meaning that having the hidden information is always beneficial. In a similar spirit, we study the utility indifference value of the ESG-related non-pecuniary benefits, defined as the monetary value that a green investor is willing to accept to disregard her ESG preferences in utility maximization (and thus to have the same objective function as a mixed investor). This utility indifference value can take both positive and negative values, meaning that, compared to mixed investors, it may be either beneficial or disadvantageous for green investors to include ESG preferences.

As demonstrated by both our theoretical results and numerical study, ESG rating information impacts portfolio choices for all types of investors through the local mean-variance component (for all investors), the hedging component (for brown investors), and the ESG component (for green investors). In the numerical study in Section 4, we consider two risky assets and assume that their ESG ratings are not publicly available. We find that the dynamic evolution of ESG ratings induces instantaneous changes in the trading strategies of the green and mixed investors, but has little impact on the brown investor. Moreover, in all the ESG scenarios considered, the maximal expected utility is highest for the green investor and lowest for the brown investor.

The remainder of the paper is organized as follows. Section 2 formulates the portfolio optimization problems for the three types of investors. Section 3 presents our main results, including the solutions to the optimization problems and the corresponding optimal investment strategies. Section 4 conducts a numerical study to illustrate the theoretical results. Section 5 concludes the study. Finally, for the sake of readability, the proofs of most theoretical results are provided in the Appendix.

2 The model

Throughout the paper, all sources of randomness are defined on a probability space $(\Omega, \mathcal{F}, P)$ and a finite time horizon $T \in \mathbb{Z}^+$ is considered. For a stochastic process $\{Y_t\}_{0\leq t\leq T}$, either univariate or multivariate, we write $Y = \{Y_t\}_{0 \le t \le T}$ and $\mathbb{F}^Y = \{\mathcal{F}_t^Y\}_{0 \le t \le T} = \{\sigma(Y_u, u \le t)\}_{0 \le t \le T}$ as long as no confusion arises. Without loss of generality, all filtrations to be introduced are assumed to satisfy the usual conditions.

We consider a financial market consisting of a riskless asset and n non-dividend paying risky assets. The risk-free interest rate is a constant, denoted by $r > 0$. The price process of the risky assets is denoted by $\mathbf{S} = \{\mathbf{S}_t\}_{0 \le t \le T}$, where $\mathbf{S}_t = (S_{1,t}, \ldots, S_{n,t})^\intercal$. For $i = 1, \ldots, n$, $S_{i,t}, 0 \le t \le T$, denotes the price process of the i^{th} risky asset and is assumed to start from $S_{i,0} > 0$ and follow the stochastic differential equation (SDE)

$$\frac{dS_{i,t}}{S_{i,t}} = \mu_i(t, G_{i,t})dt + \sum_{j=1}^{n} \sigma_{ij}(t) dW_{j,t}, \tag{2.1}$$

where $\mu_i(\cdot,\cdot) \in \mathbb{R}$ and $\sigma_{ij}(\cdot) \ge 0$ are deterministic functions, $G_{i,t}$ denotes the ESG rating of the i^{th} asset at time t and is a random variable taking values in a finite set of real numbers,[1] and $\mathbf{W} = \{\mathbf{W}_t\}_{0 \le t \le T}$ with $\mathbf{W}_t = (W_{1,t}, \ldots, W_{n,t})^\intercal$ is an n-dimensional standard Brownian motion. Note that the riskless asset is not associated with an ESG rating. Moreover, we suppose that the ESG ratings of the first $n_1 \le n-1$ risky assets, denoted as $\mathbf{G}^1$, are publicly available while the remaining ones, denoted as $\mathbf{G}^2$, are not. For later use, let

$$\begin{aligned}
\mathbf{G}_t &= (\mathbf{G}_t^1; \mathbf{G}_t^2) = (G_{1,t}, \ldots, G_{n_1,t}; G_{n_1+1,t}, \ldots, G_{n,t}), \\
\boldsymbol{\mu}(t, \mathbf{G}_t) &= \boldsymbol{\mu}(t, \mathbf{G}_t^1, \mathbf{G}_t^2) = (\mu_1(t, G_{1,t}), \ldots, \mu_n(t, G_{n,t}))^\intercal, \\
\boldsymbol{\sigma}_t &= [\sigma_{ij}(t)]_{n \times n}.
\end{aligned}$$

We consider regularly updated ESG ratings and assume that the updates take place at predetermined discrete times $k = 1, \ldots, T$. Specifically, for $l = 1, 2$, the partial ESG rating process $\mathbf{G}^l$ can be written in the form

$$\mathbf{G}_t^l = \sum_{k=1}^{T} 1_{\{k-1 \le t < k\}} \mathbf{G}_{k-1}^l, \qquad 0 \le t < T, \tag{2.2}$$

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[1] ESG rating is an ordinal variable, whose value is usually expressed as one or multiple letters. For example, MSCI ESG ratings range from the leaders (AAA, AA) to the laggard ones (B, CCC); see Figure 1. In this work, we express ESG ratings in terms of real numbers to facilitate the mathematical formulation of our model.

where $\{\mathbf{G}_k^l\}_{k\in\mathbb{N}_0}$ is a finite-state time-homogeneous Markov chain with state space $\mathfrak{g}_l$ and transition probabilities $p_{l,\mathbf{g}_i,\mathbf{g}_j}$ for $\mathbf{g}_i, \mathbf{g}_j \in \mathfrak{g}_l$. The cardinality of the state space $\mathfrak{g}_l$ is denoted by $|\mathfrak{g}_l|$. Moreover, the state space of the ESG rating process $\mathbf{G}$ is given by

$$\mathfrak{g} = \{(\mathbf{g}_1; \mathbf{g}_2) \mid \mathbf{g}_l \in \mathfrak{g}_l, l = 1, 2\},$$

and denote its transition probabilities by $p_{\mathbf{g}_i,\mathbf{g}_j}$ for $\mathbf{g}_i, \mathbf{g}_j \in \mathfrak{g}$. The initial asset prices $\mathbf{S}_0$, the Brownian motion $\mathbf{W}$, the process $\mathbf{G}^1$, and the process $\mathbf{G}^2$ are assumed to be mutually independent.

Assume that the deterministic functions $\mu_i(\cdot,\cdot)$ and $\sigma_{ij}(\cdot), i, j = 1, \ldots, n$, satisfy existence and uniqueness conditions so that the SDE (2.1) has a unique strong solution, the $n \times n$ covariance matrix

$$\boldsymbol{\Sigma}_t = \boldsymbol{\sigma}_t \boldsymbol{\sigma}_t^\intercal$$

is invertible (and hence, positive definite) for every $t \in [0, T]$, and that the following inequalities hold for all $\mathbf{g} \in \mathfrak{g}$:

$$\int_0^T \mathbf{g}^\intercal \boldsymbol{\Sigma}_t^{-1} \mathbf{g} dt \quad < \quad \infty, \tag{2.3}$$

$$\int_0^T \mathbf{1}^\intercal \boldsymbol{\Sigma}_t^{-1} \mathbf{1} dt \quad < \quad \infty, \tag{2.4}$$

$$\int_0^T \boldsymbol{\mu}(t, \mathbf{g})^\intercal \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\mu}(t, \mathbf{g}) dt \quad < \quad \infty, \tag{2.5}$$

where $\mathbf{1} \in \mathbb{R}^n$ is a vector of ones. Clearly, (2.4) and (2.5) imply that

$$\int_0^T \boldsymbol{\theta}(t, \mathbf{g})^\intercal \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\theta}(t, \mathbf{g}) dt < \infty, \tag{2.6}$$

where $\boldsymbol{\theta}(t, \mathbf{g}) = \boldsymbol{\mu}(t, \mathbf{g}) - r\mathbf{1}$. Throughout the paper, as long as no confusion arises, we often use simplifications such as $\boldsymbol{\mu}_t$ for $\boldsymbol{\mu}(t, \mathbf{G}_t)$ or $\boldsymbol{\mu}(t, \mathbf{g})$ and $\boldsymbol{\theta}_t$ for $\boldsymbol{\theta}(t, \mathbf{G}_t)$ or $\boldsymbol{\theta}(t, \mathbf{g})$.

We now introduce the investment strategy of an investor, which is a stochastic process $\boldsymbol{\pi} = \{\boldsymbol{\pi}_t\}_{0\le t\le T}$ with $\boldsymbol{\pi}_t = (\pi_{1,t}, \ldots, \pi_{n,t})^\intercal$. For $i = 1, \cdots, n$, $\pi_{i,t}$ represents the proportion of capital invested in the i^{th} risky asset at time t. The portfolio is self-financing and includes, as mentioned previously, a riskless asset that grows at a constant rate r. Then the value process of the investor's portfolio, denoted by $\{X_t\}_{0\leq t\leq T}$, evolves according to

$$\begin{aligned}
dX_t &= \sum_{i=1}^{n} \pi_{i,t} X_t \frac{dS_{i,t}}{S_{i,t}} + r\left(X_t - \sum_{i=1}^{n} \pi_{i,t} X_t\right) dt \\
&= rX_t dt + \sum_{i=1}^{n} (\mu_i(t, G_{i,t}) - r)\pi_{i,t} X_t dt + \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij}(t)\pi_{i,t} X_t dW_{j,t} \\
&= X_t\left(r + \boldsymbol{\pi}_t^\intercal \boldsymbol{\theta}_t\right) dt + X_t\left(\boldsymbol{\pi}_t^\intercal \boldsymbol{\sigma}_t d\mathbf{W}_t\right). \qquad (2.7)
\end{aligned}$$

2.1 Portfolio optimization for different investors

We study portfolio optimization problems for the three aforementioned types of investors—*brown, green, and mixed* investors. Consider a standard exponential utility function $U(x) = -e^{-\gamma x}$ with a risk aversion coefficient $\gamma > 0$. For *brown* investors who only value their terminal wealth, the optimization problem is given by

$$V_0^b = \sup_{\boldsymbol{\pi}\in\mathcal{A}^b} E\left[-e^{-\gamma_1 X_T}\right], \qquad (2.8)$$

where $\gamma_1 > 0$ and $\mathcal{A}^b$ is the set of admissible strategies for *brown* investors which is defined according to their information accessibility and will be made precise later.

For *green* investors, the utility relies not only on terminal wealth but also on the ESG performance of the assets held over the investment horizon. Inspired by Pástor et al. [18], we consider the optimization problem

$$V_0^g = \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E\left[-e^{-\gamma_1 X_T - \gamma_2 \int_0^T (\boldsymbol{\pi}_s^\intercal \mathbf{G}_s) e^{r(T-s)} X_s ds}\right], \qquad (2.9)$$

where $\mathcal{A}^g$ is the set of admissible strategies for green investors, $\gamma_2 > 0$ is the risk aversion coefficient on the ESG performance, $\boldsymbol{\pi}_s^\intercal \mathbf{G}_s$ is the weighted average ESG rating of the portfolio at time s, and $e^{r(T-s)}X_s$ is the future value of the portfolio. The integral $\int_0^T (\boldsymbol{\pi}_s^\intercal \mathbf{G}_s) e^{r(T-s)} X_s ds$ is interpreted as an aggregate wealth-weighted ESG rating, which serves to measure the overall ESG performance of the investment strategy $\boldsymbol{\pi}$ over the time period $[0, T]$. If a green investor has time preference for the ESG performance of her investment strategy, we may replace the factor $e^{r(T-s)}$ in (2.9) by $e^{\rho(T-s)}$ for some positive constant ρ to reflect such preference.

For *mixed* investors, who have the same utility function as brown investors whereas the same ESG information accessibility as green ones, the optimization problem is given by

$$V_0^m = \sup_{\boldsymbol{\pi} \in \mathcal{A}^m} E\left[-e^{-\gamma_1 X_T}\right], \tag{2.10}$$

where $\mathcal{A}^m$ is the set of admissible strategies for mixed investors.

To define the admissible strategy sets for the three types of investors, we begin by making precise the accessible information flow for each of them. The information flow of each type of investors is represented by a filtration defined below. Recall that all investors observe the price process $\mathbf{S}$ of all assets but have different information accessibility on ESG ratings. Thus, the filtrations may be different for different types of investors.

