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Alpha-CIR model with branching processes in sovereign interest rate modeling

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Abstract We introduce a class of interest rate models, called the α-CIR model, which is a natural extension of the standard CIR model by adding a jump part driven by α-stable Lévy processes with index $\alpha \in (1, 2]$. We deduce an explicit expression for the bond price by using the fact that the model belongs to the family of CBI and affine processes, and analyze the bond price and bond yield behaviors. The α-CIR model allows us to describe in a unified and parsimonious way several recent observations on the sovereign bond market such as the persistency of low interest rates together with the presence of large jumps. Finally, we provide a thorough analysis of the jumps, and in particular the large jumps.

Keywords α-Stable Lévy process · CBI process · Affine term structure model · Low interest rate · Sovereign bond

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1 Introduction

On the current European sovereign bond market, there exists a number of well-established and seemingly puzzling facts. On the one hand, the interest rate has reached a historically low level in the Euro countries. On the other hand, sovereign bonds can have large variations when uncertainty about unpredictable political or economic events increases such as in the Greek case. The aim of this paper is to present a new model for the short interest rate, called the α-CIR model, in which we give a natural extension of the well-known Cox–Ingersoll–Ross (CIR, see [7]) model by using α-stable branching processes in order to describe these recent observations on the bond market.

In the literature, large fluctuations in financial data motivate naturally the introduction of jumps in the interest rate dynamics such as in Eberlein and Raible [14], Filipović et al. [20]. Nevertheless, the presence of jumps conflicts in general with the trend of low interest rates. One way to reconcile large fluctuations with low rates is to use a regime change framework, but this may increase the dimension of stochastic processes in order to preserve the Markov property. Recently, Hawkes processes introduced in [23] have been widely adopted since they exhibit self-exciting properties which are suitable for such modeling. A large and growing literature is devoted to the financial application of Hawkes processes; see, for example, Aït-Sahalia et al. [1], Errais et al. [16], Dassios and Zhao [8], and Rambaldi et al. [33]. In the above mentioned papers, as naturally in the Hawkes framework, the driving process is at least two-dimensional since both the dynamics of the jump process and its intensity are taken into account.

In this paper, we introduce a short interest rate model by using the α-stable Lévy process, which provides a relatively simple jump model to respond to these modeling challenges in a concise way. The α-CIR model consists of a spectrally positive α-stable Lévy process besides the Brownian motion, where the parameter $\alpha \in (1, 2]$ characterizes the tail fatness and the jump behavior. When α equals 2, the α-stable process reduces to a Brownian motion and we recover the classic CIR model. In the general case when $\alpha \in (1, 2)$, there may appear infinitely many jumps in a finite time interval, which represent the fluctuations related to sovereign risks. We exploit an integral representation of the α-CIR model with random fields. From the theoretical point of view, this general representation has been thoroughly studied by Dawson and Li [9, 10] and Li and Ma [32] in the framework of CBI (continuous state branching processes with immigration) processes. In the financial literature, the random field modeling has been adopted to describe interest rate term structures; see, for example, Kennedy [30] and Albeverio et al. [3]. In our model, we adopt the integral representation to emphasize the property of branching processes since they arise as the limit of Hawkes processes and exhibit, by their nature, the self-exciting property implying that the jump frequency increases or decreases with the value of the process itself. In the modeling of interest rates, the link between CBI processes and nonnegative affine models has been established by the pioneering paper of Filipović [18] where the exponential-affine term structure of bond prices for general CBI processes has been highlighted. The CBI processes have proved to be a prolific subject in probability having interesting applications in finance; see, for instance, Duffie et al. [11].

The most simple and popular CBI process is the continuous CIR model. However, empirical studies underline that the behavior of bond prices cannot be fully explained by the CIR model which systematically overestimates short interest rates, as pointed out by Brown and Dybvig [6] and Gibbons and Ramaswamy [22]. In our framework, the CIR model is the departing model, and the inclusion of α-stable processes allows us to better describe the low interest rate behavior.

Despite the simplicity and the small number of extra parameters compared to the standard CIR, the α-CIR model shows several advantages from the financial point of view. First, the α-CIR model exhibits positive jumps and in particular, by combining a heavy-tailed jump distribution with infinite activity, can describe in a unified way both the large fluctuations observed in recent sovereign bond markets and the usual small oscillations. Second, the interest rate can be split into different components in a branching process framework which can eventually be interpreted as spreads, each one following the same dynamics. Third, by the link between the α-CIR model and the CBI processes, we deduce the bond prices in an explicit way by using the joint Laplace transform of the affine model in Filipović [18], and we analyze the bond yield behaviors following the paper of Keller-Ressel and Steiner [29].

