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What happens after a default: The conditional density approach[☆]

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Abstract

We present a general model for default times, making precise the role of the intensity process, and showing that this process allows for a knowledge of the conditional distribution of the default only "before the default". This lack of information is crucial while working in a multi-default setting. In a single default case, the knowledge of the intensity process does not allow us to compute the price of defaultable claims, except in the case where the immersion property is satisfied. We propose in this paper a density approach for default times. The density process will give a full characterization of the links between the default time and the reference filtration, in particular "after the default time". We also investigate the description of martingales in the full filtration in terms of martingales in the reference filtration, and the impact of Girsanov transformation on the density and intensity processes, and on the immersion property.

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1. Introduction

Modelling default time for a single credit event has been widely studied in the literature, the main approaches being the structural, the reduced form and the intensity ones. In this context, most works are concentrated (for pricing purpose) on the computation of conditional expectations of payoffs, given that the default has not occurred, and assuming some conditional independence between the default time and the reference filtration. In this paper, we are interested in what happens after a default occurs: we find it important to investigate the impact of a default event on the rest of the market and what goes on afterwards. The first motivation of this work was the multi-default setting.

Furthermore, in a market with multiple defaults, it will be important to compute the prices of a portfolio derivative on the disjoint sets before the first default, after the first and before the second one and so on. Our work will allow us to use a recurrence procedure to provide these computations, which will be presented in a companion paper [7].

We start with the knowledge of the conditional distribution of the default time τ, with respect to a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and we assume that this conditional distribution admits a density (see the next section for a precise definition). We firstly reformulate the classical computation result of conditional expectations with respect to the observation σ-algebra $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau \wedge t)$ before the default time τ, i.e., on the set $\{t < \tau\}$. The main purpose is then to deduce what happens after the default occurs, i.e., on the set $\{\tau \leq t\}$. We present computation results of $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ conditional expectations on the set $\{\tau \leq t\}$ by using the conditional density of τ and point out that the whole term structure of the density is needed. By establishing an explicit link between (part of) density and intensity, which corresponds to the additive and multiplicative decomposition related to the survival process (Azéma supermartingale of τ), we make clear that the intensity can be deduced from the density, but that the reverse does not hold, except when certain strong assumption, as the H-hypothesis, holds. These results show that the density approach is suitable in this after-default study and explain why the intensity approach is inadequate for this case.

Note that, even though the "density" point of view is inspired by the enlargement of filtration theory, we shall not use classical results on progressive enlargements of filtration. In fact, we take the opposite point of view: we are interested in $\mathbb{G}$-martingales and their characterization in terms of $\mathbb{F}$-(local) martingales. Moreover, these characterization results allow us to give a proof of a decomposition of $\mathbb{F}$-(local) martingales in terms of $\mathbb{G}$-semimartingales.

We show that changes of probability are obtained from the knowledge of a family of positive $\mathbb{F}$-martingales $(\beta_t(\theta), t \geq \theta)$ by using a normalization procedure and we study how the parameters of the default (i.e., the survival process, the intensity, the density) are modified by a change of probability in a general setting (we do not assume that we are working in a Brownian filtration, except for some examples), and we characterize changes of probability that do not affect the intensity process. We pay attention to the specific case where the dynamics of underlying default-free processes are changed only after the default, which shows the impact of the default event.

The paper is organized as follows. We first introduce in Section 2 the different types of information we are dealing with and the key hypothesis of density. In Section 3, we establish results on computation of conditional expectations, on the "before-default" and "after-default" sets. The immersion property is then discussed. The dynamic properties of the density process are presented in Section 4 where we make precise the links between this density process and the intensity process. We present the characterization of $\mathbb{G}$-martingales in terms of $\mathbb{F}$-local martingales in Section 5 and we give a Girsanov type property and discuss the stability of the immersion property and the invariance of intensity in Section 6. Finally, we make some conclusion remarks in the last section.

2. The different sources of information

In this section, we specify the link between the two filtrations $\mathbb{F}$ and $\mathbb{G}$, and make some hypotheses on the default time.

A strictly positive and finite random variable τ (the default time) is given on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that the law of τ admits a density with respect to η, a non-negative non-atomic σ-finite measure on $\mathbb{R}^+$. The space Ω is supposed to be endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions, that is, the filtration $\mathbb{F}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{A}$. Before the default time τ, i.e., on the set $\{t < \tau\}$, the σ-algebra $\mathcal{F}_t$ represents the information accessible to the investors at time t. When the default occurs, the investors will add this new information (i.e., the knowledge of τ) to the σ-algebra $\mathcal{F}_t$ in a progressive enlargement setting.

One of our goals is to show how the knowledge of the conditional law of τ with respect to $\mathcal{F}_t$ allows to compute the conditional expectations in the new filtration. We assume that, for any $t \geq 0$, the conditional distribution of τ with respect to $\mathcal{F}_t$ is smooth, i.e., that the $\mathcal{F}_t$-conditional distribution of τ admits a density with respect to η.

In other words, we introduce the following hypothesis, that we call density hypothesis. This hypothesis will be in force in this paper. We begin by the static case where the conditional probabilities are considered at a fixed time; in the next section, we shall discuss the dynamic version for stochastic processes.

Hypothesis 2.1 (*Density Hypothesis*)**.** We assume that η is a non-negative non-atomic measure on $\mathbb{R}^+$ and that for any time $t \geq 0$, there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to \alpha_t(\omega, \theta)$ such that for any (bounded) Borel function f,

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_0^\infty f(u)\alpha_t(u)\eta(\mathrm{d}u), \quad a.s. \tag{1}$$

The family $\alpha_t(.)$ is called the *conditional density* of τ with respect to η given $\mathcal{F}_t$ (in short the *density* of τ if no ambiguity). Then, the distribution of τ is given by $\mathbb{P}(\tau > \theta) = \int_\theta^\infty \alpha_0(u)\eta(\mathrm{d}u)$. Note that, from (1), for $f = 1$, for any t, $\int_0^\infty \alpha_t(\theta)\eta(\mathrm{d}\theta) = 1,\ a.s.$ The conditional distribution of τ is characterized by

$$S_t(\theta) := \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)\eta(\mathrm{d}u), \quad \mathbb{P} - a.s. \tag{2}$$

The family of random variables

$$S_t := S_t(t) = \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t(u)\eta(\mathrm{d}u) \tag{3}$$

plays a key role in what follows. Observe that one has

$$\{\tau > t\} \subset \{S_t > 0\} =: A_t \quad \mathbb{P} - a.s. \tag{4}$$

since $\mathbb{P}(A_t^c \cap \{\tau > t\}) = 0$. Note also that $S_t(\theta) = \mathbb{E}[S_\theta|\mathcal{F}_t]$ for any $\theta \geq t$.

More generally, if an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(\omega, \theta) \to Y_t(\omega, \theta)$ is given, the $\mathcal{F}_t$-conditional expectation of the r.v. $Y_t(\tau) := Y_t(\omega, \tau(\omega))$, assumed to be integrable, is given by

$$\mathbb{E}[Y_t(\tau)|\mathcal{F}_t] = \int_0^\infty Y_t(u)\alpha_t(u)\eta(\mathrm{d}u). \tag{5}$$

Notation: In what follows, we shall simply say that $Y_t(\theta)$ is an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-random variable and even that $Y_t(\tau)$ is an $\mathcal{F}_t \otimes \sigma(\tau)$-random variable as a short cut for $Y_t(\theta)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable.

Note that if τ is an $\mathbb{F}$-stopping time, then it may admit a density for t smaller than θ. However, since $S_t(\theta) = \mathbb{1}_{\{\tau>\theta\}}$ for $t \geq \theta$, it never admits a density on this set.

Corollary 2.2. *The default time τ avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \xi) = 0$ for every $\mathbb{F}$-stopping time ξ.*

Proof. Let ξ be an $\mathbb{F}$-stopping time bounded by a constant T. Then, the random variable $H_\xi(t) = \mathbb{1}_{\{\xi=t\}}$ is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, and, η being non-atomic

$$\mathbb{E}[H_\xi(\tau)|\mathcal{F}_t] = \mathbb{E}[\,\mathbb{E}[H_\xi(\tau)|\mathcal{F}_T]\,|\mathcal{F}_t] = \mathbb{E}\left[\int_0^\infty H_\xi(u)\alpha_T(u)\eta(\mathrm{d}u)|\mathcal{F}_t\right] = 0.$$

Hence, $\mathbb{E}[H_\xi(\tau)] = \mathbb{P}(\xi = \tau) = 0$. $\square$

Remark 2.3. By using the density approach, we adopt an additive point of view to represent the conditional probability of τ: the conditional survival function $S_t(\theta) = \mathbb{P}(\tau > \theta \mid \mathcal{F}_t)$ is written in the form $S_t(\theta) = \int_\theta^\infty \alpha_t(u)\eta(\mathrm{d}u)$. In the default framework, the "intensity" point of view is often preferred, and one uses the multiplicative representation $S_t(\theta) = \exp(-\int_0^\theta \lambda_t(u)\eta(\mathrm{d}u))$. In the particular case where η denotes the Lebesgue measure, the family of $\mathcal{F}_t$-random variables $\lambda_t(\theta) = -\partial_\theta \ln S_t(\theta)$ is called the "forward intensity". We shall discuss and compare these two points of view further on.

3. Computation of conditional expectations in a default setting

The specific information related to the default time is the knowledge of this time when it occurs. It is defined in mathematical terms as follows: let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the smallest right-continuous filtration such that τ is a $\mathbb{D}$-stopping time; in other words, $\mathcal{D}_t$ is given by $\mathcal{D}_t = \mathcal{D}^0_{t+}$ where $\mathcal{D}^0_t = \sigma(\tau \wedge t)$ and represents the default information. By definition, any $\mathcal{D}_t$-r.v. can be written in the form $f(t)\mathbb{1}_{\{t<\tau\}} + f(\tau)\mathbb{1}_{\{\tau\leq t\}}$ where f is a Borel function. This filtration $\mathbb{D}$ will be "added" to the reference filtration to define the filtration $\mathbb{G} := \mathbb{F} \vee \mathbb{D}$, which is made right-continuous and complete. The filtration $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ and making τ a stopping time. Any $\mathcal{G}_t$-r.v. $H^{\mathbb{G}}_t$ may be represented as

$$H^{\mathbb{G}}_t = H^{\mathbb{F}}_t \mathbb{1}_{\{\tau>t\}} + H_t(\tau)\mathbb{1}_{\{\tau\leq t\}} \tag{6}$$

where $H^{\mathbb{F}}_t$ is an $\mathcal{F}_t$-r.v. and $H_t(\tau)$ is an $\mathcal{F}_t \otimes \sigma(\tau)$-random variable.[1] In particular,

$$H^{\mathbb{G}}_t \mathbb{1}_{\{\tau>t\}} = H^{\mathbb{F}}_t \mathbb{1}_{\{\tau>t\}} \quad a.s., \tag{7}$$

[1] This is a direct consequence of Lemma 4.4 in [13].

where the random variable $H_t^{\mathbb{F}}$ is the $\mathcal{F}_t$-conditional expectation of $H_t^{\mathbb{G}}$ given the event $\{\tau > t\}$, well defined on the set $A_t = \{S_t > 0\}$, hence on the set $\{\tau > t\}$, as

$$H_t^{\mathbb{F}} = \frac{\mathbb{E}[H_t^{\mathbb{G}} \mathbb{1}_{\{\tau>t\}} | \mathcal{F}_t]}{\mathbb{P}(\tau > t | \mathcal{F}_t)} = \frac{\mathbb{E}[H_t^{\mathbb{G}} \mathbb{1}_{\{\tau>t\}} | \mathcal{F}_t]}{S_t} \quad a.s. \text{ on } A_t; \quad H_t^{\mathbb{F}} = 0 \quad \text{otherwise}. \tag{8}$$

3.1. Conditional expectations

The definition of $\mathbb{G}$ allows us to compute conditional expectations with respect to $\mathcal{G}_t$ in terms of conditional expectations with respect to $\mathcal{F}_t$. This will be done in two steps, depending whether or not the default has occurred: as we explained above, before the default, the only information contained in $\mathcal{G}_t$ is $\mathcal{F}_t$; after the default, the information contained in $\mathcal{G}_t$ is, roughly speaking, $\mathcal{F}_t \vee \sigma(\tau)$.