Brown investors have partial information on ESG ratings, which includes only publicly available ratings. So their information set is generated by the asset price process $\mathbf{S}$ and the partial ESG rating process $\mathbf{G}^1$, which is $\mathcal{F}_t^b := \mathcal{F}_t^{\mathbf{S}} \vee \mathcal{F}_t^{\mathbf{G}^1}$ at time t. The filtration $\mathbb{F}^b := \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G}^1} = \{\mathcal{F}_t^{\mathbf{S}} \vee \mathcal{F}_t^{\mathbf{G}^1}\}_{0 \le t \le T}$ can be viewed as a successive initial enlargement of the filtration $\mathbb{F}^{\mathbf{S}}$ by the sequence of ESG ratings $\{\mathbf{G}_k^1\}_{k \in \mathbb{N}_0}$ (see, e.g., Jacod [14] and Protter [21, Chapter VI]). *Green* investors have complete information on ESG ratings and their information flow is generated by both the asset price process $\mathbf{S}$ and the entire ESG rating process $\mathbf{G}$, which is $\mathbb{F}^g := \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G}} = \{\mathcal{F}_t^{\mathbf{S}} \vee \mathcal{F}_t^{\mathbf{G}}\}_{0 \le t \le T}$. We note that $\mathbb{F}^g = \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G^1}} \vee \mathbb{F}^{\mathbf{G^2}} = \mathbb{F}^b \vee \mathbb{F}^{\mathbf{G^2}}$. Similar to $\mathbb{F}^b$ above, the filtration $\mathbb{F}^g$ can be viewed as a successive initial enlargement of the filtration $\mathbb{F}^{\mathbf{S}}$ by the sequence of all ESG ratings $\{\mathbf{G}_k\}_{k \in \mathbb{N}_0}$. *Mixed* investors have the same information flow as green investors, so the corresponding information flow is represented by the filtration $\mathbb{F}^m = \mathbb{F}^g$.

We define the sets of admissible strategies for the three types of investors below.

Definition 2.1 *(Admissible trading strategies) The set of admissible strategies for an investor is denoted by $\mathcal{A}^H$, where $H = b, g$, and m for brown, green, and mixed investors, respectively. Any $\boldsymbol{\pi} \in \mathcal{A}^H$ satisfies the following conditions:*

(i) $\boldsymbol{\pi}$ is an $\mathbb{H}$-predictable process, where $\mathbb{H} = \mathbb{F}^b, \mathbb{F}^g$, and $\mathbb{F}^m$ for brown, green, and mixed investors, respectively;

(ii) $E\left[\int_0^T X_t^2 \left(\boldsymbol{\pi}_t^\intercal \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t\right) dt\right] < \infty$;

(iii) for any $0 \le t \le T, x \in \mathbb{R}$ *and* $\gamma_1 > 0$, $E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}}\left[\exp\left\{-\gamma_1 \inf_{t \le s \le T} X_s\right\}\right] < \infty$ *for any* $\mathbf{g}_1 \in \mathfrak{g}_1$ *and* $\boldsymbol{\beta} \in B$*, with* B *defined by* (3.3)*, for brown investors, and* $E^{t,x,\mathbf{g}}\left[\exp\left\{-\gamma_1 \inf_{t \le s \le T} X_s\right\}\right] < \infty$ *for any* $\mathbf{g} \in \mathfrak{g}$ *for green and mixed investors;*

(iv) for green investors, $\int_0^T |(\boldsymbol{\pi}_t^\intercal \mathbf{G}_t) X_t|\, dt \le \Lambda$ *almost surely for some constant* $\Lambda > 0$*.*

Remark 2.1 *For a brown investor or a mixed investor, the specification of the state space* $\mathfrak{g}$ *of the ESG process* $\mathbf{G}$ *does not affect her portfolio optimization problem. However, a green investor should assign a suitable number to each ESG rating to reflect her ESG utility in the value function* V_0^g *defined by* (2.9)*.*

3 Main results

In this section, we solve the optimization problems for brown, green and mixed investors. Our main goal is to provide characterization results for the value functions (2.8)–(2.10) of the three types of investors and the corresponding optimal portfolio strategies.

3.1 Optimization for brown investors

We first consider the optimization problem for brown investors under the partial information flow $\mathbb{F}^b = \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G}^1}$ with the nonpublic ESG rating process $\mathbf{G}^2$ being unobservable. The value function at time $t \in (0, T)$ is given by

$$V_t^b = \operatorname*{ess\,sup}_{\boldsymbol{\pi} \in \mathcal{A}^b} E\left[-e^{-\gamma_1 X_T} \,\middle|\, \mathcal{F}_t^b\right]. \tag{3.1}$$

We adopt the filtering method, by following Gennotte [9], Honda [13], and Platen and Runggaldier [20] among others, to solve the optimization problem in a backward recursive way as follows.

- *First step: filtered probability given the partial information*

The filtering problem in the discrete-time case depends on the possible states and trajectories of the Markov chain. The key quantity in this setting is the so-called filtered probability with respect to the filtration $\mathbb{F}^b = \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G}^1}$, which is defined as

$$\beta_t^{\mathbf{g}} = E\left[1_{\{\mathbf{G}_t^2 = \mathbf{g}\}} \,\middle|\, \mathcal{F}_t^{\mathbf{S}} \vee \mathcal{F}_t^{\mathbf{G}^1}\right] = P(\mathbf{G}_t^2 = \mathbf{g} | \mathcal{F}_t^{\mathbf{S}}), \qquad 0 \le t \le T, \mathbf{g} \in \mathfrak{g}_2, \tag{3.2}$$

where the second equality holds because the sigma-algebras $\mathcal{F}_T^{\mathbf{G}^1}$ and $\mathcal{F}_T^{\mathbf{G}^2}$ are independent.

We consider a sigma-algebra equivalent economy under the information flow $\mathbb{F}^b$ with the filtered probabilities defined above, and we shall rewrite the optimization problem (3.1) in this setting. Note that the filtered probabilities satisfy $\sum_{\mathbf{g}\in\mathfrak{g}_2}\beta_t^{\mathbf{g}} = 1$. Due to this constraint, we arbitrarily fix some $\mathbf{g}_0 \in \mathfrak{g}_2$ and work with the set $\hat{\mathfrak{g}}_2 = \mathfrak{g}_2 \setminus \{\mathbf{g}_0\}$. Denote by $\boldsymbol{\beta}_t$ a random vector containing elements $\beta_t^{\mathbf{g}}, \mathbf{g} \in \hat{\mathfrak{g}}_2$. Define a Borel set $B \subset [0,1]^{|\hat{\mathfrak{g}}_2|}$ such that

$$B = \left\{ \boldsymbol{\beta} \in [0,1]^{|\hat{\mathfrak{g}}_2|} : \sum_{\mathbf{g}\in\hat{\mathfrak{g}}_2} \boldsymbol{\beta}^{\mathbf{g}} \leq 1 \right\}. \tag{3.3}$$

The vector of conditional mean return rates is given by

$$\begin{aligned}
\bar{\boldsymbol{\mu}}_t &= E[\boldsymbol{\mu}(t,\mathbf{G}_t)|\mathcal{F}_t^{\mathbf{S}} \vee \mathcal{F}_t^{\mathbf{G}^1}] \\
&= \sum_{\mathbf{g}_2\in\mathfrak{g}_2} E\left[1_{\{\mathbf{G}_t^2=\mathbf{g}_2\}} \middle| \mathcal{F}_t^{\mathbf{S}} \vee \mathcal{F}_t^{\mathbf{G}^1} \right] \boldsymbol{\mu}(t,\mathbf{G}_t^1,\mathbf{g}_2) \\
&= \sum_{\mathbf{g}_2\in\hat{\mathfrak{g}}_2} \beta_t^{\mathbf{g}_2}\boldsymbol{\mu}(t,\mathbf{G}_t^1,\mathbf{g}_2) + \beta_t^{\mathbf{g}_0}\boldsymbol{\mu}(t,\mathbf{G}_t^1,\mathbf{g}_0) \\
&:= \bar{\boldsymbol{\mu}}(t,\mathbf{G}_t^1,\boldsymbol{\beta}_t).
\end{aligned} \tag{3.4}$$

Let $\mathbf{D}(\mathbf{S}_t)$ denote an $n \times n$ diagonal matrix with $S_{i,t}$ being its i^{th} diagonal element. Define a stochastic process $\bar{\mathbf{W}} = \{\bar{\mathbf{W}}_t\}_{0\leq t\leq T}$ with $\bar{\mathbf{W}}_t = (\bar{W}_{1,t},\ldots,\bar{W}_{n,t})^\intercal$ such that $\bar{\mathbf{W}}_0 = \mathbf{0}$ and

$$\begin{aligned}
d\bar{\mathbf{W}}_t &= \boldsymbol{\sigma}_t^\intercal\boldsymbol{\Sigma}_t^{-1}\mathbf{D}(\mathbf{S}_t)^{-1}d\mathbf{S}_t - \boldsymbol{\sigma}_t^\intercal\boldsymbol{\Sigma}_t^{-1}\bar{\boldsymbol{\mu}}_t dt && (3.5)\\
&= \boldsymbol{\sigma}_t^\intercal\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{\mu}_t dt + \boldsymbol{\sigma}_t d\mathbf{W}_t\right) - \boldsymbol{\sigma}_t^\intercal\boldsymbol{\Sigma}_t^{-1}\bar{\boldsymbol{\mu}}_t dt \\
&= d\mathbf{W}_t + \boldsymbol{\sigma}_t^\intercal\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\mu}_t - \bar{\boldsymbol{\mu}}_t)dt, && (3.6)
\end{aligned}$$

where the last step holds because $\boldsymbol{\sigma}_t^\intercal\boldsymbol{\Sigma}_t^{-1}\boldsymbol{\sigma}_t$ is an identity matrix (see the proof of Lemma A.1 of Liu et al. [16]). It is easy to see that $\bar{W}_{i,t} = W_{i,t}$ for $0 \leq t \leq T$ and $i = 1,\ldots,n_1$.

Lemma 3.1 *The process $\bar{\mathbf{W}}$ is an n-dimensional standard Brownian motion with respect to the filtration $\mathbb{F}^b$ under the probability measure P.*

For the ease of reading, we postpone the proofs of all Lemmas 3.1–3.5 of Section 3.1 to Appendix A.1.

From (2.7) and (3.6), the value process of the investor's investment portfolio follows the SDE

$$\begin{aligned} dX_t &= X_t \left(r + \boldsymbol{\pi}_t^\intercal \boldsymbol{\theta}_t\right) dt + X_t \left(\boldsymbol{\pi}_t^\intercal \boldsymbol{\sigma}_t d\mathbf{W}_t\right) \\ &= X_t \left(r + \boldsymbol{\pi}_t^\intercal \bar{\boldsymbol{\theta}}_t\right) dt + X_t \left(\boldsymbol{\pi}_t^\intercal \boldsymbol{\sigma}_t d\bar{\mathbf{W}}_t\right), \end{aligned} \tag{3.7}$$

where $\bar{\boldsymbol{\theta}}_t = \bar{\boldsymbol{\mu}}_t - r\mathbf{1}$. Using the above notation, we can express the value function V_t^b given by (3.1) in this $\mathbb{F}^b$-equivalent economy as

$$V^b(t, X_t, \mathbf{G}_t^1, \boldsymbol{\beta}_t) = \operatorname*{ess\,sup}_{\boldsymbol{\pi} \in \mathcal{A}^b} E\left[-e^{-\gamma_1 X_T} \,\middle|\, \mathcal{F}_t^b\right]. \tag{3.8}$$

• *Second step: recursive resolution method for a system of one-period optimization problems*

We employ a backward induction approach to solve the optimization problem (3.8), which involves decomposing the value function into a recursive system based on the transition times of the Markov chain (see, e.g., Jiao et al. [15]). This approach reduces the initial global problem to solving a system of one-period optimization problems defined by (3.9) below.