The main, and perhaps most interesting, forecast of the α-CIR model is that the bond prices decrease with respect to the parameter α, with those given by the standard CIR model being the lowest prices. The parameter α is inversely related to the tail fatness. In general, the standard behavior of bond prices increases with respect to the fatness of tails, as is the case in the extended CIR model with jumps in Duffie and Gârleanu [12] or in the Lévy–Ornstein–Uhlenbeck (LOU) dynamics (e.g. Barndorff-Nielsen and Shephard [5]) in which the jump part is a subordinator. The explanation of this seemingly paradoxical result is based on the features of the α-CIR model. The use of fat-tail-distributed positive jumps will imply a large negative compensator so that between two jumps, the mean reversion term is magnified whenever α decreases. Moreover, for a given value of α, the branching property adds a new phenomenon in the α-CIR model. The frequency of large jumps decreases when interest rates are low thanks to the self-exciting structure which allows some “freezing” effect of low short rates for relatively longer time periods compared to the standard CIR model.

From the mathematical point of view, we use the CBI characterization to deduce some useful properties of the α-CIR model such as the positivity condition and the limit distribution. We are particularly interested in the jump behavior, notably for the large jumps which signify in the interest rate dynamics a sudden increasing sovereign risk. We focus on the number of large jumps that occur during a given time interval and deduce its Laplace transform, with which we obtain the probability law and the expectation for the first large jump time. The impact of the tail index α is emphasized. Numerical illustrations show that the first large jump is more likely to occur for a smaller α. In addition, we make a comparison with a locally equivalent CIR model with jumps in which the jump frequency is not adapted to the actual level of the interest rate, but is fixed according to the initial short rate value.

The paper is organized as follows. Section 2 presents the mathematical framework of the α-CIR model and its connections to the Hawkes process. Section 3 is devoted to the characterization of the model as a CBI process and hence as an affine process, and the properties derived from this link. In Sect. 4, we apply our model to term structure modeling and present in particular the closed-form bond price and its behaviors. Section 5 deals with the analysis of jumps.

2 Model framework

This section introduces the α-CIR interest rate model and its basic properties. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

2.1 Two representations for the α-CIR model

We begin by presenting the following root representation for the short interest rate $r = (r_t)_{t\geq 0}$, which is a direct extension of the standard CIR model as

$$r_t = r_0 + \int_0^t a(b - r_s)\,ds + \sigma \int_0^t \sqrt{r_s}\,dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha}\,dZ_s, \tag{2.1}$$

where $B = (B_t)_{t\geq 0}$ is a Brownian motion and $Z = (Z_t)_{t\geq 0}$ a spectrally positive α-stable compensated Lévy process with parameter $\alpha \in (1, 2]$ which is independent of B and whose Laplace transform is given, for $q \geq 0$, by

$$\mathbb{E}[e^{-qZ_t}] = \exp\left(-\frac{tq^\alpha}{\cos(\pi\alpha/2)}\right).$$

When $\alpha \in (1, 2)$, the corresponding Lévy measure is given by

$$-\frac{1_{\{z>0\}}\,dz}{\cos(\pi\alpha/2)\Gamma(-\alpha)z^{1+\alpha}}.$$

In other words, Z_t follows the α-stable distribution with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift, i.e., $Z_t \sim S_\alpha(t^{1/\alpha}, 1, 0)$.

The existence of a unique strong solution to (2.1) follows from Fu and Li [21, Theorem 5.3]. We call the process defined by (2.1) the α-CIR process with parameters $(a, b, \sigma, \sigma_Z, \alpha)$ and denote it by α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$.

It is easy to see that the CIR model belongs to the class (2.1) by taking $\sigma_Z = 0$. Another case where we recover a CIR process is when $\alpha = 2$. In this case, the process Z becomes a standard Brownian motion scaled by the coefficient $\sqrt{2}$ which is independent of B. Hence an α-CIR process satisfying (2.1) is actually a CIR process of the form

$$r_t = r_0 + \int_0^t a(b - r_s)\,ds + \sqrt{\sigma^2 + 2\sigma_Z^2} \int_0^t \sqrt{r_s}\,d\widetilde{B}_s,$$

where $\widetilde{B} = (\sigma B + \sigma_Z Z)/\sqrt{\sigma^2 + 2\sigma_Z^2}$ is a standard Brownian motion.