The $\mathcal{G}_t$-conditional expectation of an integrable $\sigma(\tau)$-r.v. (of the form $f(\tau)$) is given by

$$\alpha_t^{\mathbb{G}}(f) := \mathbb{E}[f(\tau)|\mathcal{G}_t] = \alpha_t^{\mathrm{bd}}(f)\, \mathbb{1}_{\{\tau>t\}} + f(\tau)\, \mathbb{1}_{\{\tau \le t\}}$$

where α_t^{bd} is the value of the $\mathcal{G}_t$-conditional distribution **b**efore the **d**efault, given by

$$\alpha_t^{\mathrm{bd}}(f) := \frac{\mathbb{E}[f(\tau)\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} \quad a.s. \text{ on } \{\tau > t\}.$$

Recall the notation $S_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. On the set $\{\tau > t\}$, the "before-default" conditional distribution α_t^{bd} admits a density with respect to η, and is given by

$$\alpha_t^{\mathrm{bd}}(f) = \frac{1}{S_t} \int_t^\infty f(u)\alpha_t(u)\eta(\mathrm{d}u) \quad a.s.$$

The same calculation as in (5) can be performed in this framework and extended to the computation of $\mathcal{G}_t$-conditional expectations for a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-r.v.

Theorem 3.1. *Let $Y_T(\tau)$ be a bounded $\mathcal{F}_T \otimes \sigma(\tau)$-random variable. Then, for $t \le T$,*

$$E[Y_T(\tau)|\mathcal{G}_t] = Y_t^{\mathrm{bd}} \mathbb{1}_{\{t<\tau\}} + Y_t^{\mathrm{ad}}(T,\tau)\mathbb{1}_{\{\tau \le t\}} \quad \mathbb{P}-a.s. \tag{9}$$

where

$$Y_t^{\mathrm{bd}} = \frac{\mathbb{E}\left[\int_t^\infty Y_T(u)\alpha_T(u)\eta(\mathrm{d}u)|\mathcal{F}_t\right]}{S_t} \mathbb{1}_{\{S_t>0\}} \quad \mathbb{P}-a.s.$$

$$Y_t^{\mathrm{ad}}(T,\theta) = \frac{\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t\right]}{\alpha_t(\theta)} \mathbb{1}_{\{\alpha_t(\theta)>0\}} \quad \mathbb{P}-a.s. \tag{10}$$

Proof. The computation on the set $\{t < \tau\}$ (the pre-default case) is obtained following (7), (8) and using (5) and the fact that $T \ge t$.

For the after-default case, we note that, by (6), any $\mathcal{G}_t$-r.v. can be written on the set $\{\tau \le t\}$ as $H_t(\tau)\mathbb{1}_{\{\tau \le t\}}$. Assuming that the test r.v. $H_t(\tau)$ is positive (or bounded), then from (9),

$$\mathbb{E}[H_t(\tau)\mathbb{1}_{\{\tau\le t\}}Y_T(\tau)] = \mathbb{E}\Big[H_t(\tau)\mathbb{1}_{\{\tau\le t\}}Y_t^{\mathrm{ad}}(T,\tau)\Big].$$

Using the fact that $T \geq t$ and introducing the density $\alpha_T(\theta)$, we obtain

$$\begin{aligned}\mathbb{E}[H_t(\tau)\mathbb{1}_{\{\tau\leq t\}}Y_T(\tau)] &= \int_0^\infty \eta(\mathrm{d}\theta)\mathbb{E}[H_t(\theta)\mathbb{1}_{\{\theta\leq t\}}Y_T(\theta)\alpha_T(\theta)] \\ &= \int_0^\infty \eta(\mathrm{d}\theta)\mathbb{E}\big[H_t(\theta)\mathbb{1}_{\{\theta\leq t\}}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\big],\end{aligned}$$

$$\mathbb{E}\Big[H_t(\tau)\mathbb{1}_{\{\tau\leq t\}}Y_t^{\mathrm{ad}}(T,\tau)\Big] = \int_0^\infty \eta(\mathrm{d}\theta)\mathbb{E}\Big[H_t(\theta)\mathbb{1}_{\{\theta\leq t\}}Y_t^{ad}(T,\theta)\alpha_t(\theta)\Big].$$

The above two equalities imply (10). □

3.2. *Immersion property or H-hypothesis*

In the expression of the density $\alpha_t(\theta)$, the parameter θ plays the role of the default time. Hence, it is natural to consider the particular case where for any $\theta \geq 0$,

$$\alpha_t(\theta) = \alpha_\theta(\theta), \quad \forall t \geq \theta \ \mathrm{d}\mathbb{P} - a.s., \tag{11}$$

i.e., the case where the information contained in the reference filtration after the default time gives no new information on the conditional distribution of the default. In that case,

$$S_\theta = \mathbb{P}(\tau > \theta|\mathcal{F}_\theta) = 1 - \int_0^\theta \alpha_\theta(u)\eta(\mathrm{d}u) = 1 - \int_0^\theta \alpha_u(u)\eta(\mathrm{d}u).$$

Furthermore, for any $t > \theta$,

$$S_t(\theta) = \mathbb{P}(\tau > \theta|\mathcal{F}_t) = 1 - \int_0^\theta \alpha_t(u)\eta(\mathrm{d}u) = S_\theta, \quad a.s.$$

Hence, the knowledge of S implies that of the conditional distribution of τ for all positive t and θ: indeed, one has $S_t(\theta) = \mathbb{E}[S_\theta|\mathcal{F}_t]$ (note that, for $\theta < t$, this equality reduces to $S_t(\theta) = S_\theta$). In particular, S is decreasing and continuous, this last property will be useful later.

It follows that $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$. This last equality is known to be equivalent to the immersion property [4], also known as the H-hypothesis, stated as: for any fixed t and any bounded $\mathcal{G}_t$-r.v. $Y_t^{\mathbb{G}}$,

$$\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_\infty] = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] \quad a.s. \tag{12}$$

This is an important hypothesis in the credit risk analysis, notably for what happens before the default. However, it becomes restrictive since it allows no further information for what happens after the default time. In particular, in a multi-default setting, the immersion property between $\mathbb{F} \vee \mathbb{D}^1$ and $\mathbb{F} \vee \mathbb{D}^1 \vee \mathbb{D}^2$ is a very strong assumption where $\mathbb{D}^1$ and $\mathbb{D}^2$ are filtrations associated to two default times.

Conversely, if the immersion property holds, then (11) holds. In that case, the conditional survival functions $S_t(\theta)$ are constant in time on $[\theta, \infty)$, i.e., $S_t(\theta) = S_\theta(\theta) = S_\theta$ for $t > \theta$. Observe that the previous result (10) on the conditional expectation on the after-default set $\{\tau \leq t\}$ takes a simpler form: $Y_t^{\mathrm{ad}}(T,\theta) = \mathbb{E}[Y_T(\theta)|\mathcal{F}_t], a.s.$ for $\theta \leq t \leq T$, on the set $\{\alpha_\theta(\theta) > 0\}$. This means that under the immersion property, the information concerning the default distribution disappears when it concerns the after-default computation.

Remark 3.2. The most important example where the immersion property holds is the widely studied Cox-process model introduced by Lando [15].

4. A dynamic point of view and density process

Our aim is here to give a dynamic study of the previous results. We shall call $(S_t, t \geq 0)$ the survival process, which is an $\mathbb{F}$-supermartingale. We have obtained equalities for fixed t, we would like to study the conditional expectations as stochastic processes. One of the goals is to recover the value of the intensity of the random time, and the (additive and multiplicative) decompositions of the supermartingale S. Another one is to study the link between $\mathbb{G}$- and $\mathbb{F}$-martingales: this is for interest of pricing.

In this section, we present the dynamic version of the previous results in terms of $\mathbb{F}$- or $\mathbb{G}$-martingales or supermartingales. To be more precise, we need some "universal" regularity on the paths of the density process. We treat some technical problems in Section 4.1 which can be skipped for the first reading.

4.1. Regular version of martingales

One of the major difficulties is to prove the existence of a universal càdlàg martingale version of the family of densities. Fortunately, results of Jacod [11] or Stricker and Yor [19] help us to solve this technical problem. See also Amendinger's thesis [2] for a presentation of the problem, and detailed proofs of some results used in our paper.

Jacod ([11], Lemme 1.8) establishes the existence of a universal càdlàg version of the density process in the following sense: there exists a non-negative function $\alpha_t(\omega, \theta)$ càdlàg in t, optional w.r.t. the filtration $\widehat{\mathbb{F}}$ on $\widehat{\Omega} = \Omega \times \mathbb{R}^+$, generated by $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$, such that

- for any θ, $\alpha_.(\theta)$ is an $\mathbb{F}$-martingale; moreover, denoting $\zeta^\theta = \inf\{t : \alpha_{t-}(\theta) = 0\}$, then $\alpha_.(\theta) > 0$, and $\alpha_-(\theta) > 0$ on $[0, \zeta^\theta)$, and $\alpha_.(\theta) = 0$ on $[\zeta^\theta, \infty)$.
- For any bounded family $(Y_t(\omega, \theta), t \geq 0)$ measurable w.r.t. $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$, (where $\mathcal{P}(\mathbb{F})$ is the $\mathbb{F}$-predictable σ-field), the $\mathbb{F}$-predictable projection of the process $Y_t(\omega, \tau(\omega))$ is the process $Y_t^{(p)} = \int \alpha_{t-}(\theta) Y_t(\theta) \eta(\mathrm{d}\theta)$.
  In particular, for any t, $\mathbb{P}(\zeta^\tau < t) = \mathbb{E}[\int_0^\infty \alpha_{t-}(\theta) \mathbb{1}_{\{\zeta^\theta < t\}} \eta(\mathrm{d}\theta)] = 0$. So, ζ^τ is infinite a.s.
- We say that the process $(Y_t(\omega, \theta), t \geq 0)$ is $\mathbb{F}$-optional if it is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable, where $\mathcal{O}(\mathbb{F})$ is the optional σ-field of $\mathbb{F}$. In particular, the process $(Y_t(\omega, t), t \geq 0)$ is optional.

We are also concerned with the càdlàg version of the martingale $(S_t(u), t \geq 0)$ for any $u \in \mathbb{R}^+$. By the previous result, we have a universal version of their predictable projections, $S_{t-}(u) = S_t^{(p)}(u) = \int_u^\infty \alpha_{t-}(\theta) \eta(\mathrm{d}\theta)$. It remains to define $S_t(u) = \lim_{q \in \mathbb{Q}^+, q \downarrow t} S_q^{(p)}(u)$ to obtain a universal càdlàg version of the martingales $S_.(u)$. Remark that to show directly that $\int_u^\infty \alpha_t(\theta) \eta(\mathrm{d}\theta)$ is a càdlàg process, we need stronger assumption on the process $\alpha_t(\theta)$ which allows us to apply the Lebesgue theorem w.r.t. $\eta(\mathrm{d}\theta)$.