For $t \in [k-1, k)$ for some $k \in \{1, \ldots, T\}$, consider a one-period optimization problem for brown investors given by

$$V^{b,k}(t, x, \mathbf{g}_1, \boldsymbol{\beta}) = \sup_{\boldsymbol{\pi} \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}}\left[-e^{-\gamma_1 e^{r(T-k)} X_k - h_k(\boldsymbol{\beta}_{k-})}\right], \tag{3.9}$$

where $h_k : B \to \mathbb{R}_+$ is a bounded function.

The condition $\mathbf{G}_t^1 = \mathbf{g}_1$ in (3.9) implies that $\mathbf{G}_s^1 = \mathbf{g}_1$ for all $k-1 \le s < k$, but the value of $\mathbf{G}_s^2$ for any $k-1 \le s < k$ is unknown to brown investors. Thus, the vector of conditional mean return rates (3.4) becomes

$$\bar{\boldsymbol{\mu}}_s = \bar{\boldsymbol{\mu}}(s, \mathbf{g}_1, \boldsymbol{\beta}_s), \qquad k-1 \le s < k. \tag{3.10}$$

Define a stochastic process $\hat{W}_s = \bar{W}_{k-1+s} - \bar{W}_{k-1}, 0 \le s < 1$, where the process $\bar{\mathbf{W}}$ is given by (3.6).

Lemma 3.2 *Given that $\mathbf{G}_t^1 = \mathbf{g}_1$ for some $t \in [k-1, k)$, the process $\hat{\mathbf{W}}$ is an n-dimensional $\{\mathcal{F}_{k-1+s}^{\mathbf{S}}\}_{0 \le s < 1}$ standard Brownian motion under the probability measure P. Moreover, for any*

$\mathcal{F}$-measurable and integrable random variable ξ independent of the Brownian motion $\mathbf{W}$, we have

$$E[\xi|\mathcal{F}^{\mathbf{S}}_{k-1+s}] = E[\xi|\mathcal{F}^{\mathbf{S}}_{k-1}] + \int_{k-1}^{k-1+s} \left(\boldsymbol{\sigma}_u^{\intercal}\boldsymbol{\Sigma}_u^{-1}\left(E\left[\boldsymbol{\mu}_u\xi|\,\mathcal{F}^{\mathbf{S}}_u\right] - \bar{\boldsymbol{\mu}}_u E[\xi|\mathcal{F}^{\mathbf{S}}_u]\right)\right)^{\intercal} d\hat{\mathbf{W}}_{u-k+1} \tag{3.11}$$

for $0 \le s < 1$.

For any $\mathbf{g}_2 \in \hat{\mathfrak{g}}_2$ and $0 \le s < 1$, applying Lemma 3.2 with $\xi = 1_{\{\mathbf{G}^2_{k-1}=\mathbf{g}_2\}}$ yields

$$\beta^{\mathbf{g}_2}_{k-1+s} \;=\; \beta^{\mathbf{g}_2}_{k-1} + \int_{k-1}^{k-1+s} \left(\boldsymbol{\sigma}_u^{\intercal}\boldsymbol{\Sigma}_u^{-1}\left(\boldsymbol{\mu}(u,\mathbf{g}_1,\mathbf{g}_2) - \bar{\boldsymbol{\mu}}_u\right)\beta^{\mathbf{g}_2}_u\right)^{\intercal} d\hat{\mathbf{W}}_{u-k+1}. \tag{3.12}$$

For a measurable function $w = w(t,x,\mathbf{g}_1,\boldsymbol{\beta})$ on $[k-1,k)\times\mathbb{R}\times\mathfrak{g}_1\times B$ such that $w(\cdot,\cdot,\mathbf{g}_1,\cdot) \in C^{1,2,2}([k-1,k)\times\mathbb{R}\times B)$ for any $\mathbf{g}_1 \in \mathfrak{g}_1$, define an operator $\mathcal{L}^{\boldsymbol{\pi}}_1$ by

$$\begin{aligned}\mathcal{L}^{\boldsymbol{\pi}}_1 w \;=\;& w_t + rxw_x + \frac{1}{2}\left(\boldsymbol{\pi}^{\intercal}\boldsymbol{\Sigma}\boldsymbol{\pi}\right)x^2 w_{xx} + \left(\boldsymbol{\pi}^{\intercal}\bar{\boldsymbol{\theta}}\right)xw_x + x\sum_{\mathbf{g}_2\in\hat{\mathfrak{g}}_2}\left(\boldsymbol{\pi}^{\intercal}\boldsymbol{\sigma}\boldsymbol{\alpha}(t,\boldsymbol{\beta},\mathbf{g}_1,\mathbf{g}_2)\right)w_{x\beta^{\mathbf{g}_2}} \\ &+\frac{1}{2}\sum_{\mathbf{g}_i\in\hat{\mathfrak{g}}_2}\sum_{\mathbf{g}_j\in\hat{\mathfrak{g}}_2}\boldsymbol{\alpha}(t,\boldsymbol{\beta},\mathbf{g}_1,\mathbf{g}_i)^{\intercal}\boldsymbol{\alpha}(t,\boldsymbol{\beta},\mathbf{g}_1,\mathbf{g}_j)w_{\beta^{\mathbf{g}_i}\beta^{\mathbf{g}_j}}.\end{aligned}$$

where

$$\boldsymbol{\alpha}(t,\boldsymbol{\beta},\mathbf{g}_1,\mathbf{g}_2) = \boldsymbol{\sigma}_t^{\intercal}\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{\mu}(t,\mathbf{g}_1,\mathbf{g}_2) - \bar{\boldsymbol{\mu}}(t,\mathbf{g}_1,\boldsymbol{\beta})\right)\beta^{\mathbf{g}_2}. \tag{3.13}$$

To derive an analytical expression for the value function (3.9), we first establish the following verification lemma:

Lemma 3.3 *Let $w = w(t,x,\mathbf{g}_1,\boldsymbol{\beta})$ be a measurable function on $[k-1,k)\times\mathbb{R}\times\mathfrak{g}_1\times B$ such that $w(\cdot,\cdot,\mathbf{g}_1,\cdot) \in C^{1,2,2}([k-1,k)\times\mathbb{R}\times B)$ for any $\mathbf{g}_1 \in \mathfrak{g}_1$, and that*

$$|w(t,x,\mathbf{g}_1,\boldsymbol{\beta})| \le e^{-\gamma_1 e^{r(T-t)}x}, \qquad (t,x,\mathbf{g}_1,\boldsymbol{\beta}) \in [k-1,k)\times\mathbb{R}\times\mathfrak{g}_1\times B. \tag{3.14}$$

(i) Suppose that

$$\begin{aligned}\max_{\boldsymbol{\pi}}\mathcal{L}^{\boldsymbol{\pi}}_1 w \;&\le\; 0, \quad (t,x,\mathbf{g}_1,\boldsymbol{\beta}) \in [k-1,k)\times\mathbb{R}\times\mathfrak{g}_1\times B, &(3.15)\\ \lim_{t\uparrow k} w(t,x,\mathbf{g}_1,\boldsymbol{\beta}) \;&:=\; w(k-,x,\mathbf{g}_1,\boldsymbol{\beta}) \\ &\ge\; -e^{-\gamma_1 e^{r(T-k)}x - h_k(\boldsymbol{\beta})}, \quad (x,\mathbf{g}_1,\boldsymbol{\beta}) \in \mathbb{R}\times\mathfrak{g}_1\times B. &(3.16)\end{aligned}$$

Then we have

$$w(t,x,\mathbf{g}_1,\boldsymbol{\beta}) \geq \sup_{\boldsymbol{\pi}\in\mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}}\left[-e^{-\gamma_1 e^{r(T-k)}X_k - h_k(\boldsymbol{\beta}_{k-})}\right] \tag{3.17}$$

for all $(t,x,\mathbf{g}_1,\boldsymbol{\beta}) \in [k-1,k) \times \mathbb{R} \times \mathfrak{g}_1 \times B$.

(ii) Suppose further that $w(k-,x,\mathbf{g}_1,\boldsymbol{\beta}) = -e^{-\gamma_1 e^{r(T-k)}x - h_k(\boldsymbol{\beta})}$*, there exists a measurable function* $\boldsymbol{\pi}^*(t,x,\mathbf{g}_1,\boldsymbol{\beta})$ *on* $[0,T]\times\mathbb{R}\times\mathfrak{g}_1\times B$ *such that*

$$\max_{\boldsymbol{\pi}} \mathcal{L}_1^{\boldsymbol{\pi}} w = \mathcal{L}_1^{\boldsymbol{\pi}^*} w = 0 \tag{3.18}$$

for all $(t,x,\mathbf{g}_1,\boldsymbol{\beta}) \in [k-1,k)\times\mathbb{R}\times\mathfrak{g}_1\times B$*, and that* $\boldsymbol{\pi}^* = \{\boldsymbol{\pi}^*(t,X_t,\mathbf{G}_t^1,\boldsymbol{\beta}_t)\}_{0\leq t\leq T} \in \mathcal{A}^b$*. Then we have*

$$w(t,x,\mathbf{g}_1,\boldsymbol{\beta}) = E^{t,x,\mathbf{g}_1,\boldsymbol{\beta},\boldsymbol{\pi}^*}\left[-e^{-\gamma_1 e^{r(T-k)}X_k - h_k(\boldsymbol{\beta}_{k-})}\right] \tag{3.19}$$

for all $(t,x,\mathbf{g}_1,\boldsymbol{\beta}) \in [k-1,k)\times\mathbb{R}\times\mathfrak{g}_1\times B$.

The lemma below gives an analytical expression for the value function (3.9) along with the optimal portfolio strategy. The probability measure $\hat{P}$ therein will be used to derive an analytical expression for the value function of brown investors.

Lemma 3.4 *Define a probability measure* $\hat{P}$ *on* $\mathcal{F}_T^b = \mathcal{F}_T^{\mathbf{S}} \vee \mathcal{F}_T^{\mathbf{G}^1}$ *via the Radon–Nikodym derivative*

$$\frac{d\hat{P}}{dP} = \exp\left\{-\int_0^T \bar{\boldsymbol{\theta}}_t^\intercal \boldsymbol{\Sigma}_t^{-1}\boldsymbol{\sigma}_t d\bar{\mathbf{W}}_t - \frac{1}{2}\int_0^T \bar{\boldsymbol{\theta}}_t^\intercal \boldsymbol{\Sigma}_t^{-1}\bar{\boldsymbol{\theta}}_t dt\right\}, \tag{3.20}$$

and, for $k-1 \leq t < k, k = 1,\ldots,T$*, let*

$$A^k(t,\mathbf{g}_1,\boldsymbol{\beta}) = E^{\hat{P}}\left[\frac{1}{2}\int_t^k \bar{\boldsymbol{\theta}}_s^\intercal \boldsymbol{\Sigma}_s^{-1}\bar{\boldsymbol{\theta}}_s ds + h_k(\boldsymbol{\beta}_{k-})\middle| \mathbf{G}_t^1 = \mathbf{g}_1, \boldsymbol{\beta}_t = \boldsymbol{\beta}\right].$$

Suppose $A^k(t,\mathbf{g}_1,\cdot) \in C^2(B)$ *for any* $t\in[k-1,k)$ *and* $\mathbf{g}_1 \in \mathfrak{g}_1$*, and there exists* $L > 0$ *such that*[2]

$$\left\|\frac{\partial A^k(t,\mathbf{g}_1,\boldsymbol{\beta})}{\partial\boldsymbol{\beta}}\right\|_\infty := \left\|A_{\boldsymbol{\beta}}^k(t,\mathbf{g}_1,\boldsymbol{\beta})\right\|_\infty \leq L, \qquad k-1\leq t<k, \mathbf{g}_1\in\mathfrak{g}_1, \boldsymbol{\beta}\in B, \tag{3.21}$$

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[2] The verification of the boundedness condition on the partial derivative $A_{\boldsymbol{\beta}}^k(t,\mathbf{g}_1,\boldsymbol{\beta})$ is left as an open question. We would like to point out that this does not affect the application of our theoretical results. In particular, for numerical studies, we can draw a plot of $A^k(t,\mathbf{g}_1,\boldsymbol{\beta})$ against each filtered probability $\beta^{\mathbf{g}_2}$, holding other quantities fixed, to check whether the condition is satisfied, and we can numerically approximate the partial derivative $A_{\boldsymbol{\beta}}^k(t,\mathbf{g}_1,\boldsymbol{\beta})$ to compute the optimal portfolio of a brown investor.

where $\|\cdot\|_\infty$ *is the supremum norm. Then the value function (3.9) for* $(t, x, \mathbf{g}_1, \boldsymbol{\beta}) \in [k-1, k) \times \mathbb{R} \times \mathfrak{g}_1 \times B$ *is given by*

$$V^{b,k}(t, x, \mathbf{g}_1, \boldsymbol{\beta}) = -e^{-\gamma_1 e^{r(T-t)} x - A^k(t, \mathbf{g}_1, \boldsymbol{\beta})}, \qquad k = 1, \ldots, T. \tag{3.22}$$

Moreover, the optimal portfolio strategy is given by

$$\boldsymbol{\pi}_t^b = \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \left(\bar{\boldsymbol{\theta}}_t - \sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} \left(\boldsymbol{\mu}(t, \mathbf{G}_t^1, \mathbf{g}_2) - \bar{\boldsymbol{\mu}}_t \right) \beta_t^{\mathbf{g}_2} A^k_{\beta \mathbf{g}_2}(t, \mathbf{G}_t^1, \boldsymbol{\beta}_t) \right) \tag{3.23}$$

for $k-1 \le t < k, k = 1, \ldots, T$.