To distinguish the before-default and after-default analysis, we consider naturally the two families of density processes $(\alpha_t(\theta), t < \theta)$ and $(\alpha_t(\theta), t \geq \theta)$. Given the martingale property, we have $\alpha_t(\theta) = \mathbb{E}[\alpha_{\theta-}(\theta) | \mathcal{F}_t]$ for $t < \theta$ and we choose the diagonal terms $\alpha_\theta(\theta) = \alpha_{\theta-}(\theta)$.

4.2. Density and intensity processes

We are now interested in the relationship between the density and the intensity processes of τ. As we shall see, this is closely related to the (additive and multiplicative) decompositions of the supermartingale S, called the survival process.

4.2.1. $\mathbb{F}$-decompositions of the survival process S

In this section, we characterize the martingale and the predictable increasing part of the additive and multiplicative Doob–Meyer decomposition of the supermartingale S in terms of the density.

Proposition 4.1. (1) *The Doob–Meyer decomposition of the survival process S is given by $S_t = 1 + M_t^{\mathbb{F}} - \int_0^t \alpha_u(u)\eta(\mathrm{d}u)$ where $M^{\mathbb{F}}$ is the càdlàg square-integrable $\mathbb{F}$-martingale defined by*

$$M_t^{\mathbb{F}} = -\int_0^t \big(\alpha_t(u) - \alpha_u(u)\big)\eta(\mathrm{d}u) = \mathbb{E}\left[\int_0^\infty \alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t\right] - 1, \quad a.s.$$

(2) *Let $\zeta^{\mathbb{F}} := \inf\{t : S_{t-} = 0\}$ and define $\lambda_t^{\mathbb{F}} := \frac{\alpha_t(t)}{S_{t-}}$ on $\{t < \zeta^{\mathbb{F}}\}$ and $\lambda_t^{\mathbb{F}} := \lambda_{t\wedge\zeta^{\mathbb{F}}}^{\mathbb{F}}$ on $\{t \geq \zeta^{\mathbb{F}}\}$.*

The multiplicative decomposition of S is given by

$$S_t = L_t^{\mathbb{F}} \mathrm{e}^{-\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)} \tag{13}$$

where $L^{\mathbb{F}}$ is the $\mathbb{F}$-local martingale solution of $\mathrm{d}L_t^{\mathbb{F}} = \mathrm{e}^{\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)}\mathrm{d}M_t^{\mathbb{F}}$, $L_0^{\mathbb{F}} = 1$.

Proof. (1) First notice that $(\int_0^t \alpha_u(u)\eta(\mathrm{d}u), t \geq 0)$ is an $\mathbb{F}$-adapted continuous increasing process (the measure η does not have any atom). By the martingale property of $(\alpha_t(\theta), t \geq 0)$, for any fixed t, one has

$$S_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t(u)\eta(\mathrm{d}u) = \mathbb{E}\Big[\int_t^\infty \alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t\Big], \quad a.s.$$

Therefore, the non-negative process $S_t + \int_0^t \alpha_u(u)\eta(\mathrm{d}u) = \mathbb{E}[\int_0^\infty \alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t]$ is a square-integrable martingale since

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^\infty \alpha_u(u)\eta(\mathrm{d}u)\Big)^2\Big] &= 2\mathbb{E}\Big[\int_0^\infty \alpha_u(u)\eta(\mathrm{d}u)\int_u^\infty \alpha_s(s)\eta(\mathrm{d}s)\Big] \\ &= 2\mathbb{E}\Big[\int_0^\infty S_u\alpha_u(u)\eta(\mathrm{d}u)\Big] \leq 2.\end{aligned}$$

We shall choose its càdlàg version if needed. Using the fact that $\int_0^\infty \alpha_t(u)\eta(\mathrm{d}u) = 1$ and that $\alpha(u)$ is a martingale, we obtain

$$\forall t, \quad \mathbb{E}\Big[\int_0^\infty \alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t\Big] = 1 - \int_0^t (\alpha_t(u) - \alpha_u(u))\eta(\mathrm{d}u), \ a.s.$$

and the result follows.

Note that the square integrability property of $M^{\mathbb{F}}$ is a general property, which holds for any Azéma supermartingale. See e.g. [17, p. 380].

(2) Setting $L_t^{\mathbb{F}} = S_t\mathrm{e}^{\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)}$, integration by parts formula and (1) yield to

$$\mathrm{d}L_t^{\mathbb{F}} = \mathrm{e}^{\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)}\mathrm{d}S_t + \mathrm{e}^{\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)}\lambda_t^{\mathbb{F}}S_t\eta(\mathrm{d}t) = \mathrm{e}^{\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)}\mathrm{d}M_t^{\mathbb{F}},$$

which implies the result. □

Remarks 4.2. (1) Note that, from (4), $\mathbb{P}(\zeta^{\mathbb{F}} \geq \tau) = 1$.

(2) The survival process S is a decreasing process if and only if the martingale $M^{\mathbb{F}}$ is constant ($M^{\mathbb{F}} \equiv 0$) or equivalently if and only if the local martingale $L^{\mathbb{F}}$ is constant ($L^{\mathbb{F}} \equiv 1$). In that

case, by Proposition 4.1, S is the continuous decreasing process $S_t = e^{-\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)}$. Hence, for any pair (t, θ), $t \le \theta$, the conditional distribution is given by $S_t(\theta) = \mathbb{E}[e^{-\int_0^\theta \lambda_s^{\mathbb{F}} \eta(ds)} | \mathcal{F}_t]$.

(3) The condition $M^{\mathbb{F}} \equiv 0$ can be written as $\int_0^t (\alpha_t(u) - \alpha_u(u))\eta(du) = 0$ and is satisfied if, for $t \ge u$, $\alpha_t(u) - \alpha_u(u) = 0$ (immersion property), but the converse is not true.

4.2.2. Relationship with the $\mathbb{G}$-intensity

The intensity approach has been used extensively in the credit literature. We study now in more details the relationship between the density and the intensity, and notably between the $\mathbb{F}$-density process of τ and its intensity process with respect to $\mathbb{G}$. We first recall some definitions.

Definition 4.3. Let τ be a $\mathbb{G}$-stopping time. The $\mathbb{G}$-*compensator* of τ is the $\mathbb{G}$-predictable increasing process $\Lambda^{\mathbb{G}}$ such that the process $(N_t^{\mathbb{G}} = \mathbb{1}_{\{\tau \le t\}} - \Lambda_t^{\mathbb{G}}, t \ge 0)$ is a $\mathbb{G}$-martingale. If $\Lambda^{\mathbb{G}}$ is absolutely continuous with respect to the measure η, the $\mathbb{G}$-adapted process $\lambda^{\mathbb{G}}$ such that $\Lambda_t^{\mathbb{G}} = \int_0^t \lambda_s^{\mathbb{G}} \eta(ds)$ is called the $(\mathbb{G}, \eta)$-*intensity process* or the $\mathbb{G}$-*intensity* if there is no ambiguity. The $\mathbb{G}$-compensator is stopped at τ, i.e., $\Lambda_t^{\mathbb{G}} = \Lambda_{t\wedge\tau}^{\mathbb{G}}$. Hence, $\lambda_t^{\mathbb{G}} = 0$ on $\{t > \tau\}$.

The following results give the $\mathbb{G}$-intensity of τ in terms of $\mathbb{F}$-density, and conversely the $\mathbb{F}$-density $\alpha_t(\theta)$ in terms of the $\mathbb{G}$-intensity, but only for $\theta \ge t$.

Proposition 4.4. (1) *The random time τ admits a $(\mathbb{G}, \eta)$-intensity given by*

$$\lambda_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} \lambda_t^{\mathbb{F}} = \mathbb{1}_{\{\tau > t\}} \frac{\alpha_t(t)}{S_t}, \quad \mathbb{P} - a.s. \tag{14}$$

The process $(N_t^{\mathbb{G}} := \mathbb{1}_{\{\tau \le t\}} - \int_0^t \lambda_s^{\mathbb{G}} \eta(ds), t \ge 0)$ is a $\mathbb{G}$-martingale, and $(L_t^{\mathbb{G}} := \mathbb{1}_{\{\tau > t\}} e^{\int_0^t \lambda_s^{\mathbb{G}} \eta(ds)}, t \ge 0)$ is a $\mathbb{G}$-local martingale.

(2) *For any $\theta \ge t$, we have $\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{G}} | \mathcal{F}_t]$ on the set $\{t < \zeta^{\mathbb{F}}\}$.*
Furthermore, the $\mathbb{F}$-optional projections of the martingale $N^{\mathbb{G}}$ and of the local martingale $L^{\mathbb{G}}$ are the $\mathbb{F}$-martingale $-M^{\mathbb{F}}$ and the $\mathbb{F}$-local martingale $L^{\mathbb{F}}$.

Proof. (1) The $\mathbb{G}$-local martingale property of $N^{\mathbb{G}}$ is equivalent to the $\mathbb{G}$-local martingale property of $L_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} e^{\int_0^t \lambda_s^{\mathbb{G}} \eta(ds)} = \mathbb{1}_{\{\tau > t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)}$, since

$$dL_t^{\mathbb{G}} = -L_{t-}^{\mathbb{G}} d\mathbb{1}_{\{\tau \le t\}} + \mathbb{1}_{\{\tau > t\}} e^{\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)} \lambda_t^{\mathbb{F}} \eta(dt) = -L_{t-}^{\mathbb{G}} dN_t^{\mathbb{G}}.$$

The process $\int_0^t \lambda_s^{\mathbb{F}} \eta(ds)$ is continuous, so we can proceed by localization, introducing the $\mathbb{G}$-stopping times $\tau_n = \tau \mathbb{1}_{\{\tau \le T_n\}} + \infty \mathbb{1}_{\{\tau > T_n\}}$ where $T_n = \inf\{t : \int_0^t \lambda_s^{\mathbb{F}} \eta(ds) > n\}$. Then, the local martingale property of the stopped process $L_t^{\mathbb{G},n} := L_{t\wedge\tau^n}^{\mathbb{G}}$ follows from the $\mathbb{F}$-local martingale property of $L_{t\wedge T^n}^{\mathbb{F}} = L_t^{\mathbb{F},n}$, since for any $s \le t$,

$$\begin{aligned}
\mathbb{E}[L_t^{\mathbb{G},n} | \mathcal{G}_s] &= \mathbb{E}[\mathbb{1}_{\{\tau > t\wedge T^n\}} e^{\int_0^{t\wedge T^n} \lambda_u^{\mathbb{F}} \eta(du)} | \mathcal{G}_s] \\
&= \mathbb{1}_{\{\tau > s\wedge T^n\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau > t\wedge T^n\}} e^{\int_0^{t\wedge T^n} \lambda_u^{\mathbb{F}} \eta(du)} | \mathcal{F}_s]}{S_s} \\
&= \mathbb{1}_{\{\tau > s\wedge T^n\}} \frac{\mathbb{E}[S_{t\wedge T^n} e^{\int_0^{t\wedge T^n} \lambda_u^{\mathbb{F}} \eta(du)} | \mathcal{F}_s]}{S_s} = \mathbb{1}_{\{\tau > s\wedge T^n\}} \frac{L_s^{\mathbb{F},n}}{S_s}
\end{aligned}$$

where the second equality comes from Theorem 3.1 and the last equality follows from the $\mathbb{F}$-local martingale property of $L^{\mathbb{F},n}$.

Then, the form of the intensities follows from the definition. Since the bounded supermartingale $\mathbb{1}_{\{\tau\le t\}}$ is obviously of class (D), the compensated local martingale $N^{\mathbb{G}}$ is a uniformly integrable martingale.

(2) By the martingale property of density, for any $\theta\ge t$, $\alpha_t(\theta)=\mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]$. Using the definition of S, and the value of $\lambda^{\mathbb{G}}$ given in (1), we obtain

$$\alpha_t(\theta)=\mathbb{E}\left[\alpha_\theta(\theta)\,\frac{\mathbb{1}_{\{\tau>\theta\}}}{S_\theta}\middle|\,\mathcal{F}_t\right]=\mathbb{E}[\lambda^{\mathbb{G}}_\theta|\mathcal{F}_t],\quad \text{on } \{t<\zeta^{\mathbb{F}}\},\ a.s.$$

Hence, the value of the density can be partially deduced from the intensity.

The $\mathbb{F}$-projection of the local martingale $L^{\mathbb{G}}_t=\mathbb{1}_{\{\tau>t\}}e^{\int_0^t\lambda^{\mathbb{G}}_s\eta(ds)}$ is the local martingale $S_te^{\int_0^t\lambda^{\mathbb{F}}_s\eta(ds)}=L^{\mathbb{F}}_t$ by definition of the survival process S. Similarly, since $\alpha_t(\theta)=\mathbb{E}[\lambda^{\mathbb{G}}_\theta|\mathcal{F}_t]$, the $\mathbb{F}$-projection of the martingale $N^{\mathbb{G}}_t=\mathbb{1}_{\{\tau\le t\}}-\int_0^t\lambda^{\mathbb{G}}_s\eta(ds)$ is $1-S_t-\int_0^t\alpha_s(s)\eta(ds)=-M^{\mathbb{F}}_t$. □

Remarks 4.5. (1) Since the intensity process exists, τ is a totally inaccessible $\mathbb{G}$-stopping time.

(2) The density hypothesis, and the fact that η is non-atomic allow us to choose $\alpha_s(s)/S_s$ as an intensity, instead of $\alpha_s(s)/S_{s-}$ as it is usually done (see [8] in the case where the numerator $\alpha_s(s)$ represents the derivative of the compensator of S).

(3) Proposition 4.4 shows that the intensity $\lambda^{\mathbb{G}}_t$ can be completely deduced from $\alpha_t(t)$ since $S_t=\int_t^\infty\alpha_t(\theta)\eta(d\theta)=\int_t^\infty\mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]\eta(d\theta)$. However, given $\lambda^{\mathbb{G}}_t$, we can only obtain the partial knowledge of $\alpha_t(\theta)$ for $\theta\ge t$. This is the reason why the intensity approach is not sufficient to study what goes on after a default event (except under the immersion property), since we observe in (10) that the other part of $\alpha_t(\theta)$ where $\theta<t$ plays an important role in the after-default computation.

(4) Proposition 4.1 shows that density and intensity approaches correspond respectively to the additive and the multiplicative decomposition point of view of the survival process S.

We now use the density–intensity relationship to characterize the pure jump $\mathbb{G}$-martingales having only one jump at τ. The items (2) and (3) in the following corollary can be viewed as some representation theorems for such martingales, in an additive and a multiplicative form respectively. Note also that (1) is a classical result for $\mathbb{G}$-predictable processes (see Remark 4.5 in [13]), we here are interested in $\mathbb{G}$-optional processes.

Corollary 4.6. (1) *For any locally bounded $\mathbb{G}$-optional process $H^{\mathbb{G}}$, the process*

$$N^{H,\mathbb{G}}_t:=H^{\mathbb{G}}_\tau\mathbb{1}_{\{\tau\le t\}}-\int_0^{t\wedge\tau}\frac{\alpha_s(s)}{S_s}H^{\mathbb{G}}_s\eta(ds)=\int_0^t H^{\mathbb{G}}_s\,dN^{\mathbb{G}}_s,\quad t\ge 0 \tag{15}$$

is a $\mathbb{G}$-local martingale.

(2) *Any pure jump $\mathbb{G}$-martingale $M^{\mathbb{G}}$ which has only one locally bounded jump at τ can be written on the form (15), with $H^{\mathbb{G}}_\tau=M^{\mathbb{G}}_\tau-M^{\mathbb{G}}_{\tau-}$.*

(3) *If, in addition, $M^{\mathbb{G}}$ is positive, then it satisfies $dM^{\mathbb{G}}_t/M^{\mathbb{G}}_{t-}=(u_t-1)dN^{\mathbb{G}}_t$ where u is a positive $\mathbb{F}$-optional process associated with the relative jump such that $u_\tau=M^{\mathbb{G}}_\tau/M^{\mathbb{G}}_{\tau-}$. The martingale $M^{\mathbb{G}}$ has the equivalent representation*

$$M^{\mathbb{G}}_t=\big((u_\tau-1)\mathbb{1}_{\{\tau\le t\}}+1\big)e^{-\int_0^{t\wedge\tau}(u_s-1)\lambda^{\mathbb{F}}_s\eta(ds)}. \tag{16}$$

Proof. (1) The $\mathbb{G}$-martingale property of $N^{\mathbb{G}}$ implies that $N^{H,\mathbb{G}}$ defined in (15) is a $\mathbb{G}$-martingale for any bounded predictable process $H^{\mathbb{G}}$ (typically $H_t^{\mathbb{G}} = H_s^{\mathbb{G}}\mathbb{1}_{]s,\infty]}(t)$). For (1), it suffices to verify the $\mathbb{G}$-martingale property for the process $N^{H,\mathbb{G}}$ where $H^{\mathbb{G}}$ is the $\mathbb{G}$-optional process of the form $H_t^{\mathbb{G}} = H_s^{\mathbb{G}}\mathbb{1}_{[s,\infty)}(t)$. By comparing the two cases, we know that this is also true since τ avoids $\mathbb{F}$-stopping times.

(2) The locally bounded jump $H_\tau^{\mathbb{G}}$ of the martingale $M^{\mathbb{G}}$ at time τ is the value at time τ of some locally bounded $\mathbb{F}$-optional process $H^{\mathbb{F}}$. Then the difference $M^{\mathbb{G}} - N^{H,\mathbb{G}}$ is a continuous local martingale. In addition, it is also a finite variation process, and hence is a constant process.

(3) We calculate the differential of the finite variation process $M^{\mathbb{G}}$ as

$$\begin{aligned}\mathrm{d}M_t^{\mathbb{G}} &= -M_t^{\mathbb{G}}(u_t - 1)\lambda_t^{\mathbb{G}}\eta(\mathrm{d}t) + M_{t-}^{\mathbb{G}}(u_t - 1)(\mathrm{d}N_t^{\mathbb{G}} + \lambda_t^{\mathbb{G}}\eta(\mathrm{d}t))\\ &= M_{t-}^{\mathbb{G}}(u_t - 1)\mathrm{d}N_t^{\mathbb{G}}.\end{aligned}$$

Then $M^{\mathbb{G}}$ is the exponential martingale of the pure jump martingale $(u_t - 1)\mathrm{d}N_t^{\mathbb{G}}$. □

4.3. An example of HJM type

We now give some examples, where we point out similarities with Heath–Jarrow–Morton models. Here, our aim is not to present a general framework, therefore, we reduce our attention to the case where the reference filtration $\mathbb{F}$ is generated by a multi-dimensional standard Brownian motion W. The following two propositions, which model the dynamics of the conditional probability $S(\theta)$, correspond respectively to the additive and multiplicative points of view. From the predictable representation theorem in the Brownian filtration, applied to the family of bounded martingales $(S_t(\theta), t \geq 0)$, $\theta \geq 0$, there exists a family of $\mathbb{F}$-predictable processes $(Z_t(\theta), t \geq 0)$ satisfying $Z_t(0) = 0$, $\forall t \geq 0$, such that

$$\mathrm{d}S_t(\theta) = Z_t(\theta)\mathrm{d}W_t, \tag{17}$$

In this section, we assume "smoothness conditions" without giving details to avoid a lengthy redaction. We refer to Musiela and Rutkowski [16, Chap. 11] for conditions which ensure that the stochastic integrals are well defined in a HJM model and [17, p. 312] for conditions allowing to differentiate stochastic differential equations with respect to a parameter.

Proposition 4.7. *Let $\mathrm{d}S_t(\theta) = Z_t(\theta)\mathrm{d}W_t$ be the martingale representation of $(S_t(\theta), t \geq 0)$ and assume that the processes $(Z_t(\theta); t \geq 0)$ are differentiable in the following sense: there exists a family of processes $(z_t(\theta), t \geq 0)$ such that $Z_t(\theta) = \int_0^\theta z_t(u)\eta(\mathrm{d}u)$, $Z_t(0) = 0$. Then, under regularity conditions,*

(1) *the density processes have the following dynamics $\mathrm{d}\alpha_t(\theta) = -z_t(\theta)\mathrm{d}W_t$ where $z(\theta)$ is subjected to the constraint $\int_0^\infty z_t(\theta)\eta(\mathrm{d}\theta) = 0$ for any $t \geq 0$.*

(2) *The survival process S evolves as $\mathrm{d}S_t = -\alpha_t(t)\eta(\mathrm{d}t) + Z_t(t)\mathrm{d}W_t$.*

(3) *With more regularity assumptions, if $(\partial_\theta\alpha_t(\theta))_{\theta=t}$ is simply denoted by $\partial_\theta\alpha_t(t)$, then the process $\alpha_t(t)$ follows:*

$$\mathrm{d}\alpha_t(t) = \partial_\theta\alpha_t(t)\eta(\mathrm{d}t) - z_t(t)\mathrm{d}W_t.$$

Proof. (1) Observe that $Z(0) = 0$ since $S(0) = 1$, hence the existence of z is related with some smoothness conditions. Then using the stochastic Fubini theorem (Theorem IV.65 [17]), one has

$$S_t(\theta) = S_0(\theta) + \int_0^t Z_u(\theta)\mathrm{d}W_u = S_0(\theta) + \int_0^\theta \eta(\mathrm{d}v) \int_0^t z_u(v)\mathrm{d}W_u.$$

So (1) follows. Using the fact that for any $t \geq 0$,

$$\begin{aligned} 1 &= \int_0^\infty \alpha_t(u)\eta(\mathrm{d}u) = \int_0^\infty \eta(\mathrm{d}u)\Big(\alpha_0(u) - \int_0^t z_s(u)\mathrm{d}W_s\Big) \\ &= 1 - \int_0^t \mathrm{d}W_s \int_0^\infty z_s(u)\eta(\mathrm{d}u), \end{aligned}$$

one gets $\int_0^\infty z_t(u)\eta(\mathrm{d}u) = 0$.

(2) By using Proposition 4.1 and integration by parts,

$$\begin{aligned} M_t^{\mathbb{F}} &= -\int_0^t (\alpha_t(u) - \alpha_u(u))\eta(\mathrm{d}u) = \int_0^t \eta(\mathrm{d}u) \int_u^t z_s(u)\mathrm{d}W_s \\ &= \int_0^t \mathrm{d}W_s\Big(\int_0^s z_s(u)\eta(\mathrm{d}u)\Big), \end{aligned}$$

which implies (2).