The following result regarding $\beta_t^{\mathbf{g}_2}$ in (3.23) will be used to prove Proposition 3.1.

Lemma 3.5 *For any* $\mathbf{g} \in \mathfrak{g}_2$*, we have*

$$\beta_k^{\mathbf{g}} = \sum_{\tilde{\mathbf{g}} \in \mathfrak{g}_2} \beta_{k-}^{\tilde{\mathbf{g}}} p_{2, \tilde{\mathbf{g}}, \mathbf{g}}, \qquad k = 1, \ldots, T. \tag{3.24}$$

- *Third step: value function for brown investors*

We can now apply the above one-step result recursively in a backward manner from the terminal time T to the initial time 0, and deduce the value function V_0^b together with the optimal portfolio strategy. Recall that the auxiliary probability measure $\hat{P}$ is defined on $\mathcal{F}_T^b$ via the Radon–Nikodym derivative (3.20). For $T-1 \le t \le T$, let

$$f_T(t, \mathbf{g}_1, \boldsymbol{\beta}) = E^{\hat{P}} \left[\frac{1}{2} \int_t^T \bar{\boldsymbol{\theta}}_s^\intercal \boldsymbol{\Sigma}_s^{-1} \bar{\boldsymbol{\theta}}_s ds \,\middle|\, \mathbf{G}_t^1 = \mathbf{g}_1, \boldsymbol{\beta}_t = \boldsymbol{\beta} \right].$$

For $k-1 \le t < k, k = 1, \ldots, T-1$, let

$$f_k(t, \mathbf{g}_1, \boldsymbol{\beta}) = E^{\hat{P}} \left[\frac{1}{2} \int_t^k \bar{\boldsymbol{\theta}}_s^\intercal \boldsymbol{\Sigma}_s^{-1} \bar{\boldsymbol{\theta}}_s ds - \ln \left(\sum_{\tilde{\mathbf{g}} \in \mathfrak{g}_1} p_{1, \mathbf{g}_1, \tilde{\mathbf{g}}} e^{-f_{k+1}(k, \tilde{\mathbf{g}}, \boldsymbol{\beta}_k)} \right) \,\middle|\, \mathbf{G}_t^1 = \mathbf{g}_1, \boldsymbol{\beta}_t = \boldsymbol{\beta} \right].$$

Now we are ready to present our first main result:

Proposition 3.1 *For $k = 1, \ldots, T$, suppose $f_k(t, \mathbf{g}_1, \cdot) \in C^2(B)$ for any $t \in [k-1, k)$ and $\mathbf{g}_1 \in \mathfrak{g}_1$, and there exists $L > 0$ such that*

$$\left\| \frac{\partial f_k(t, \mathbf{g}_1, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \right\|_\infty \leq L, \qquad k-1 \leq t < k, \mathbf{g}_1 \in \mathfrak{g}_1, \boldsymbol{\beta} \in B,$$

then the value function (3.8) for $(t, x, \mathbf{g}_1, \boldsymbol{\beta}) \in [k-1, k) \times \mathbb{R} \times \mathfrak{g}_1 \times B$ is given by

$$V^b(t, x, \mathbf{g}_1, \boldsymbol{\beta}) = -e^{-\gamma_1 e^{r(T-t)} x - f_k(t, \mathbf{g}_1, \boldsymbol{\beta})}. \tag{3.25}$$

Moreover, the optimal portfolio strategy is given by

$$\boldsymbol{\pi}_t^b = \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \left(\bar{\boldsymbol{\theta}}_t - \sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} \left(\boldsymbol{\mu}(t, \mathbf{G}_t^1, \mathbf{g}_2) - \bar{\boldsymbol{\mu}}_t \right) \beta_t^{\mathbf{g}_2} \frac{\partial f_k(t, \mathbf{G}_t^1, \boldsymbol{\beta}_t)}{\partial \beta^{\mathbf{g}_2}} \right) \tag{3.26}$$

for $k-1 \leq t < k$ and $k = 1, \ldots, T$.

Proof. First the expression (3.25) for $T-1 \leq t < T$ and the corresponding optimal portfolio strategy (3.26) follow immediately from Lemma 3.4 with $k = T$ and $h_T \equiv 1$.

Next we consider the time interval $[T-2, T-1)$. From (3.24), $\boldsymbol{\beta}_{T-1} = \mathbf{H}(\boldsymbol{\beta}_{(T-1)-})$, where $\mathbf{H}(\boldsymbol{\beta})$ is a deterministic function of $\boldsymbol{\beta}$. Note that the value process X is continuous almost surely. By the dynamic programming principle, we have

$$\begin{aligned}
& V(t, x, \mathbf{g}_1, \boldsymbol{\beta}) \\
= \; & \sup_{\pi \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}} \left[V(T-1, X_{T-1}, \mathbf{G}_{T-1}^1, \boldsymbol{\beta}_{T-1}) \right] \\
= \; & \sup_{\pi \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}} \left[-e^{-\gamma_1 e^r X_{T-1}} \exp\left\{ -E^{\hat{P}} \left[\int_{T-1}^{T} \frac{1}{2} \bar{\boldsymbol{\theta}}_s^\intercal \boldsymbol{\Sigma}_s^{-1} \bar{\boldsymbol{\theta}}_s ds \middle| \mathbf{G}_{T-1}^1, \boldsymbol{\beta}_{T-1} \right] \right\} \right] \\
= \; & \sup_{\pi \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}} \left[-e^{-\gamma_1 e^r X_{T-1} - f_T\left(T-1, \mathbf{G}_{T-1}^1, \mathbf{H}(\boldsymbol{\beta}_{(T-1)-})\right)} \right] \\
= \; & \sup_{\pi \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}} \left[E \left[-e^{-\gamma_1 e^r X_{(T-1)-} - f_T\left(T-1, \mathbf{G}_{T-1}^1, \mathbf{H}(\boldsymbol{\beta}_{(T-1)-})\right)} \middle| \mathcal{F}_{(T-1)-}^b \right] \right] \\
= \; & \sup_{\pi \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}} \left[-e^{-\gamma_1 e^r X_{(T-1)-}} E \left[e^{-f_T\left(T-1, \mathbf{G}_{T-1}^1, \mathbf{H}(\boldsymbol{\beta}_{(T-1)-})\right)} \middle| \mathbf{G}_{(T-1)-}^1 = \mathbf{g}_1, \boldsymbol{\beta}_{(T-1)-} \right] \right] \\
= \; & \sup_{\pi \in \mathcal{A}^b} E^{t,x,\mathbf{g}_1,\boldsymbol{\beta}} \left[-e^{-\gamma_1 e^r X_{T-1}} \left(\sum_{\tilde{\mathbf{g}} \in \mathfrak{g}_1} p_{1,\mathbf{g}_1,\tilde{\mathbf{g}}} e^{-f_T\left(T-1, \tilde{\mathbf{g}}, \mathbf{H}(\boldsymbol{\beta}_{(T-1)-})\right)} \right) \right].
\end{aligned}$$

The function $f_T(\cdot,\cdot)$ is bounded because of (A.6). Apply Lemma 3.4 again with $k = T-1$, we obtain

$$\begin{aligned}
& V(t, x, \mathbf{g}_1, \boldsymbol{\beta}) \\
= \; & -e^{-\gamma_1 e^{r(T-t)} x} \\
& \times \exp\left\{ E^{\hat{P}} \left[-\frac{1}{2}\int_t^{T-1} \bar{\boldsymbol{\theta}}_s^{\intercal} \boldsymbol{\Sigma}_s^{-1} \bar{\boldsymbol{\theta}}_s ds + \ln\left(\sum_{\tilde{\mathbf{g}} \in \mathfrak{g}_1} p_{1,\mathbf{g}_1,\tilde{\mathbf{g}}} e^{-f_T\left(T-1, \tilde{\mathbf{g}}, \mathbf{H}(\boldsymbol{\beta}_{(T-1)-})\right)} \right) \middle| \mathbf{G}_t^1 = \mathbf{g}_1, \boldsymbol{\beta}_t = \boldsymbol{\beta} \right] \right\} \\
= \; & -e^{-\gamma_1 e^{r(T-t)} x - f_{T-1}(t, \mathbf{g}_1, \boldsymbol{\beta})}.
\end{aligned}$$

Moreover, the optimal portfolio strategy is given by (3.26).

Lastly, we complete the proof for all $t \in [0, T)$ by repeating the above procedure. ■

When all of the ESG ratings are unobservable to brown investors, i.e., $n_1 = 0$, the value function (3.25) can be simplified to

$$V^b(t, x, \boldsymbol{\beta}) = -e^{-\gamma_1 e^{r(T-t)} x - f(t, \boldsymbol{\beta})}, \qquad 0 \le t < T, \boldsymbol{\beta} \in B,$$

where

$$f(t, \boldsymbol{\beta}) = E^{\hat{P}} \left[\frac{1}{2} \int_t^T \bar{\boldsymbol{\theta}}_s^{\intercal} \boldsymbol{\Sigma}_s^{-1} \bar{\boldsymbol{\theta}}_s ds \middle| \boldsymbol{\beta}_t = \boldsymbol{\beta} \right].$$

Moreover, the optimal portfolio strategy is given by

$$\boldsymbol{\pi}_t^b = \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \left(\bar{\boldsymbol{\theta}}_t - \sum_{\mathbf{g} \in \hat{\mathfrak{g}}_2} \left(\boldsymbol{\mu}(t, \mathbf{g}) - \bar{\boldsymbol{\mu}}_t \beta_t^{\mathbf{g}} \right) \frac{\partial f(t, \boldsymbol{\beta}_t)}{\partial \beta^{\mathbf{g}}} \right), \qquad 0 \le t < T.$$

In the following remarks, we investigate the components of the optimal portfolio of a brown investor.

Remark 3.1 *(Local mean-variance portfolio) Given the information set $\mathcal{F}_t^b$, the conditional mean and the conditional variance of dX_t in* (3.7) *are respectively $X_t \left(r + \boldsymbol{\pi}_t^{\intercal} \bar{\boldsymbol{\theta}}_t\right) dt$ and $X_t^2 \left(\boldsymbol{\pi}_t^{\intercal} \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t\right) dt$. Consider the mean-variance optimization problem*

$$\max_{\boldsymbol{\pi}_t} \left(X_t \left(r + \boldsymbol{\pi}_t^{\intercal} \bar{\boldsymbol{\theta}}_t\right) dt - \frac{\gamma_1}{2} X_t^2 \left(\boldsymbol{\pi}_t^{\intercal} \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t\right) dt \right).$$

It is easy to verify that the maximizer is

$$\bar{\boldsymbol{\pi}}_t^* = \frac{1}{\gamma_1 X_t} \boldsymbol{\Sigma}_t^{-1} \bar{\boldsymbol{\theta}}_t. \tag{3.27}$$

Similarly, using the expression (2.7) *for* dX_t *and conditional on* $\mathcal{F}_t^g$ *or* $\mathcal{F}_t^m$ *we consider*

$$\max_{\boldsymbol{\pi}_t} \Big(X_t \left(r + \boldsymbol{\pi}_t^\intercal \boldsymbol{\theta}_t \right) dt - \frac{\gamma_1}{2} X_t^2 \left(\boldsymbol{\pi}_t^\intercal \boldsymbol{\Sigma}_t \boldsymbol{\pi}_t \right) dt \Big),$$

which yields the maximizer

$$\boldsymbol{\pi}_t^* = \frac{1}{\gamma_1 X_t} \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\theta}_t. \tag{3.28}$$

We refer to the portfolios (3.27) *and* (3.28) *as local mean-variance portfolios, following the Markowitz framework.*

Remark 3.2 *(Hedging portfolio) Proposition 3.1 shows that the optimal portfolio* (3.26) *of a brown investor is a combination of the riskless asset, the local mean-variance portfolio* (3.27)*, and the following portfolio*

$$\begin{aligned} & -\frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \left(\sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} \left(\boldsymbol{\mu}(t, \mathbf{G}_t^1, \mathbf{g}_2) - \bar{\boldsymbol{\mu}}_t \right) \beta_t^{\mathbf{g}_2} \frac{\partial f_k(t, \mathbf{G}_t^1, \boldsymbol{\beta}_t)}{\partial \beta^{\mathbf{g}_2}} \right) \\ = \; & \boldsymbol{\Sigma}_t^{-1} \left(\sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} \left(\boldsymbol{\mu}(t, \mathbf{G}_t^1, \mathbf{g}_2) - \bar{\boldsymbol{\mu}}_t \right) \beta_t^{\mathbf{g}_2} \left(-\frac{V^b_{x\beta^{\mathbf{g}_2}}(t, X_t, \mathbf{G}_t^1, \boldsymbol{\beta}_t)}{X_t V^b_{xx}(t, X_t, \mathbf{G}_t^1, \boldsymbol{\beta}_t)} \right) \right), \end{aligned} \tag{3.29}$$

where the above equality follows from (3.25)*.*

Given that $X_t = x, \mathbf{G}_t^1 = \mathbf{g}_1$*, and* $\boldsymbol{\beta}_t = \boldsymbol{\beta}$*, the demand for the* i^{th} *risky asset in the portfolio* (3.29) *can be written as*

$$x \sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} l_{i,\mathbf{g}_2} \left(-\frac{V^b_{x\beta^{\mathbf{g}_2}}(t, x, \mathbf{g}_1, \boldsymbol{\beta})}{x V^b_{xx}(t, x, \mathbf{g}_1, \boldsymbol{\beta})} \right) = \sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} l_{i,\mathbf{g}_2} \left(-\frac{\partial V_x^b / \partial \beta^{\mathbf{g}_2}}{\partial V_x^b / \partial x} \right)$$

for some constants $l_{i,\mathbf{g}_2}, \mathbf{g}_2 \in \hat{\mathfrak{g}}_2$*. From the above expression, we observe that the portfolio* (3.29) *reflects the demand for hedging against unfavorable shifts in the filtered probabilities (see, e.g., the analysis in Merton [17, Section 5]). Here, an unfavorable shift in the filtered probability* $\beta^{\mathbf{g}_2}$ *is understood as a change in* $\beta^{\mathbf{g}_2}$ *such that the marginal utility of wealth* V_x^b *will decrease for a given* *level of wealth. We refer to the portfolio* (3.29) *as a hedging portfolio.*

3.2 Optimization for green investors

The optimization problem for green investors, who utilize the complete information on both asset prices and ESG ratings, is conducted under the filtration $\mathbb{F}^g = \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G}}$. We introduce the value function as

$$V^g(t, X_t, \mathbf{G}_t) = \operatorname*{ess\,sup}_{\boldsymbol{\pi} \in \mathcal{A}^g} E\left[-e^{-\gamma_1 X_T - \gamma_2 \int_t^T (\boldsymbol{\pi}_s^\intercal \mathbf{G}_s) e^{r(T-s)} X_s ds} \middle| \mathcal{F}_t^g \right]. \qquad (3.30)$$

Similar to the case of brown investors, we will employ a backward induction approach which involves solving a system of one-period optimization problems defined by (3.31) below. Consider the time interval $[k-1, k)$ for some $k \in \{1, \dots, T\}$. For green investors, we study a one-period optimization problem given by

$$\begin{aligned} V^{g,k}(t, x, \mathbf{g}) &= \sup_{\boldsymbol{\pi} \in \mathcal{A}^g} E^{t,x,\mathbf{g}} \left[-e^{-\gamma_1 e^{r(T-k)} X_k - \gamma_2 \int_t^k (\boldsymbol{\pi}_s^\intercal \mathbf{G}_s) e^{r(T-s)} X_s ds} \right] \\ &= \sup_{\boldsymbol{\pi} \in \mathcal{A}^g} E^{t,x,\mathbf{g}} \left[-e^{-\gamma_1 e^{r(T-k)} X_k - \gamma_2 \int_t^k (\boldsymbol{\pi}_s^\intercal \mathbf{g}) e^{r(T-s)} X_s ds} \right]. \end{aligned} \qquad (3.31)$$

Given the complete information on ESG ratings with $\mathbf{G}_t = \mathbf{g}$, the value process of the investment portfolio follows the SDE

$$dX_s = X_s \left(r + \boldsymbol{\pi}_s^\intercal \boldsymbol{\theta}(s, \mathbf{g}) \right) ds + X_s \left(\boldsymbol{\pi}_s^\intercal \boldsymbol{\sigma}_s d\mathbf{W}_s \right), \qquad k-1 \le s < k. \qquad (3.32)$$

For a measurable function $w = w(t, x, \mathbf{g})$ on $[k-1, k) \times \mathbb{R} \times \mathfrak{g}$ such that $w(\cdot, \cdot, \mathbf{g}) \in C^{1,2}([k-1, k) \times \mathbb{R})$ for any $\mathbf{g} \in \mathfrak{g}$, define an operator $\mathcal{L}_2^{\boldsymbol{\pi}}$ by

$$\mathcal{L}_2^{\boldsymbol{\pi}} w = w_t + rxw_x + \boldsymbol{\pi}^\intercal \boldsymbol{\theta}(t, \mathbf{g}) w_x + \frac{1}{2} \boldsymbol{\pi}^\intercal \boldsymbol{\Sigma}_t \boldsymbol{\pi} w_{xx}.$$

To solve for the value function (3.31), we first establish the following verification lemma. The proofs of Lemmas 3.6 and 3.7 can be found in Appendix A.2.

Lemma 3.6 *Let* $w = w(t, x, \mathbf{g})$ *be a measurable function on* $[k-1, k) \times \mathbb{R} \times \mathfrak{g}$ *such that* $w(\cdot, \cdot, \mathbf{g}) \in$

$C^{1,2}([k-1,k)\times\mathbb{R})$ *for any* $\mathbf{g}\in\mathfrak{g}$, *and that*

$$|w(t,x,\mathbf{g})| \le e^{-\gamma_1 e^{r(T-t)}x}, \qquad (t,x,\mathbf{g})\in[k-1,k)\times\mathbb{R}\times\mathfrak{g}. \tag{3.33}$$

(i) Suppose that

$$\begin{aligned} \max_{\boldsymbol{\pi}}\left\{\mathcal{L}_2^{\boldsymbol{\pi}} w - \gamma_2\left(\boldsymbol{\pi}^\intercal\mathbf{g}\right)e^{r(T-t)}xw\right\} &\le 0, \qquad (t,x,\mathbf{g})\in[k-1,k)\times\mathbb{R}\times\mathfrak{g}, \quad (3.34)\\ \lim_{t\uparrow k} w(t,x,\mathbf{g}) = w(k-,x,\mathbf{g}) &\ge -e^{-\gamma_1 e^{r(T-k)}x}, \qquad (x,\mathbf{g})\in\mathbb{R}\times\mathfrak{g}. \quad (3.35) \end{aligned}$$

Then we have

$$w(t,x,\mathbf{g}) \ge \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[-e^{-\gamma_1 e^{r(T-k)}X_k-\gamma_2\int_t^k(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}\right] \tag{3.36}$$

for all $(t,x,\mathbf{g})\in[k-1,k)\times\mathbb{R}\times\mathfrak{g}$.

(ii) Suppose further that $w(k-,x,\mathbf{g}) = -e^{-\gamma_1 e^{r(T-k)}x}$, *there exists a measurable function* $\boldsymbol{\pi}^*(t,x,\mathbf{g})$ *on* $[0,T]\times\mathbb{R}\times\mathfrak{g}$ *such that*

$$\max_{\boldsymbol{\pi}}\left\{\mathcal{L}_2^{\boldsymbol{\pi}} w - \gamma_2\left(\boldsymbol{\pi}^\intercal\mathbf{g}\right)e^{r(T-t)}xw\right\} = \mathcal{L}_2^{\boldsymbol{\pi}^*} w - \gamma_2\left((\boldsymbol{\pi}^*)^\intercal\mathbf{g}\right)e^{r(T-t)}xw = 0 \tag{3.37}$$

for all $(t,x,\mathbf{g})\in[k-1,k)\times\mathbb{R}\times\mathfrak{g}$, *and that* $\boldsymbol{\pi}^* = \{\boldsymbol{\pi}^*(t,X_t,\mathbf{G}_t)\}_{0\le t\le T}\in\mathcal{A}^g$. *Then we have*

$$w(t,x,\mathbf{g}) = E^{t,x,\mathbf{g},\boldsymbol{\pi}^*}\left[-e^{-\gamma_1 e^{r(T-k)}X_k-\gamma_2\int_t^k((\boldsymbol{\pi}_s^*)^\intercal\mathbf{g})e^{r(T-s)}X_s ds}\right]$$

for all $(t,x,\mathbf{g})\in[k-1,k)\times\mathbb{R}\times\mathfrak{g}$.

The lemma below gives an analytical expression for the value function (3.31). The results are verified by Lemma 3.6; see Appendix A.2.