(3) We follow the same way as for the decomposition of S, by studying the process

$$\alpha_t(t) - \int_0^t (\partial_\theta\alpha_s)(s)\eta(\mathrm{d}s) = \alpha_t(0) + \int_0^t (\partial_\theta\alpha_t)(s)\eta(\mathrm{d}s) - \int_0^t (\partial_\theta\alpha_s)(s)\eta(\mathrm{d}s)$$

where the notation $\partial_\theta\alpha_t(t)$ is defined in 3). Using the martingale representation of $\alpha_t(\theta)$ and integration by parts (assuming that smoothness hypothesis allows these operations), the integral in the right-hand side is a stochastic integral,

$$\begin{aligned} &\int_0^t \Big((\partial_\theta\alpha_t)(s) - (\partial_\theta\alpha_s)(s)\Big)\eta(\mathrm{d}s) = -\int_0^t \eta(\mathrm{d}s)\partial_\theta\Big(\int_s^t z_u(\theta)\mathrm{d}W_u\Big) \\ &= -\int_0^t \eta(\mathrm{d}s)\int_s^t \partial_\theta z_u(s)\mathrm{d}W_u = -\int_0^t \mathrm{d}W_u \int_0^u \eta(\mathrm{d}s)\partial_\theta z_u(s) \\ &= -\int_0^t \mathrm{d}W_u(z_u(u) - z_u(0)) \end{aligned}$$

The stochastic integral $\int_0^t z_u(0)\mathrm{d}W_u$ is the stochastic part of the martingale $\alpha_t(0)$, and so the property (3) holds true. $\square$

Remark 4.8. If the density admits the dynamics of a multiplicative form instead of an additive one, that is, if $\mathrm{d}\alpha_t(\theta) = \alpha_t(\theta)\gamma_t(\theta)\mathrm{d}W_t$, then the constraint condition in (1) of Proposition 4.7 becomes $\int_0^\infty \alpha_t(u)\gamma_t(u)\eta(\mathrm{d}u) = 0$, as given in [5].

We now consider $(S_t(\theta), t \geq 0)$ as in the classical HJM models (see [18]) where its dynamics is given in multiplicative form. By using the forward intensity $\lambda_t(\theta)$ of τ (see Remark 2.3), the density can then be calculated as $\alpha_t(\theta) = \lambda_t(\theta)S_t(\theta)$. It follows that the forward intensity is non-negative. As noted in Remark 2.3, $\lambda(\theta)$ plays the same role as the spot forward rate in the interest rate models.

Classically, HJM framework is studied for time smaller than maturity, i.e. $t \leq T$. Here we consider all positive pairs (t, θ).

Proposition 4.9. *We keep the notation and the assumptions in Proposition 4.7. For any $t, \theta \geq 0$, let $\Psi_t(\theta) = \frac{Z_t(\theta)}{S_t(\theta)}$. We assume that there exists a family of processes ψ such that $\Psi_t(\theta) = \int_0^\theta \psi_t(u)\eta(\mathrm{d}u)$. Then*

(1) $S_t(\theta) = S_0(\theta)\exp\left(\int_0^t \Psi_s(\theta)\mathrm{d}W_s - \frac{1}{2}\int_0^t |\Psi_s(\theta)|^2\mathrm{d}s\right)$;

(2) *the forward intensity $\lambda(\theta)$ has the following dynamics:*

$$\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)\mathrm{d}W_s + \int_0^t \psi_s(\theta)\Psi_s(\theta)\mathrm{d}s; \tag{18}$$

(3) $S_t = \exp\left(-\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s) + \int_0^t \Psi_s(s)\mathrm{d}W_s - \frac{1}{2}\int_0^t |\Psi_s(s)|^2\mathrm{d}s\right)$;

Proof. By choice of notation, (1) holds since the process $S_t(\theta)$ is the solution of the equation

$$\frac{\mathrm{d}S_t(\theta)}{S_t(\theta)} = \Psi_t(\theta)\mathrm{d}W_t, \quad \forall t, \theta \geq 0. \tag{19}$$

(2) is the consequence of (1) and the definition of $\lambda(\theta)$.
(3) This representation is the multiplicative version of the additive decomposition of S in Proposition 4.7. We recall that $\lambda_t^{\mathbb{F}} = \alpha_t(t)S_t^{-1}$. There are no technical difficulties because S is continuous. □

Remarks 4.10. If $\Psi_s(s) = 0$, then $S_t = \exp(-\int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s))$ and is decreasing. For the (H)-hypothesis to hold, one needs that $\Psi_s(\theta) = 0$ for any $s \geq \theta$.

When $\theta > t$, the non-negativity property of the forward intensity is implied by the weaker condition $\lambda_t(t) \geq 0$. That is similar to the case of zero coupon bond prices. But when $\theta < t$, an additional assumption on the dynamics (18) is necessary. We do not characterize this condition, we shall only provide an example (see below).

Remark 4.11. The above results are not restricted to the Brownian filtration and can be easily extended to more general filtrations under similar representation $\mathrm{d}S_t(\theta) = Z_t(\theta)\mathrm{d}M_t$ where M is a martingale which can include jumps. In this case, Proposition 4.7 can be generalized in a similar form; for Proposition 4.9, more attention should be payed to Doléans–Dade exponential martingales with jumps.

Example. We now give a particular example which provides a large class of non-negative forward intensity processes. The non-negativity property of λ is satisfied, by (2) of Proposition 4.9, if the two following conditions hold:

- for any θ, the process $\psi(\theta)\Psi(\theta)$ is non-negative, (in particular if the family $\psi(\theta)$ has constant sign);
- for any θ, the local martingale $\zeta_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)\mathrm{d}W_s$ is a Doléans–Dade exponential of some martingale, i.e., if there exists a family of adapted processes $b(\theta)$ such that $\zeta(\theta)$ is the solution of

$$\zeta_t(\theta) = \lambda_0(\theta) + \int_0^t \zeta_s(\theta)b_s(\theta)\mathrm{d}W_s,$$

that is, if $-\int_0^t \psi_s(\theta)\mathrm{d}W_s = \int_0^t \zeta_s(\theta)b_s(\theta)\mathrm{d}W_s$. Here the initial condition is a positive constant $\lambda_0(\theta)$. Hence, we choose

$$\psi_t(\theta) = -b_t(\theta)\zeta_t(\theta) = -b_t(\theta)\lambda_0(\theta)\exp\left(\int_0^t b_s(\theta)\mathrm{d}W_s - \frac{1}{2}\int_0^t b_s^2(\theta)\mathrm{d}s\right)$$

where λ_0 is a positive intensity function and $b(\theta)$ is a family of $\mathbb{F}$-adapted processes with constant sign. Then,

$$\alpha_t(\theta) = \lambda_t(\theta) \exp\left(- \int_0^\theta \lambda_t(v) \mathrm{d}v \right),$$

where

$$\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)\, \mathrm{d}W_s + \int_0^t \psi_s(\theta)\, \Psi_s(\theta)\, \mathrm{d}s,$$

is a family of density processes.

5. Characterization of $\mathbb{G}$-martingales in terms of $\mathbb{F}$-martingales

In the theory of pricing and hedging, martingale properties play a very important role. In this section, we study the martingale characterization when taking into account information of the default occurrence. The classical question in the enlargement of filtration theory is to give decomposition of $\mathbb{F}$-martingales in terms of $\mathbb{G}$-semimartingales. For the credit problems, we are concerned with the problem in a converse sense, that is, with the links between $\mathbb{G}$-martingales and $\mathbb{F}$-(local) martingales. In the literature, $\mathbb{G}$-martingales which are stopped at τ have been investigated, particularly in the credit context. For our analysis of after-default events, we are interested in the martingales which start at the default time τ and in martingales having one jump at τ, as the ones introduced in Corollary 4.6. The goal of this section is to present characterization results for these types of $\mathbb{G}$-martingales.

5.1. $\mathbb{G}$-local martingale characterization

Any $\mathbb{G}$-local martingale may be split into two local martingales, the first one stopped at time τ and the second one starting at time τ, that is

$$Y_t^{\mathbb{G}} = Y_{t\wedge\tau}^{\mathbb{G}} + (Y_t^{\mathbb{G}} - Y_\tau^{\mathbb{G}})\mathbb{1}_{\{\tau\le t\}}.$$

The density hypothesis allows us to provide a characterization[2] of $\mathbb{G}$-local martingales stopped at time τ.

Proposition 5.1. *A $\mathbb{G}$-adapted càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale stopped at time τ if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y defined on $[0, \zeta^{\mathbb{F}})$ and a locally bounded $\mathbb{F}$-optional process Z such that $Y_t^{\mathbb{G}} = Y_t \mathbb{1}_{\{\tau>t\}} + Z_\tau \mathbb{1}_{\{\tau\le t\}}$ a.s. and that*

$$\left(U_t := Y_t S_t + \int_0^t Z_s \alpha_s(s) \eta(\mathrm{d}s),\ t \ge 0 \right) \text{ is an } \mathbb{F}\text{-local martingale on } [0, \zeta^{\mathbb{F}}). \tag{20}$$

Moreover, if $Y^{\mathbb{G}}$ is a uniformly integrable $\mathbb{G}$-martingale stopped at time τ, the process U is a martingale.

Equivalently, using the multiplicative decomposition of S as $S_t = L_t^{\mathbb{F}} \mathrm{e}^{-\int_0^t \lambda_s^{\mathbb{F}} \eta(\mathrm{d}s)}$ on $[0, \zeta^{\mathbb{F}})$, the condition (20) is equivalent to

$$\left(L_t^{\mathbb{F}} \left[Y_t + \int_0^t (Z_s - Y_s) \lambda_s^{\mathbb{F}} \eta(\mathrm{d}s) \right],\ t \ge 0 \right) \text{ is an } \mathbb{F}\text{-local martingale on } [0, \zeta^{\mathbb{F}}). \tag{21}$$

2 The following proposition was established in [3, Lemma 4.1.3] in a hazard process setting.

Proof. Assume that $Y^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. The conditional expectation of $Y_t^{\mathbb{G}}$ given $\mathcal{F}_t$ is the $\mathbb{F}$-local martingale $Y^{\mathbb{F}}$ defined on $[0, \zeta^{\mathbb{F}})$ as $Y_t^{\mathbb{F}} = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = Y_t S_t + \int_0^t Z_s \alpha_t(s)\eta(\mathrm{d}s)$ where the last equality is obtained by using the $\mathcal{F}_t$-density of τ. From the definition of U, one has $Y_t^{\mathbb{F}} - U_t = \int_0^t Z_s(\alpha_t(s) - \alpha_s(s))\eta(\mathrm{d}s)$. Using the fact that Z is locally bounded and that $(\alpha_t(s), t \geq 0)$ is an $\mathbb{F}$-martingale, it is easy to check that $Y^{\mathbb{F}} - U$ is an $\mathbb{F}$-local martingale, hence U is also an $\mathbb{F}$-local martingale. Moreover, if $Y^{\mathbb{G}}$ is uniformly integrable, $\mathbb{E}[|Y_\vartheta^{\mathbb{G}}|] < \infty$ for any $\mathbb{F}$-stopping time ϑ, and the quantity $Y_\vartheta \mathbb{1}_{\{\tau>\vartheta\}}$ is integrable. Hence $Y_\vartheta S_\vartheta$ is also integrable, and

$$\mathbb{E}\left[\int_0^{\zeta^{\mathbb{F}}} |Z_s|\alpha_s(s)\eta(\mathrm{d}s)\right] = \mathbb{E}[|Y_\tau^{\mathbb{G}}|] < \infty,$$

which establishes that U is a martingale. The local martingale case follows by localization.

Conversely, for any $\mathbb{F}$-adapted process Z and any $t > s$,

$$\mathbb{E}[Z_\tau \mathbf{1}_{\tau\leq t}|\mathcal{G}_s] = Z_\tau \mathbf{1}_{\{\tau\leq s\}} + \mathbf{1}_{\{s<\tau\}}\frac{1}{S_s}\mathbb{E}\left[\int_s^t Z_u\alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_s\right].$$

Then using Theorem 3.1 and the fact that U is an $\mathbb{F}$-local martingale, we obtain that $\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{G}_s] = Y_s^{\mathbb{G}}$ a.s., hence $Y^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale.

The second formulation is based on the multiplicative representation (13), $S_t = L_t^{\mathbb{F}} \mathrm{e}^{-\Lambda_t^{\mathbb{F}}}$ where $\Lambda_t^{\mathbb{F}} = \int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)$ is a continuous increasing process. Hence

$$\begin{aligned}\mathrm{d}(Y_t L_t^{\mathbb{F}}) &= \mathrm{d}(Y_t S_t \mathrm{e}^{\Lambda_t^{\mathbb{F}}}) = \mathrm{e}^{\Lambda_t^{\mathbb{F}}}\mathrm{d}(Y_t S_t) + \mathrm{e}^{\Lambda_t^{\mathbb{F}}} Y_t S_t \lambda_t^{\mathbb{F}}\eta(\mathrm{d}t)\\ &= \mathrm{e}^{\Lambda_t^{\mathbb{F}}}\mathrm{d}U_t + (Y_t - Z_t)\lambda_t^{\mathbb{F}} L_t^{\mathbb{F}}\eta(\mathrm{d}t),\end{aligned}$$

where the last equality follows from (14) that $\alpha_t(t) = \lambda_t^{\mathbb{F}} S_t$. The local-martingale property of the process U is then equivalent to that of $(Y_t L_t^{\mathbb{F}} - \int_0^t (Y_s - Z_s)\lambda_s^{\mathbb{F}} L_s^{\mathbb{F}}\eta(\mathrm{d}s), t \geq 0)$, and then to the condition (21). □

Corollary 5.2. *A $\mathbb{G}$-martingale stopped at time τ and equal to* 1 *on* $[0, \tau)$ *is constant on* $[0, \tau]$.

Proof. The integration by parts formula proves that $(L_t^{\mathbb{F}} \int_0^t (1 - Z_s)\lambda_s^{\mathbb{F}}\eta(\mathrm{d}s), t \geq 0)$ is a local martingale if and only if the continuous bounded variation process $(\int_0^t L_s^{\mathbb{F}}(1 - Z_s)\lambda_s^{\mathbb{F}}\eta(\mathrm{d}s), t \geq 0)$ is a local martingale, that is, if $L_s^{\mathbb{F}}(1 - Z_s)\lambda_s^{\mathbb{F}} = 0$, which implies that $Z_s = 1$ on $[0, \zeta^{\mathbb{F}})$. □

The before-default $\mathbb{G}$-martingale $Y^{\mathrm{bd,G}}$ can always be separated into two parts: a martingale which is stopped at τ and is continuous at τ, and a martingale which has a jump at τ.

Corollary 5.3. *Let $Y^{\mathrm{bd,G}}$ be a $\mathbb{G}$-local martingale stopped at τ of the form $Y_t^{\mathrm{bd,G}} = Y_t \mathbb{1}_{\{\tau>t\}} + Z_\tau \mathbb{1}_{\{\tau\leq t\}}$. Then there exist two $\mathbb{G}$-local martingales $Y^{\mathrm{c,bd}}$ and $Y^{\mathrm{d,bd}}$ such that $Y^{\mathrm{bd,G}} = Y^{\mathrm{c,bd}} + Y^{\mathrm{d,bd}}$, which satisfy the following conditions:*

(1) $(Y_t^{\mathrm{d,bd}} = (Z_\tau - Y_\tau)\mathbb{1}_{\{\tau\leq t\}} - \int_0^{t\wedge\tau}(Z_s - Y_s)\lambda_s^{\mathbb{F}}\eta(\mathrm{d}s), t \geq 0)$ *is a $\mathbb{G}$-local martingale with a single jump at τ;*

(2) $(Y_t^{\mathrm{c,bd}} = \widetilde{Y}_{\tau\wedge t}, t \geq 0)$ *is a $\mathbb{G}$-local martingale continuous at τ, where $\widetilde{Y}_t = Y_t + \int_0^t (Z_s - Y_s)\lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)$ and $(L_t^{\mathbb{F}}\widetilde{Y}_t, t \geq 0)$ is an $\mathbb{F}$-local martingale.*

Proof. From Corollary 4.6, $Y^{\mathrm{d,bd}}$ is a local martingale. The result follows. □

Corollary 5.4. *Under the immersion property, a process $Y^{\mathbb{G}}$ stopped at τ and continuous at time τ is a $\mathbb{G}$-local martingale if and only if Y is an $\mathbb{F}$-local martingale. In other words, any $\mathbb{G}$-local martingale stopped at time τ and continuous at τ is an $\mathbb{F}$-local martingale stopped at τ.*

We now concentrate on the $\mathbb{G}$-local martingales starting at τ, which, as we can see below, are easier to characterize.

Proposition 5.5. *Any càdlàg integrable process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale starting at τ with $Y_\tau = 0$ if and only if there exists an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-process $(Y_t(.), t \geq 0)$ such that $Y_t(t) = 0$ and $Y_t^{\mathbb{G}} = Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}$ and that, for any $\theta > 0$, $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^\theta)$, where ζ^θ is defined as in Section 4.1. The same result follows for local martingales by localization.*

Proof. From Theorem 3.1, for any $t > s \geq 0$, one has

$$\begin{aligned}
&\mathbb{E}(Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}|\mathcal{G}_s) \\
&= \mathbb{1}_{\{s<\tau\}}\frac{1}{S_s}\mathbb{E}\left(\int_s^\infty \mathbf{1}_{u\leq t} Y_t(u)\alpha_t(u)\eta(\mathrm{d}u)|\mathcal{F}_s\right) \\
&\quad + \mathbb{1}_{\{\tau \leq s\}}\frac{1}{\alpha_s(\tau)}\left(\mathbb{E}(Y_t(\theta)\alpha_t(\theta)|\mathcal{F}_s)|_{\theta=\tau}\right) \\
&= \mathbb{1}_{\{s<\tau\}}\frac{1}{S_s}\mathbb{E}\left(\int_s^t Y_t(u)\alpha_t(u)\eta(\mathrm{d}u)|\mathcal{F}_s\right) + \mathbb{1}_{\{\tau \leq s\}}\frac{1}{\alpha_s(\tau)}\left(\mathbb{E}(Y_t(\theta)\alpha_t(\theta)|\mathcal{F}_s)|_{\theta=\tau}\right). \qquad (22)
\end{aligned}$$

Assume that $Y_t^{\mathbb{G}} = Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}$ is a $\mathbb{G}$-martingale starting at τ with $Y_\tau = 0$, so that $Y_t(t) = 0$. Then, $\mathbb{E}(Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}|\mathcal{G}_s) = Y_s(\tau)\mathbb{1}_{\{\tau \leq s\}}$ and (22) implies that

$$\mathbb{E}(Y_t(\theta)\alpha_t(\theta)|\mathcal{F}_s) = Y_s(\theta)\alpha_s(\theta) \quad \text{for } t > s > \theta.$$

It follows that $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is a martingale.
Conversely, using the fact that $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-martingale, and that $Y_t(t) = 0$, (22) leads to

$$\begin{aligned}
\mathbb{E}(Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}|\mathcal{G}_s) &= \mathbb{1}_{\{s<\tau\}}\frac{1}{S_s}\int_s^t \mathbb{E}(Y_t(u)\alpha_t(u)|\mathcal{F}_s)\eta(\mathrm{d}u) + \mathbb{1}_{\{\tau \leq s\}}Y_s(\tau) \\
&= \mathbb{1}_{\{s<\tau\}}\frac{1}{S_s}\int_s^t \mathbb{E}(Y_t(u)\alpha_t(u)|\mathcal{F}_u|\mathcal{F}_s)\eta(\mathrm{d}u) + \mathbb{1}_{\{\tau \leq s\}}Y_s(\tau) \\
&= \mathbb{1}_{\{\tau \leq s\}}Y_s(\tau).
\end{aligned}$$

So $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale. □

Combining Propositions 5.1 and 5.5, we give the characterization of a general $\mathbb{G}$-martingale.

Proposition 5.6. *A càdlàg process $Y^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale if and only if there exist an $\mathbb{F}$-adapted càdlàg process Y and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-optional process $Y_t(.)$ such that $Y_t^{\mathbb{G}} = Y_t\mathbb{1}_{\{\tau>t\}} + Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}$ and*

(1) *the process $(Y_tS_t + \int_0^t Y_s(s)\alpha_s(s)\eta(\mathrm{d}s),\ t \geq 0)$ or equivalently $(L_t^{\mathbb{F}}[Y_t + \int_0^t (Y_s(s) - Y_s)\lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)], t \geq 0)$ is an $\mathbb{F}$-local martingale;*
(2) *for any $\theta > 0$, $(Y_t(\theta)\alpha_t(\theta), t \geq \theta)$ is an $\mathbb{F}$-local martingale on $[\theta, \zeta^\theta)$.*

Proof. Notice that $Y_t^{\mathrm{ad},\mathbb{G}} = (Y_t(\tau) - Y_\tau(\tau))\mathbb{1}_{\{\tau\le t\}}$. Then the theorem follows directly by applying Propositions 5.1 and 5.5 on $Y^{\mathrm{bd},\mathbb{G}}$ and $Y^{\mathrm{ad},\mathbb{G}}$ respectively. □

The above proposition allows for the following result. However, the converse is not always true except under some extra conditions.

Theorem 5.7. *Let $(Y_t(\theta), t \ge \theta)$ be a family of non-negative processes such that*

(1) *$(Y_t(\theta)\alpha_t(\theta), t \ge \theta)$ is an $\mathbb{F}$-martingale on $[\theta, \zeta^\theta)$,*
(2) *$\int_0^\infty Y_s(s)\alpha_s(s)\eta(\mathrm{d}s) < \infty$,*

and define Y as $Y_t S_t = \mathbb{E}[\int_t^\infty Y_s(s)\alpha_s(s)\eta(\mathrm{d}s)|\mathcal{F}_t]$. Then the process $Y_t^{\mathbb{G}} = Y_t\mathbb{1}_{\{t<\tau\}} + Y_t(\tau)\mathbb{1}_{\{t\ge\tau\}}$ is a $\mathbb{G}$-martingale.

Proof. From Proposition 5.6, the process $Y^{\mathbb{G}}$ is a positive $\mathbb{G}$-local martingale, hence a supermartingale. Since $\mathbb{E}[Y_t^{\mathbb{G}}] = \mathbb{E}[\int_0^\infty Y_s(s)\alpha_s(s)\eta(\mathrm{d}s)]$ is constant, the process $Y^{\mathbb{G}}$ is a martingale. □

In order to obtain the converse, one needs to add condition so that the martingale property of $Y_t S_t + \int_0^t Y_s(s)\alpha_s(s)\eta(\mathrm{d}s)$ implies that $Y_t S_t = \mathbb{E}[\int_t^\infty Y_s(s)\alpha_s(s)\eta(\mathrm{d}s)|\mathcal{F}_t]$, mainly some uniformly integrable conditions. Notice that if $Y_t(t)S_t = \mathbb{E}[\int_t^\infty Y_s(s)\alpha_s(s)\eta(\mathrm{d}s)|\mathcal{F}_t]$, then $Y^{\mathbb{G}}$ has no jump at τ.