Lemma 3.7 *The value function (3.31) for* $(t,x,\mathbf{g})\in[k-1,k)\times\mathbb{R}\times\mathfrak{g}, k=1,\ldots,T$, *is given by*

$$V^{g,k}(t,x,\mathbf{g}) = -e^{-\gamma_1 e^{r(T-t)}x-\int_t^k\frac{1}{2}\left(\boldsymbol{\theta}(s,\mathbf{g})+\frac{\gamma_2}{\gamma_1}\mathbf{g}\right)^\intercal\boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\theta}(s,\mathbf{g})+\frac{\gamma_2}{\gamma_1}\mathbf{g}\right)ds}. \tag{3.38}$$

Moreover, the optimal portfolio strategy is given by

$$\boldsymbol{\pi}_t^g = \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\theta}_t + \frac{\gamma_2}{\gamma_1^2 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \mathbf{G}_t, \qquad 0 \le t \le T. \tag{3.39}$$

For green investors, the value function (3.30) and the optimal portfolio strategy are given by:

Proposition 3.2 *The value function (3.30) for* $(t, x, \mathbf{g}) \in [0, T) \times \mathbb{R} \times \mathfrak{g}$ *is given by*

$$V^g(t, x, \mathbf{g}) = -e^{-\gamma_1 e^{r(T-t)} x} E\left[e^{-\int_t^T \frac{1}{2}\left(\boldsymbol{\theta}_s + \frac{\gamma_2}{\gamma_1}\mathbf{G}_s\right)^\intercal \boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\theta}_s + \frac{\gamma_2}{\gamma_1}\mathbf{G}_s\right) ds} \,\middle|\, \mathbf{G}_t = \mathbf{g} \right]. \tag{3.40}$$

The optimal portfolio strategy is given by

$$\boldsymbol{\pi}_t^g = \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\theta}_t + \frac{\gamma_2}{\gamma_1^2 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \mathbf{G}_t, \qquad 0 \le t \le T. \tag{3.41}$$

Proof. First consider $t \in [T-1, T)$. Recall the ESG rating process (2.2), we have $\mathbf{G}_t = \mathbf{G}_{T-1}$ for $T - 1 \le t < T$. Then the value function (3.30) can be written as

$$V^g(t, x, \mathbf{g}) = \sup_{\boldsymbol{\pi} \in \mathcal{A}^g} E^{t,x,\mathbf{g}} \left[-e^{-\gamma_1 X_T - \gamma_2 \int_t^T (\boldsymbol{\pi}_s^\intercal \mathbf{g}) e^{r(T-s)} X_s ds} \right],$$

which reduces to the one-period optimization problem (3.31) with $k = T$. Apply Lemma 3.7 with $k = T$, we get

$$V^g(t, x, \mathbf{g}) = -e^{-\gamma_1 e^{r(T-t)} x - \int_t^T b_{\mathbf{g}}(s) ds},$$

where

$$b_{\mathbf{g}}(t) = \frac{1}{2} \left(\boldsymbol{\theta}(t, \mathbf{g}) + \frac{\gamma_2}{\gamma_1} \mathbf{g} \right)^\intercal \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{\theta}(t, \mathbf{g}) + \frac{\gamma_2}{\gamma_1} \mathbf{g} \right). \tag{3.42}$$

Using the identity $\mathbf{G}_t = \mathbf{G}_{T-1}$ for $T - 1 \le t < T$ again, the above expression for $V^g(t, x, \mathbf{g})$ can be written as (3.40). Moreover, the optimal portfolio strategy is given by (3.41). This completes the proof for $t \in [T-1, T)$.

Next consider $t \in [T-2, T-1)$. Recall (2.2), in which $\{\mathbf{G}_k\}_{k \in \mathbb{N}_0}$ is a Markov chain. It is easy to see from (2.7) that $X_{T-1} = X_{(T-1)-}$ almost surely. From these observations, the dynamic programming principle, and (3.40) with $t = T-1$, we derive

$$\begin{aligned}
& V^g(t,x,\mathbf{g}) \\
= \ & \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[e^{-\gamma_2\int_t^{T-1}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}V^g(T-1,X_{T-1},\mathbf{G}_{T-1})\right] \\
= \ & \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[e^{-\gamma_2\int_t^{T-1}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}\left(-e^{-\gamma_1 e^r X_{T-1}-\int_{T-1}^T b_{\mathbf{G}_s}(s)ds}\right)\right] \\
= \ & \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[E\left[-e^{-\gamma_1 e^r X_{(T-1)-}-\gamma_2\int_t^{(T-1)-}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds-\int_{T-1}^T b_{\mathbf{G}_{T-1}}(s)ds}\middle|\mathcal{F}^g_{(T-1)-}\right]\right] \\
= \ & \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[-e^{-\gamma_1 e^r X_{(T-1)-}-\gamma_2\int_t^{(T-1)-}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}E\left[e^{-\int_{T-1}^T b_{\mathbf{G}_{T-1}}(s)ds}\middle|\mathcal{F}^g_{(T-1)-}\right]\right] \\
= \ & \sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[-e^{-\gamma_1 e^r X_{T-1}-\gamma_2\int_t^{T-1}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}E\left[e^{-\int_{T-1}^T b_{\mathbf{G}_{T-1}}(s)ds}\middle|\mathbf{G}_{T-2}=\mathbf{g}\right]\right] \\
= \ & E\left[e^{-\int_{T-1}^T b_{\mathbf{G}_s}(s)ds}\middle|\mathbf{G}_{T-2}=\mathbf{g}\right]\sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[-e^{-\gamma_1 e^r X_{T-1}-\gamma_2\int_t^{T-1}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}\right]. \qquad (3.43)
\end{aligned}$$

Applying Lemma 3.7 again with $k = T-1$, we obtain

$$\sup_{\boldsymbol{\pi}\in\mathcal{A}^g} E^{t,x,\mathbf{g}}\left[-e^{-\gamma_1 e^r X_{T-1}-\gamma_2\int_t^{T-1}(\boldsymbol{\pi}_s^\intercal\mathbf{g})e^{r(T-s)}X_s ds}\right] = -e^{-\gamma_1 e^{r(T-t)}x-\int_t^{T-1}b_{\mathbf{g}}(s)ds}, \qquad (3.44)$$

where $b_{\mathbf{g}}(\cdot)$ is given by (3.42). From (3.43), (3.44), and that $\mathbf{G}_t = \mathbf{G}_{T-2}$ for all $T-2 \le t < T-1$, we obtain (3.40). Moreover, the optimal portfolio strategy is given by (3.41). This completes the proof for $t \in [T-2, T-1)$.

Lastly, we complete the proof for all $t \in [0,T]$ by repeating the above procedure. ■

Remark 3.3 *(ESG portfolio) For a green investor, her optimal portfolio* (3.41) *includes, in addition to the riskless asset and the local mean-variance portfolio (see Remark 3.1), an ESG portfolio with weights proportional to* $\boldsymbol{\Sigma}_t^{-1}\mathbf{G}_t$. *This portfolio provides an additional weight toward higher ESG-rated assets, reflecting the green investor's preference for such assets.*

3.3 Optimization for mixed investors

In terms of information considered, the case of mixed investors is similar to that of green investors, and the corresponding filtration is $\mathbb{F}^m = \mathbb{F}^{\mathbf{S}} \vee \mathbb{F}^{\mathbf{G}} = \mathbb{F}^g$. We consider the value function

$$V^m(t,X_t,\mathbf{G}_t) = \operatorname*{ess\,sup}_{\boldsymbol{\pi}\in\mathcal{A}^m} E\left[-e^{-\gamma_1 X_T}\middle|\mathcal{F}_t^m\right]. \qquad (3.45)$$

The optimization problem can be solved by going along the same lines as the proof of Proposition 3.2 with γ_2 set to 0, leading to the following:

Proposition 3.3 *The value function defined by (3.45) for* $(t,x,\mathbf{g}) \in [0,T) \times \mathbb{R} \times \mathfrak{g}$ *is given by*

$$V^m(t,x,\mathbf{g}) = -e^{-\gamma_1 e^{r(T-t)}x} E\left[e^{-\int_t^T \frac{1}{2}\boldsymbol{\theta}_s^\intercal \boldsymbol{\Sigma}_s^{-1}\boldsymbol{\theta}_s ds} \middle| \mathbf{G}_t = \mathbf{g}\right]. \tag{3.46}$$

Moreover, the optimal portfolio strategy is given by

$$\boldsymbol{\pi}_t^m = \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \boldsymbol{\theta}_t, \qquad 0 \le t < T. \tag{3.47}$$

For a mixed investor, her optimal portfolio consists of the riskless asset and the local mean-variance portfolio, making her portfolio choice resemble that of a classical Markowitz framework. Moreover, the three types of investors perceive the mean return rates of the risky assets differently, as explained below.

Remark 3.4 *(Mean return rates) Rewrite respectively the optimal portfolios (3.26), (3.41), and (3.47) of the three types of investors as*

$$\begin{aligned}
\boldsymbol{\pi}_t^b &= \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} (\boldsymbol{\mu}_t^b - r\mathbf{1}), \\
\boldsymbol{\pi}_t^g &= \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} (\boldsymbol{\mu}_t^g - r\mathbf{1}), \\
\boldsymbol{\pi}_t^m &= \frac{1}{\gamma_1 X_t} e^{-r(T-t)} \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{\mu}_t - r\mathbf{1}\right),
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{\mu}_t^b &= \bar{\boldsymbol{\mu}}_t - \sum_{\mathbf{g}_2 \in \hat{\mathfrak{g}}_2} \left(\boldsymbol{\mu}(t, \mathbf{G}_t^1, \mathbf{g}_2) - \bar{\boldsymbol{\mu}}_t\right) \beta_t^{\mathbf{g}_2} \frac{\partial f_k(t, \mathbf{G}_t^1, \boldsymbol{\beta}_t)}{\partial \beta^{\mathbf{g}_2}}, \\
\boldsymbol{\mu}_t^g &= \boldsymbol{\mu}_t + \frac{\gamma_2}{\gamma_1} \mathbf{G}_t.
\end{aligned}$$

For a brown investor, the vector of mean return rates $\boldsymbol{\mu}_t^b$ *is adjusted by a term reflecting the hedging demand explained in Remark 3.2. For a green investor, the vector of mean return rates* $\boldsymbol{\mu}_t^g$ *is ESG-adjusted. In particular, the ESG-adjusted mean return rate of an asset increases when its*

ESG rating improves and decreases when its ESG rating worsens. Moreover, the amount of this adjustment depends on the green investor's relative risk aversion, measured by γ_2/γ_1.

3.4 Utility indifference values

In this section, we study the equivalent monetary value of the hidden ESG information for a brown investor, expressed in terms of its utility indifference value, following Amendinger et al. [1]. Suppose the wealth of the brown investor at time 0 is a constant x, the utility indifference value of the hidden ESG information is defined as the lump sum amount that she is willing to pay to acquire the information so that her expected utility remains unchanged. This value can also be interpreted as the maximum amount that the brown investor is willing to pay for the hidden ESG information.

Definition 3.1 *The utility indifference value of the hidden ESG information, denoted by* $v^{\mathbf{G}^2}$*, is defined as the solution of the equation*

$$u^b(x) = u^m(x - v^{\mathbf{G}^2}),$$

where $u^b(x) = E[V^b(0, x, \mathbf{G}_0^1, \boldsymbol{\beta}_0)]$ *and* $u^m(x) = E[V^m(0, x, \mathbf{G}_0)]$.

For any $\mathbf{g} \in \mathfrak{g}_2$, the filtered probability

$$\beta_0^{\mathbf{g}} = P(\mathbf{G}_0^2 = \mathbf{g} | \sigma(\mathbf{S}_0)) = P(\mathbf{G}_0^2 = \mathbf{g})$$

is nonrandom. By (3.25) and (3.46), we have

$$\begin{aligned} u^b(x) &= -e^{-\gamma_1 e^{rT} x} E\left[e^{-f_1(t, \mathbf{G}_0^1, \boldsymbol{\beta}_0)}\right], \\ u^m(x) &= -e^{-\gamma_1 e^{rT} x} E\left[e^{-\int_0^T \frac{1}{2} \boldsymbol{\theta}_s^\intercal \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\theta}_s ds}\right]. \end{aligned}$$

Therefore,

$$v^{\mathbf{G}^2} = \frac{1}{\gamma_1} e^{-rT} \ln \left(\frac{E\left[e^{-f_1(t, \mathbf{G}_0^1, \boldsymbol{\beta}_0)}\right]}{E\left[e^{-\int_0^T \frac{1}{2} \boldsymbol{\theta}_s^\intercal \boldsymbol{\Sigma}_s^{-1} \boldsymbol{\theta}_s ds}\right]} \right).$$

It follows from Proposition 3.1 and Definition 2.1 that $\boldsymbol{\pi}^b \in \mathcal{A}^b \subseteq \mathcal{A}^m$. By virtue of the definitions of the value functions V^b and V^m, it is trivial that $u^b(x) \leq u^m(x)$. Thus, the utility indifference value $v^{\mathbf{G}^2} \geq 0$. The implication is that the hidden ESG information is valuable in that a brown investor can increase her maximized expected utility when the information can be accessed without any cost.

Apart from the utility indifference value of the hidden ESG information, it is also interesting to study the utility indifference value of the ESG-related non-pecuniary benefits for a green investor defined below. In (3.48), v^g can be interpreted as the monetary value that the green investor is willing to accept to disregard her ESG preferences in utility maximization.