Remark 5.8. We observe again the fact that to characterize what goes on before the default, it suffices to know the survival process S or the intensity $\lambda^{\mathbb{F}}$. However, for the after-default studies, we need the whole conditional distribution of τ, i.e., $\alpha_t(\theta)$ where $\theta \le t$.

5.2. Decomposition of $\mathbb{F}$-(local) martingale

An important result in the enlargement of filtration theory is the decomposition of $\mathbb{F}$-(local) martingales as $\mathbb{G}$-semimartingales. Using the above results, we provide an alternative proof for a result established in [12], simplified by using the fact that any $\mathbb{F}$-martingale is continuous at time τ. Our method is interesting, since it gives the intuition of the decomposition without using any result on enlargement of filtrations.

Proposition 5.9. *Any $\mathbb{F}$-martingale $Y^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale which can be written as $Y_t^{\mathbb{F}} = M_t^{Y,\mathbb{G}} + A_t^{Y,\mathbb{G}}$ where $M^{Y,\mathbb{G}}$ is a $\mathbb{G}$-martingale and $(A_t^{Y,\mathbb{G}} := A_t\mathbb{1}_{\{\tau>t\}} + A_t(\tau)\mathbb{1}_{\{\tau\le t\}}, t \ge 0)$ is an $\mathbb{G}$-optional process with finite variation. Moreover*

$$A_t = \int_0^t \frac{\mathrm{d}[Y^{\mathbb{F}}, S]_s}{S_s} \quad \text{and} \quad A_t(\theta) = \int_\theta^t \frac{\mathrm{d}[Y^{\mathbb{F}}, \alpha(\theta)]_s}{\alpha_s(\theta)} + A_\theta \tag{23}$$

where [,] denotes the co-variation process.

Remark 5.10. Note that our decomposition differs from the usual one, since our process A is optional (and not predictable) and that we are using the co-variation process, instead of the predictable co-variation process. As a consequence our decomposition is not unique.

Proof. On the one hand, assuming that $Y^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale, it can be decomposed as the sum of a $\mathbb{G}$-(local)martingale and a $\mathbb{G}$-optional process $A^{Y,\mathbb{G}}$ with finite variation which can be written as $A_t\mathbb{1}_{\{\tau>t\}} + A_t(\tau)\mathbb{1}_{\{\tau\le t\}}$ where A and $A(\theta)$ are still unknown. Note that, since $Y^{\mathbb{F}}$ has no jump at τ (indeed, τ avoids $\mathbb{F}$-stopping times — see Corollary 2.2), we can choose $M^{Y,\mathbb{G}}$ such that $M^{Y,\mathbb{G}}$ and hence $A^{Y,\mathbb{G}}$ have no jump at τ. Applying the martingale characterization result obtained in Proposition 5.6 to the $\mathbb{G}$-local martingale

$$Y_t^{\mathbb{F}} - A_t^{Y,\mathbb{G}} = (Y_t^{\mathbb{F}} - A_t)\mathbb{1}_{\{\tau>t\}} + (Y_t^{\mathbb{F}} - A_t(\tau))\mathbb{1}_{\{\tau\le t\}}$$

leads to the fact that the two processes

$$((Y_t^{\mathbb{F}} - A_t)L_t^{\mathbb{F}}, t \ge 0) \quad \text{and} \quad (\alpha_t(\theta)(Y_t^{\mathbb{F}} - A_t(\theta)), t \ge \theta)$$

are $\mathbb{F}$-(local) martingales. Since

$$\begin{aligned} \mathrm{d}\big((Y_t^{\mathbb{F}} - A_t)L_t^{\mathbb{F}}\big) &= (Y_{t-}^{\mathbb{F}} - A_{t-})\mathrm{d}L_t^{\mathbb{F}} + L_{t-}^{\mathbb{F}}\mathrm{d}(Y_t^{\mathbb{F}} - A_t) + \mathrm{d}\langle Y^{\mathbb{F}}, L^{\mathbb{F}}\rangle_t^c \\ &\quad + \Delta(Y_t^{\mathbb{F}} - A_t)\Delta L_t^{\mathbb{F}} \end{aligned}$$

and

$$-L_{t-}^{\mathbb{F}}\mathrm{d}A_t - \Delta A_t \Delta L_t^{\mathbb{F}} = -L_t^{\mathbb{F}}\mathrm{d}A_t,$$

one has

$$\mathrm{d}\big((Y_t^{\mathbb{F}} - A_t)L_t^{\mathbb{F}}\big) = (Y_{t-}^{\mathbb{F}} - A_{t-})\mathrm{d}L_t^{\mathbb{F}} + L_{t-}^{\mathbb{F}}\mathrm{d}Y_t^{\mathbb{F}} + \mathrm{d}[Y^{\mathbb{F}}, L^{\mathbb{F}}]_t - L_t^{\mathbb{F}}\mathrm{d}A_t.$$

Based on the intuition given by the Girsanov theorem and on the fact that $Y^{\mathbb{F}}$ and $L^{\mathbb{F}}$ are $\mathbb{F}$-local martingales, we find that a natural candidate for the finite variation processes A is $dA_t = \mathrm{d}[Y^{\mathbb{F}}, L^{\mathbb{F}}]_t / L_t^{\mathbb{F}}$. Since $L^{\mathbb{F}}$ is the product of S and a continuous increasing process $\mathrm{e}^{\Lambda^{\mathbb{F}}}$, we have $\mathrm{d}[Y^{\mathbb{F}}, L^{\mathbb{F}}]_t / L_t^{\mathbb{F}} = \mathrm{d}[Y^{\mathbb{F}}, S]_t / S_t$ and A satisfies the first equality in (23). Similarly, a natural candidate for the family $A(\theta)$ is $dA_t(\theta) = \mathrm{d}[Y^{\mathbb{F}}, \alpha(\theta)]_t / \alpha_t(\theta)$, so that $A(\theta)$ satisfies the second equality in (23).

On the other hand, Proposition 5.6 implies that $Y^{\mathbb{F}} - A^{Y,\mathbb{G}}$ is a $\mathbb{G}$-local martingale. It follows that $Y^{\mathbb{F}}$ is indeed a $\mathbb{G}$-semimartingale. $\square$

6. Change of probabilities

Change of probability measure is a key tool in derivative pricing as in martingale theory. In credit risk framework, we are also able to calculate parameters of the conditional distribution of the default time w.r.t. a new probability measure. The links between change of probability measure and the initial enlargement of filtrations have been established, in particular, in [11,10,1]. In statistics, it is motivated by the Bayesian approach [9].

6.1. Girsanov theorem

We present a general Girsanov type result. As in Theorem 5.7, we consider a positive $\mathbb{G}$-martingale and we shall add some normalization coefficient so that it is a martingale with expectation 1. The Radon–Nikodým density is given in an additive form instead of in a multiplicative one as in the classical literature, which makes the density of τ have a simple form under the new probability measure.

Theorem 6.1 (*Girsanov's Theorem*)**.** *Let $(\Omega, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ be a probability space and $(\alpha_t(\theta), t \ge 0)$, $\theta \ge 0$ be a family of density processes. Let $(\beta_t(\theta), t \ge \theta)$ be a family of càdlàg strictly positive martingales and define $\beta_t(\theta) = \mathbb{E}[\beta_\theta(\theta)|\mathcal{F}_t]$ for any $t < \theta$, $M_t^{\beta} := \int_0^{\infty} \beta_t(\theta)\eta(\mathrm{d}\theta)$ and assume $\mathbb{E}[M_t^{\beta}] < \infty$, (so that $\int_0^{\infty} \beta_t(\theta)/M_t^{\beta}\eta(\mathrm{d}\theta) = 1$).*

Let $q_t(\theta) := (\beta_t(\theta)/\alpha_t(\theta), t \geq \theta)$ *and define* $(q_t, t \geq 0)$ *by* $q_t S_t = \mathbb{E}\big[\int_t^\infty q_u(u)\alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t\big]$.

In particular, $q_0 = \mathbb{E}[\int_0^\infty q_u(u)\alpha_u(u)\eta(\mathrm{d}u)] = M_0^\beta$. *Let*

$$Q_t^{\mathbb{G}} := \frac{q_t}{M_0^\beta}\mathbb{1}_{\{\tau>t\}} + \frac{q_t(\tau)}{M_0^\beta}\mathbb{1}_{\{\tau\leq t\}}.$$

(1) *Then* $Q^{\mathbb{G}}$ *is a positive* $(\mathbb{G}, \mathbb{P})$*-martingale with expectation equal to* 1*, which defines a probability measure* $\mathbb{Q}$ *on* $(\Omega, \mathcal{A}, \mathbb{G})$ *equivalent to* $\mathbb{P}$ *and*

$$\alpha_t^{\mathbb{Q}}(\theta) = \frac{\beta_t(\theta)}{M_t^\beta}, \quad t \geq 0,\ \theta \geq 0 \tag{24}$$

is the $(\mathbb{F}, \mathbb{Q})$*-density process of* τ. *The restriction of* $\mathbb{Q}$ *to* $\mathbb{F}$ *admits the Radon–Nikodým density* $Q_t^{\mathbb{F}} = \mathbb{E}[Q_t^{\mathbb{G}}|\mathcal{F}_t] = \frac{M_t^\beta}{M_0^\beta}$.

(2) (a) *The* $\mathbb{Q}$*-conditional survival process is defined on* $[0, \zeta^{\mathbb{F}})$ *by*

$$S_t^{\mathbb{Q}} = S_t \frac{q_t}{M_t^\beta} = \frac{\mathbb{E}[\int_t^\infty \beta_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t]}{M_t^\beta},$$

and is null after $\zeta^{\mathbb{F}}$*;*

(2) (b) *the* $(\mathbb{F}, \mathbb{Q})$*-intensity process is* $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}} \frac{q_t(t)}{q_t}$, $\eta(\mathrm{d}t)$*-a.s.*

Proof. By construction, the process $Q^{\mathbb{G}}$ verifies the conditions in Theorem 5.7, then it is a positive $(\mathbb{G}, \mathbb{P})$-martingale with expectation equal to 1. So it can be taken as the Radon–Nikodým density of a new probability measure $\mathbb{Q}$ with respect to $\mathbb{P}$ on $\mathcal{G}_t$, i.e. $\mathrm{d}\mathbb{Q} = Q_t^{\mathbb{G}}\mathrm{d}\mathbb{P}$. For any positive function h,

$$\mathbb{E}_{\mathbb{Q}}[h(\tau)|\mathcal{F}_t] = \frac{1}{Q_t^{\mathbb{F}}}\mathbb{E}[Q_\tau^{\mathbb{G}} h(\tau)|\mathcal{F}_t] = \frac{1}{M_t^\beta}(H_t^1 + H_t^2)$$

where

$$H_t^1 = \mathbb{E}[\mathbb{1}_{\{\tau>t\}} h(\tau) q_\tau(\tau)|\mathcal{F}_t] = \mathbb{E}\left[\int_t^\infty h(u) q_u(u)\alpha_u(u)\eta(\mathrm{d}u)|\mathcal{F}_t\right],$$

$$H_t^2 = \mathbb{E}[\mathbb{1}_{\{\tau\leq t\}} h(\tau) q_t(\tau)|\mathcal{F}_t] = \int_0^t h(u) q_t(u)\alpha_t(u)\eta(\mathrm{d}u).$$

Since $\beta_t(u) = \mathbb{E}[q_u(u)\alpha_u(u)|\mathcal{F}_t]$ for $u \geq t$, one has $H_t^1 + H_t^2 = \int_0^\infty h(u)\beta_t(u)\eta(\mathrm{d}u)$. Consequently, the process $\frac{\beta_t(u)}{M_t^\beta} =: \alpha_t^{\mathbb{Q}}(u)$ is a $\mathbb{Q}$-martingale density of τ. By construction, $S_t^{\mathbb{Q}} = \frac{S_t q_t}{M_t^\beta}$ and the new intensity is $\lambda_t^{\mathbb{F},\mathbb{Q}} = \frac{\alpha_t^{\mathbb{Q}}(t)}{S_t^{\mathbb{Q}}} = \frac{\beta_t(t)}{S_t q_t} = \lambda_t^{\mathbb{F}} \frac{q_t(t)}{q_t}$. $\square$

Corollary 6.2. *The Radon–Nikodým density of any change of probability can be decomposed as* $Q_t^{\mathbb{G}} = Q_t^1 Q_t^2$ *where* Q^1 *is a pure jump* $\mathbb{G}$*-martingale with a single jump at* τ *and* Q^2 *is a* $\mathbb{G}$*-martingale continuous at* τ. *Hence* $\mathrm{d}\mathbb{Q} = Q_t^{\mathbb{G}}\mathrm{d}\mathbb{P} = Q_t^2\mathrm{d}\mathbb{Q}^1$ *on* $\mathcal{G}_t$ *where* $\mathrm{d}\mathbb{Q}^1 = Q_t^1\mathrm{d}\mathbb{P}$. *The change of probability* Q^1 *affects only the intensity and the intensity of* τ *is the same under* $\mathbb{Q}^1$ *and* $\mathbb{Q}$. *If in addition the immersion property holds under* $\mathbb{P}$*, then* $\mathbb{Q}^1|_{\mathbb{F}} = \mathbb{P}|_{\mathbb{F}}$.

Proof. Let u be an $\mathbb{F}$-optional process associated with the jump of $Q^{\mathbb{G}}$ such that $u_\tau = \frac{q_\tau(\tau)}{q_{\tau-}}$, and define Q^1 by (16) in the multiplicative form $Q^1_t = \big((u_\tau - 1)\mathbb{1}_{\{\tau \le t\}} + 1\big)\mathrm{e}^{-\int_0^{t\wedge\tau}(u_s-1)\lambda_s^{\mathbb{F},\mathbb{P}}\eta(\mathrm{d}s)}$. Then Q^1 is a pure jump martingale with the same jump as $Q^{\mathbb{G}}$ at τ and is of expectation 1. Under the probability $\mathbb{Q}^1$, the intensity is $\lambda_t^{\mathbb{F},\mathbb{Q}^1} = \lambda_t^{\mathbb{F},\mathbb{P}}\frac{q_t(t)}{q_t}$ by Theorem 6.1. The $\mathbb{G}$-martingale Q^2 is continuous at τ and the intensity under $\mathbb{Q}^1$ is hence preserved under the probability $\mathbb{Q}$, i.e. $\lambda_t^{\mathbb{F},\mathbb{Q}^1} = \lambda_t^{\mathbb{F},\mathbb{Q}}$. If the immersion property holds under $\mathbb{P}$, one can verify that $Q_t^{1,\mathbb{F}} = \mathbb{E}[Q^1_t|\mathcal{F}_t] = 1$. Hence $\mathbb{Q}^1|_{\mathbb{F}} = \mathbb{P}|_{\mathbb{F}}$. □

6.2. Applications

6.2.1. Immersion property

It is known, from [10], that under density hypothesis, there exists at least a change of probability, such that the immersion property holds under this change of probability. Theorem 6.1 provides a full characterization of such changes of probability.

Proposition 6.3. *We keep the notation of Theorem 6.1. The immersion property holds true under $\mathbb{Q}$ if and only if*

$$q_t(\theta) = \frac{1}{\alpha_t(\theta)}\frac{\beta_\theta(\theta)}{M_\theta^\beta}M_t^\beta, \quad t \ge \theta. \tag{25}$$

(1) *In particular, if $\mathbb{Q}^0$ is the probability measure such that $q_t(\theta) = \frac{M_t^\beta}{\alpha_t(\theta)}$ (and $q_t = M_t^\beta$), then the random variable τ is independent of $\mathbb{F}$ under $\mathbb{Q}^0$.*

(2) *In addition, $\mathbb{Q}$ is a probability measure under which the immersion property holds and the intensity process does not change if and only if (25) holds and $qL^{\mathbb{F}}$ is a uniformly integrable $\mathbb{F}$-martingale.*

Proof. The immersion property holds true under $\mathbb{Q}$ if and only if $\alpha_t^{\mathbb{Q}}(\theta) = \alpha_\theta^{\mathbb{Q}}(\theta)$ for any $t \ge \theta$. By (24), this is equivalent to that $\frac{\beta_t(\theta)}{M_t^\beta}$ remains constant, i.e., $\frac{\beta_t(\theta)}{M_t^\beta} = \frac{\beta_\theta(\theta)}{M_\theta^\beta}$, which is exactly (25). The particular case (1) holds true with $\frac{\beta_t(\theta)}{M_t^\beta} = 1$. In addition, $\alpha_t^{\mathbb{Q}}(\theta)$ equals constant 1, which means that τ is independent of $\mathbb{F}$ under $\mathbb{Q}^0$.

(2) If $\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}}$, then $q_t(t) = q(t)$. Since $Q^{\mathbb{G}}$ is a $\mathbb{G}$-martingale, we know that $qL^{\mathbb{F}}$ is an $\mathbb{F}$-martingale by Proposition 5.1. For the reverse, if $qL^{\mathbb{F}} = qS\mathrm{e}^{\Lambda}$ where $\Lambda_t = \int_0^t \lambda_s^{\mathbb{F}}\eta(\mathrm{d}s)$ is an $\mathbb{F}$-martingale, by a similar argument as in the proof of Proposition 4.1, one can verify that $(q_tS_t + \int_0^t q_sS_s\lambda_s^{\mathbb{F}}\eta(\mathrm{d}s), t \ge 0)$ is an $\mathbb{F}$-martingale. On the other hand, by definition, $(q_tS_t + \int_0^t \beta_s(s)\eta(\mathrm{d}s), t \ge 0)$ is an $\mathbb{F}$-martingale. By the uniqueness of the decomposition of the supermartingale qS, we obtain $\beta_s(s) = q_sS_s\lambda_s^{\mathbb{F}}$, $\eta(\mathrm{d}s)$-a.s., so $q_s(s) = q_s$. □

It is well known, from Kusuoka [14], that the immersion property is not stable by a change of probability measure. In the following, we shall characterize, under the density hypothesis, changes of probability which preserve this immersion property, that is, H-hypothesis is satisfied under both $\mathbb{P}$ and $\mathbb{Q}$. (See also [6] for a different study of changes of probabilities preserving the immersion property.)

Corollary 6.4. *We keep the notation of Theorem 6.1. Assume in addition that the immersion property holds under $\mathbb{P}$. The only changes of probability measure which preserve the immersion property have Radon–Nikodým densities that are the product of a pure jump positive martingale with only one jump at time τ, and a positive $\mathbb{F}$-martingale.*

Proof. Let the Radon–Nikodým density $(Q_t^{\mathbb{G}}, t \geq 0)$ be a pure jump martingale with only one jump at time τ. Then the immersion property still holds under $\mathbb{Q}$. So we can restrict our attention to the case when in the both universes the intensity processes are the same. Then the Radon–Nikodým density is continuous at time τ and the processes $(q_t, t \geq 0)$ and $(q_t(\theta), t \geq \theta)$ are $\mathbb{F}$-(local) martingales.

Assume now that the immersion property holds also under the new probability measure $\mathbb{Q}$. Both martingales $L^{\mathbb{F},\mathbb{P}}$ and $L^{\mathbb{F},\mathbb{Q}}$ are constant, and $Q_t^{\mathbb{F}} = q_t$. Moreover the $\mathbb{Q}$-density process being constant after the default, i.e. when $\theta < t$, $q_t(\theta)/q_t = q_\theta(\theta)/q_\theta = 1, a.s.$ The processes $Q^{\mathbb{G}}$, $Q^{\mathbb{F}}$ and q are indistinguishable. $\square$

6.2.2. Probability measures that coincide on $\mathcal{G}_\tau$

We finally study changes of probability which preserve the information before the default, and give the impact of a change of probability after the default.

As shown in this paper, the knowledge of the intensity does not allow to give full information on the conditional law of the default, except if the immersion property holds. Starting with a model under which the immersion property holds, taking $q_t(t) = q_t$ in Theorem 6.1 will lead us to a model where the default time admits the same intensity whereas the immersion property does not hold, and then the impact of the default changes the dynamics of the default-free assets.

We shall present a specific case where, under the two probability measures, the dynamics of these assets are the same before the default but are changed after the default, a phenomenon that is observed in the actual crisis. We impose that the new probability $\mathbb{Q}$ coincide with $\mathbb{P}$ on the σ-algebra $\mathcal{G}_\tau$. In particular, if m is an $(\mathbb{F}, \mathbb{P})$-martingale, the process $(m_{t\wedge\tau}, t \geq 0)$ will be an $((\mathcal{G}_{t\wedge\tau})_{t\geq 0}, \mathbb{Q})$-martingale (but not necessarily an $(\mathbb{G}, \mathbb{Q})$-martingale).

Proposition 6.5. *Let $(q_t(\theta), t \geq \theta)$ be a family of positive $(\mathbb{F}, \mathbb{P})$-martingales such that $q_\theta(\theta) = 1$ and let $\mathbb{Q}$ be the probability measure with Radon–Nikodým density equal to the $(\mathbb{G}, \mathbb{P})$-martingale*

$$Q_t^{\mathbb{G}} = \mathbb{1}_{\{\tau > t\}} + q_t(\tau)\mathbb{1}_{\{\tau \leq t\}}\,. \tag{26}$$

Then, $\mathbb{Q}$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$ and the $\mathbb{P}$ and $\mathbb{Q}$ intensities of τ are the same.

Furthermore, if $S^{\mathbb{Q}}$ is the $\mathbb{Q}$-survival process, the processes $(S_t/S_t^{\mathbb{Q}}, t \geq 0)$ and the family $(\alpha_t^{\mathbb{Q}}(\theta)\, S_t/S_t^{\mathbb{Q}},\ t \geq \theta)$ are $(\mathbb{F}, \mathbb{P})$-martingales.

Proof. The first part is a direct consequence of Theorem 6.1. It remains to note that $S_t/S_t^{\mathbb{Q}} = M_t^{\beta}$ and $\alpha_t^{\mathbb{Q}}(\theta)S_t/S_t^{\mathbb{Q}} = \beta_t(\theta)$. $\square$

7. Conclusion

Our study relies on the impact of information related to the default time on the market.

Starting from a default-free model, where some assets are traded with the knowledge of a reference filtration $\mathbb{F}$, we consider the case where the participants of the market take into account the possibility of a default in view of trading default-sensitive asset. If we are only concerned by what happens up to the default time, the natural assumption is to assume the immersion property with a stochastic intensity process adapted to the default-free market evolution.

The final step is to anticipate that the default should have a large impact on the market, as now after the crisis. In particular, with the nonconstant "after-default" density, we express how the default-free market is modified after the default. In addition, hedging strategies of default-free contingent claims are not the same in the both universes.

In a following paper [7], we shall apply this methodology to several default times, making this tool powerful for correlation of defaults. In another paper, we shall provide explicit examples of density processes, and give some general construction of these processes.

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