Definition 3.2 *The utility indifference value of the non-pecuniary benefits that a green investor derives from her ESG preferences, denoted by v^g, is defined as the solution of the equation*

$$u^g(x) = u^m(x + v^g). \tag{3.48}$$

where $u^g(x) = E[V^g(0, x, \mathbf{G}_0)]$ and $u^m(x) = E[V^m(0, x, \mathbf{G}_0)]$.

Recall the value functions (3.40) and (3.46). By Definition 3.2, we have

$$-e^{-\gamma_1 e^{rT} x} E\left[e^{-\int_0^T \frac{1}{2}\left(\boldsymbol{\theta}_s + \frac{\gamma_2}{\gamma_1}\mathbf{G}_s\right)^{\intercal} \boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\theta}_s + \frac{\gamma_2}{\gamma_1}\mathbf{G}_s\right) ds}\right] = -e^{-\gamma_1 e^{rT}(x + v^g)} E\left[e^{-\int_0^T \frac{1}{2}\boldsymbol{\theta}_s^{\intercal}\boldsymbol{\Sigma}_s^{-1}\boldsymbol{\theta}_s ds}\right],$$

which leads to

$$v^g = -\frac{1}{\gamma_1} e^{-rT} \ln\left(\frac{E\left[e^{-\int_0^T \frac{1}{2}\left(\boldsymbol{\theta}_s + \frac{\gamma_2}{\gamma_1}\mathbf{G}_s\right)^{\intercal} \boldsymbol{\Sigma}_s^{-1}\left(\boldsymbol{\theta}_s + \frac{\gamma_2}{\gamma_1}\mathbf{G}_s\right) ds}\right]}{E\left[e^{-\int_0^T \frac{1}{2}\boldsymbol{\theta}_s^{\intercal}\boldsymbol{\Sigma}_s^{-1}\boldsymbol{\theta}_s ds}\right]}\right).$$

It is important to note that the utility indifference value v^g is not necessarily positive. For instance, if the ESG ratings of the vast majority of risky assets take negative values, then it is likely that v^g is negative.

4 Numerical study

In this section, we conduct a numerical study to illustrate the theoretical results presented earlier. We focus on the portfolio choices of a brown investor, a green investor, and a mixed investor, and examine the impact of ESG information on their optimal portfolio strategies.

We assume that there are $n = 2$ risky assets. As specified by (2.1), let the dynamics of the asset price processes be given by

$$\frac{dS_{1,t}}{S_{1,t}} = \mu_1(G_{1,t})dt + \sigma_1 dW_{1,t}, \tag{4.1}$$

$$\frac{dS_{2,t}}{S_{2,t}} = \mu_2(G_{2,t})dt + \sigma_2 \left(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}\right), \tag{4.2}$$

where $(W_{1,t}, W_{2,t})^\intercal, 0 \leq t \leq T$, is a 2-dimensional standard Brownian motion. The parameters in (4.1) and (4.2) are calibrated using the stock prices and MSCI ESG ratings of the Coca-Cola Company (Coca-Cola) and Uber Technologies, Inc. (Uber) from October 2020 to November 2024. During this time period, the rating of Coca-Cola was either AAA or AA, and the rating of Uber was either A or BB. We assign numeric values $1, 0.5, -0.5$, and -1 to the ratings AAA, AA, A, and BB, respectively. The state space of the ESG rating process for the two assets is therefore given by

$$\mathfrak{g} = \{\mathbf{g}_1 = (1, -0.5), \mathbf{g}_2 = (1, -1), \mathbf{g}_3 = (0.5, -0.5), \mathbf{g}_4 = (0.5, -1)\},$$

and the transition probabilities are assumed to be

$$p_{(i_1,i_2),(j_1,j_2)} = q_{i_1,j_1}q_{i_2,j_2}, \qquad (i_1, i_2), (j_1, j_2) \in \mathfrak{g},$$

where $q_{i,j} = 0.9$ if $i = j$ and 0.1 if $i \neq j$. Using the maximum likelihood method, we obtain the parameter values as $\sigma_1 = 0.16, \sigma_2 = 0.48$, and $\rho = 0.09$. Moreover, for the drift coefficients, we have $\mu_1(1) = 0.12, \mu_1(0.5) = 0.08, \mu_2(-0.5) = 0.22$, and $\mu_2(-1) = 0.27$. For illustrative purposes, we assume that the ratings of these assets are not publicly available, i.e., $n_1 = 0$.

The other model parameters are set as follows: $T = 5$ for the investment horizon, $r = 0.03$ for the risk-free interest rate, and $X_0 = 10$ for the initial wealth. The risk aversion coefficient associated with the terminal wealth is assumed to be $\gamma_1 = 0.5$,[3] and we will compare two values for the risk aversion coefficient associated with the ESG performance: $\gamma_2 = 0.005$ and $\gamma_2 = 0.01$. These parameters and the transition probabilities of the ESG rating process are not calibrated.

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[3] We choose $\gamma_1 < 1$ to ensure that the resulting portfolio weights remain within a reasonable range. Large values of γ_1 lead to overly conservative positions, which result in impractically small allocations to the risky assets. The same consideration applies to our choices of γ_2.

In the subsequent study, all (conditional) expectations involved are computed based on a simulation of 20,000 sample paths of either the process $\{\mathbf{G}_t\}_{t_0 \le t \le T}$ or the process $\{\boldsymbol{\beta}_t\}_{t_0 \le t \le T}$ for some $t_0 \le T$, where $\mathbf{G}_t = (G_{1,t}, G_{2,t})$ and $\boldsymbol{\beta}_t = \left(\beta_t^{\mathbf{g}_1}, \beta_t^{\mathbf{g}_2}, \beta_t^{\mathbf{g}_3}, \beta_t^{\mathbf{g}_4}\right)^\intercal$. Moreover, we consider the five cases below which represent different ESG scenarios:

C1. $P(\mathbf{G}_0 = \mathbf{g}_1) = 0.7$, $P(\mathbf{G}_0 = \mathbf{g}_2) = 0.1$, $P(\mathbf{G}_0 = \mathbf{g}_3) = 0.1$, and $P(\mathbf{G}_0 = \mathbf{g}_4) = 0.1$, or equivalently, $\boldsymbol{\beta}_0 = (0.7, 0.1, 0.1, 0.1)^\intercal$;

C2. $P(\mathbf{G}_0 = \mathbf{g}_1) = 0.1$, $P(\mathbf{G}_0 = \mathbf{g}_2) = 0.7$, $P(\mathbf{G}_0 = \mathbf{g}_3) = 0.1$, and $P(\mathbf{G}_0 = \mathbf{g}_4) = 0.1$, or equivalently, $\boldsymbol{\beta}_0 = (0.1, 0.7, 0.1, 0.1)^\intercal$;

C3. $P(\mathbf{G}_0 = \mathbf{g}_1) = 0.1$, $P(\mathbf{G}_0 = \mathbf{g}_2) = 0.1$, $P(\mathbf{G}_0 = \mathbf{g}_3) = 0.7$, and $P(\mathbf{G}_0 = \mathbf{g}_4) = 0.1$, or equivalently, $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.7, 0.1)^\intercal$;

C4. $P(\mathbf{G}_0 = \mathbf{g}_1) = 0.1$, $P(\mathbf{G}_0 = \mathbf{g}_2) = 0.1$, $P(\mathbf{G}_0 = \mathbf{g}_3) = 0.1$, and $P(\mathbf{G}_0 = \mathbf{g}_4) = 0.7$, or equivalently, $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.1, 0.7)^\intercal$;

C5. $P(\mathbf{G}_0 = \mathbf{g}_1) = 0.25$, $P(\mathbf{G}_0 = \mathbf{g}_2) = 0.25$, $P(\mathbf{G}_0 = \mathbf{g}_3) = 0.25$, and $P(\mathbf{G}_0 = \mathbf{g}_4) = 0.25$, or equivalently, $\boldsymbol{\beta}_0 = (0.25, 0.25, 0.25, 0.25)^\intercal$.

In the first four cases, each $\mathbf{C}i$ assigns more weight to the ESG ratings $\mathbf{g}_i$, while the last case gives equal weights to all possible ratings.

Table 1 presents the local mean-variance (MV) portfolio, the ESG portfolio, and their combined value (which corresponds to either the optimal portfolio $\boldsymbol{\pi}_0^m$ or $\boldsymbol{\pi}_0^g$) for the mixed investor and the green investor at time 0, respectively. We consider different ESG scenarios. For both investors, the ESG ratings are observable and their optimal portfolios are based on the observed ratings.

We note clearly that the ESG ratings affect both the local MV portfolio and the ESG portfolio. In the ESG portfolio, the weight of an asset is positive (respectively, negative) when its ESG rating takes a positive (respectively, negative) value. As a result, the optimal portfolio $\boldsymbol{\pi}_0^g$ of the green investor is tilted towards asset 1 with a better ESG rating and tilted away from asset 2 with a worse ESG rating, compared to $\boldsymbol{\pi}_0^m$ of the mixed investor. Moreover, when the green investor has a stronger ESG preference, indicated by a larger γ_2, she allocates more (respectively, less) wealth to asset 1 (respectively, asset 2).

Panel A: $G_{1,0} = 1, G_{2,0} = -0.5$				
		Local MV	ESG	Combined
Mixed investor	Asset 1	0.5715	n/a	0.5715
	Asset 2	0.1248	n/a	0.1248
Green investor	Asset 1	0.5715	0.0688	0.6403
$\gamma_2 = 0.005$	Asset 2	0.1248	−0.0058	0.1190
Green investor	Asset 1	0.5715	0.1376	0.7091
$\gamma_2 = 0.01$	Asset 2	0.1248	−0.0116	0.1132

Panel B: $G_{1,0} = 1$ and $G_{2,0} = -1$				
		Local MV	ESG	Combined
Mixed investor	Asset 1	0.5613	n/a	0.5613
	Asset 2	0.1625	n/a	0.1625
Green investor	Asset 1	0.5613	0.0698	0.6311
$\gamma_2 = 0.005$	Asset 2	0.1625	−0.0096	0.1529
Green investor	Asset 1	0.5613	0.1397	0.7010
$\gamma_2 = 0.01$	Asset 2	0.1625	−0.0191	0.1434

Panel C: $G_{1,0} = 0.5$ and $G_{2,0} = -0.5$				
		Local MV	ESG	Combined
Mixed investor	Asset 1	0.3003	n/a	0.3003
	Asset 2	0.1329	n/a	0.1329
Green investor	Asset 1	0.3003	0.0349	0.3352
$\gamma_2 = 0.005$	Asset 2	0.1329	−0.0048	0.1281
Green investor	Asset 1	0.3003	0.0698	0.3701
$\gamma_2 = 0.01$	Asset 2	0.1329	−0.0096	0.1233

Panel D: $G_{1,0} = 0.5$ and $G_{2,0} = -1$				
		Local MV	ESG	Combined
Mixed investor	Asset 1	0.2901	n/a	0.2901
	Asset 2	0.1706	n/a	0.1706
Green investor	Asset 1	0.2901	0.0359	0.3260
$\gamma_2 = 0.005$	Asset 2	0.1706	−0.0085	0.1621
Green investor	Asset 1	0.2901	0.0719	0.3620
$\gamma_2 = 0.01$	Asset 2	0.1706	−0.0171	0.1535

Table 1. Optimal portfolios for the mixed and green investors. This table reports the local MV portfolio, the ESG portfolio, and the optimal portfolios $\boldsymbol{\pi}_0^m$ and $\boldsymbol{\pi}_0^g$ (labeled as Combined) at time 0 for various values of the ESG ratings $G_{1,0}$ and $G_{2,0}$.

Case **C1**: $\boldsymbol{\beta}_0 = (0.7, 0.1, 0.1, 0.1)^\intercal$				
		Local MV	Hedging	Combined
Brown investor	Asset 1	0.5152	−0.0151	0.5001
	Asset 2	0.1340	0.0005	0.1345
Case **C2**: $\boldsymbol{\beta}_0 = (0.1, 0.7, 0.1, 0.1)^\intercal$				
		Local MV	Hedging	Combined
Brown investor	Asset 1	0.5091	−0.0199	0.4892
	Asset 2	0.1566	−0.0013	0.1553
Case **C3**: $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.7, 0.1)^\intercal$				
		Local MV	Hedging	Combined
Brown investor	Asset 1	0.3525	−0.0126	0.3399
	Asset 2	0.1389	−0.0009	0.1380
Case **C4**: $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.1, 0.7)^\intercal$				
		Local MV	Hedging	Combined
Brown investor	Asset 1	0.3464	−0.0088	0.3376
	Asset 2	0.1615	−0.0001	0.1614
Case **C5**: $\boldsymbol{\beta}_0 = (0.25, 0.25, 0.25, 0.25)^\intercal$				
		Local MV	Hedging	Combined
Brown investor	Asset 1	0.4308	−0.0211	0.4097
	Asset 2	0.1477	−0.0007	0.1470

Table 2. Optimal portfolios for the brown investor. This table reports the local MV portfolio, the hedging portfolio, and the optimal portfolio $\boldsymbol{\pi}_0^b$ (labeled as Combined) at time 0 for various values of the filtered probabilities $\boldsymbol{\beta}_0$.

Table 2 presents the local MV portfolio, the hedging portfolio, and their combined value as the optimal portfolio $\boldsymbol{\pi}_0^b$ for the brown investor at time 0. For the brown investor, the ESG ratings are unobservable. Her optimal portfolio at time 0 is based on the probability distribution of the ratings at time 0, which is characterized by the filtered probabilities $\boldsymbol{\beta}_0$. We observe that the weights in the hedging portfolio are generally negative, and their magnitudes are much smaller than the corresponding ones in the local MV portfolio. Moreover, the brown investor allocates more wealth to asset 1 in cases **C1** and **C3**, compared to the other cases. This is because the filtered probability of the event $\{G_{1,0} = 1\}$ is 0.8 in cases **C1** and **C3**, and asset 1 has a higher mean return rate when its rating is 1, i.e., $\mu_1(1) > \mu_1(0.5)$. Likewise, she allocates more wealth to asset 2 in cases **C2** and **C4**, compared to the other cases.

In Figure 2, we plot the optimal portfolio strategies of different investors over time. We consider the evolution of the ESG ratings as follows: $\mathbf{G}_0 = \mathbf{G}_1 = \mathbf{G}_2 = (0.5, -0.5), \mathbf{G}_3 = (0.5, -1)$, and $\mathbf{G}_4 = (1, -1)$. In particular, the ESG rating of asset 2 is downgraded at time $t = 3$, while the rating of asset 1 is upgraded at time $t = 4$.

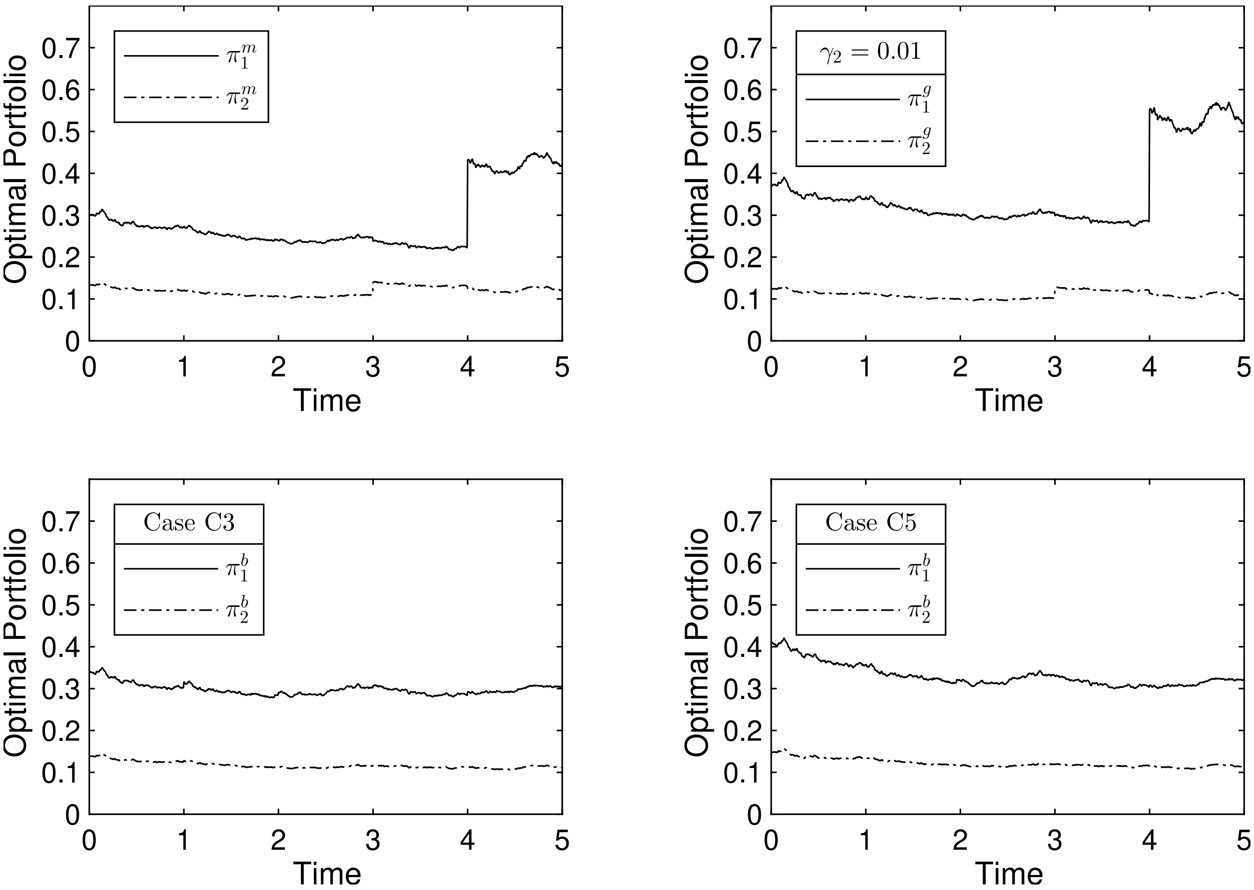

Figure 2. Optimal portfolio strategies over time for different investors: up-left for the mixed investor; up-right for the green investor, down for the brown investor with two cases **C3** and **C5**.

Despite the downgrade of asset 2 at time 3, both the mixed and green investors slightly increase their holdings of asset 2 due to its higher mean return rate, i.e., $\mu_2(-1) > \mu_2(-0.5)$. The adjustment amount is lower for the green investor, as the ESG rating of asset 2 is negative, resulting in a negative weight for asset 2 in the ESG portfolio. When asset 1 is upgraded at time 4, both investors significantly increase their holdings of asset 1 due to its higher mean return rate, i.e., $\mu_1(1) > \mu_1(0.5)$. Moreover, the adjustment amount is higher for the green investor, as the ESG rating of asset 1 is positive, resulting in a positive weight for asset 1 in the ESG portfolio. In contrast, the brown investor does not make any significant adjustments to her optimal portfolio at time 3 and time 4.

Table 3 presents the maximal expected utility values V_0^g, V_0^m, and V_0^b—defined by (2.8), (2.9), and (2.10) for a given initial wealth $X_0 = 10$ (also denoted as $u^g(X_0), u^m(X_0)$, and $u^b(X_0)$ in Section 3.4)—for the three investors at time 0 under various ESG scenarios. We observe that, in all cases **C1**–**C5**, the maximal expected utility of the green investor is greater than that of the mixed investor, which in turn is greater than that of the brown investor, i.e., $u^g(X_0) > u^m(X_0) > u^b(X_0)$. When the risk aversion coefficient associated with the ESG performance γ_2 is larger, the green investor achieves greater maximal expected utility.

			$\gamma_2 = 0.005$	$\gamma_2 = 0.01$
	$u^b(X_0)$	$u^m(X_0)$	$u^g(X_0)$	$u^g(X_0)$
Case **C1**: $\boldsymbol{\beta}_0 = (0.7, 0.1, 0.1, 0.1)^\intercal$	-0.00117	-0.00112	-0.00101	-0.00091
Case **C2**: $\boldsymbol{\beta}_0 = (0.1, 0.7, 0.1, 0.1)^\intercal$	-0.00108	-0.00103	-0.00095	-0.00086
Case **C3**: $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.7, 0.1)^\intercal$	-0.00142	-0.00138	-0.00131	-0.00124
Case **C4**: $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.1, 0.7)^\intercal$	-0.00130	-0.00126	-0.00121	-0.00116
Case **C5**: $\boldsymbol{\beta}_0 = (0.25, 0.25, 0.25, 0.25)^\intercal$	-0.00125	-0.00120	-0.00112	-0.00104

Table 3. The maximal expected utility values of different investors at time 0 under various ESG scenarios.

Finally, Table 4 presents the utility indifference value of the hidden ESG information $v^{\mathbf{G}^2}$ for the brown investor and the utility indifference value of the ESG-related non-pecuniary benefits v^g for the green investor in cases **C1**–**C5**. As explained in Subsection 3.4, $v^{\mathbf{G}^2}$ is always nonnegative, implying that the hidden ESG information is valuable. We observe that $v^{\mathbf{G}^2}$ takes the largest value in case **C5** when there is the most uncertainty about the actual ESG ratings of the assets. Moreover, v^g increases when the green investor has a stronger ESG preference, indicated by a larger γ_2. For a given γ_2, v^g is the greatest in case **C1** with $P(G_{1,0} = 1, G_{2,0} = -0.5) = 0.7$, corresponding to the most favorable ESG scenario, and the smallest in case **C5** with $P(G_{1,0} = 0.5, G_{2,0} = -1) = 0.7$, corresponding to the worst ESG scenario.

5 Concluding remarks

We consider portfolio optimization problems for three types of investors—*brown*, *green*, and *mixed* investors who have different ESG preferences and ESG information accessibility. We obtain optimal

	$v^{\mathbf{G}^2}$	$\gamma_2 = 0.005$ v^g	$\gamma_2 = 0.01$ v^g
Case **C1**: $\boldsymbol{\beta}_0 = (0.7, 0.1, 0.1, 0.1)^\intercal$	0.0704	0.1709	0.3655
Case **C2**: $\boldsymbol{\beta}_0 = (0.1, 0.7, 0.1, 0.1)^\intercal$	0.0728	0.1509	0.3229
Case **C3**: $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.7, 0.1)^\intercal$	0.0556	0.0830	0.1794
Case **C4**: $\boldsymbol{\beta}_0 = (0.1, 0.1, 0.1, 0.7)^\intercal$	0.0572	0.0613	0.1388
Case **C5**: $\boldsymbol{\beta}_0 = (0.25, 0.25, 0.25, 0.25)^\intercal$	0.0757	0.1120	0.2408

Table 4. The utility indifference value of the hidden ESG information $v^{\mathbf{G}^2}$ and the utility indifference value of the ESG-related non-pecuniary benefits v^g under various ESG scenarios.

investment strategies for these investors and examine the impact of ESG information on their portfolio choices. Our results offer insights into the different components of the optimal portfolios, including the local mean-variance component for all investors, the hedging component for brown investors, and the ESG component for green investors. We show that the utility indifference value of the hidden ESG information is always nonnegative. In all ESG scenarios considered in our numerical study, the maximal expected utility is highest for the green investor and smallest for the brown investor. We also analyze how the transition of ESG ratings can influence the portfolio choices of different types of investors.

Despite the rapid growth of ESG investing, investors often encounter a significant amount of uncertainty about the true corporate ESG profile; see Avramov et al. [2]. For example, there is a substantial ESG rating disagreement between rating agencies; see Christensen et al. [6]. Naturally, such uncertainty can affect investment decisions. In this regard, it will be interesting to extend the current study to take ESG uncertainty into account.

Moreover, our work has been carried out under the assumption that the ESG rating process is driven by a discrete-time Markov chain with deterministic transition probabilities. In reality, the transition of ESG ratings is a complex process influenced by various risk factors. Thus, it is worthwhile to extend the current study so that the transition of ESG ratings is determined by stochastic transition rates that may be correlated with asset prices.

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7 Disclosure of interest

We declare that there are no relevant financial or non-financial competing interests to report.

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