The departure of the process Z from Brownian motion is controlled by the tail index α. When $\alpha < 2$, Z is a pure jump process with heavy tails. For any fixed t, the

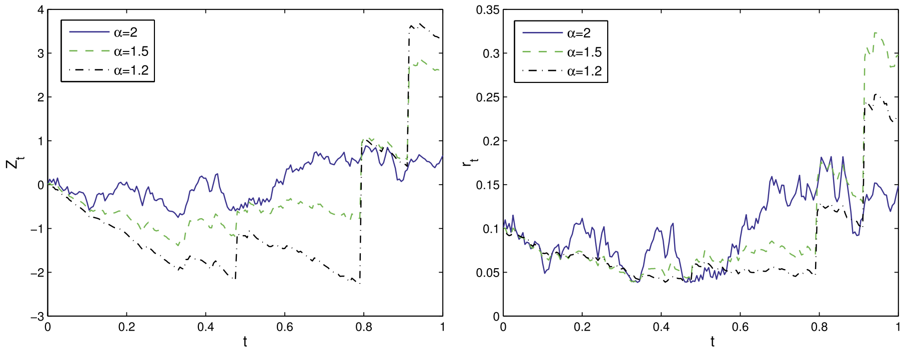


Fig. 1 Lévy process Z (left) and the corresponding short rate r (right) with different values of α: blue line for $\alpha = 2$, green line for $\alpha = 1.5$, black line for $\alpha = 1.2$

distribution of Z_t is a stable distribution and the tail of the distribution decays like a power function with index $-\alpha$. This means that a stable random variable exhibits more variability than a Gaussian one and it is more likely to take values far away from the median. Compared to a standard Poisson or compound Poisson process, this pure jump process has an infinite number of (small) jumps over any time interval, allowing it to capture the extreme activity. At the same time α-stable processes share some properties with Brownian motion such as self-similarity or the stability property, which means that the distribution of the α-stable process over any horizon has the same shape upon scaling. From the statistical point of view, the process given by (2.1) is characterized by two more parameters with respect to the CIR model, namely α and σ_Z.

Figure 1 gives a simulation for the compensated α-stable process Z and the corresponding short interest rate r defined in (2.1) with three different values of α: 2, 1.5, and 1.2. The other parameters are fixed to be $a = 0.1$, $b = 0.3$, $\sigma = 0.1$, $\sigma_Z = 0.3$, and $r_0 = 0.1$. We observe that smaller values of α imply larger jumps and deeper negative drift between the jumps in Z. As the jumps are related to the actual level of the interest rate r (see more details in Sect. 2.2), smaller values of α also correspond to a persistency of low interest rates, as shown by the path of $\alpha = 1.2$ in Fig. 1 (right).

We then introduce a general integral representation for the α-CIR model by using random fields; see, for instance, [15]. Let us consider for any $t \geq 0$ the equation

$$r_t = r_0 + \int_0^t a(b - r_s)\, ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_{\mathbb{R}_+} \zeta \widetilde{N}(ds, du, d\zeta), \tag{2.2}$$

where $W(ds, du)$ is a white noise on $\mathbb{R}_+^2$ with intensity $ds\, du$, $\widetilde{N}(ds, du, d\zeta)$ is an independent compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $ds\, du\, \mu(d\zeta)$ with $\mu(d\zeta)$ being a Lévy measure on $\mathbb{R}_+$ and satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < \infty$. It follows from Dawson and Li [10, Theorem 3.1] or Li and Ma [32, Theorem 2.1] that (2.2) has a unique strong solution. We call the process given by (2.2) the α-CIR integral type process with parameters $(a, b, \sigma, \sigma_Z, \mu)$.

When the Lévy measure μ is given by

$$\mu_\alpha(d\zeta) = -\frac{1_{\{\zeta>0\}}\,d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1<\alpha<2, \tag{2.3}$$

the solution of (2.2) has the same probability law as that of (2.1). Moreover, on an extended probability space, the solutions of the two equations are equal almost surely by using similar arguments as in [31, Theorem 9.32].

2.2 Link to Hawkes process

We explain the connection of the α-CIR model to Hawkes processes and the related self-exciting property. We begin by considering a standard CIR model with integral representation. Let $W(ds,du)$ be a white noise on $\mathbb{R}_+^2$ with intensity $ds\,du$. The CIR process r, when $\sigma_Z = 0$, is given in the form

$$r_t = r_0 + \int_0^t a(b-r_s)\,ds + \sigma\int_0^t\int_0^{r_s} W(ds,du),$$

or equivalently as

$$r_t = r^*(t) + \sigma\int_0^t\int_0^{r_s} e^{-a(t-s)}W(ds,du), \tag{2.4}$$

where $r^*(t)$ is a deterministic function given by $r^*(t) = r_0e^{-at} + ab\int_0^t e^{-a(t-s)}\,ds$. The expression (2.4), where the integral interval depends on the level of r, shows the self-exciting feature.

We then consider a simple Hawkes process with exponential kernel, which is defined as a point process J with intensity r given by

$$r_t = r^*(t) + \int_0^t e^{-a(t-s)}\,dJ_s,$$

where r^* is the background rate, i.e., the deterministic part of the process J. When a jump arrives, the intensity r increases, which also increases the probability of a next jump; this is the self-exciting property of the Hawkes process. In order to facilitate the comparison with our integral representation, we give a different characterization of the intensity r. Let N be a Poisson process on $\mathbb{R}^2$ with characteristic measure $ds\,du$, so J_t can be written in the form $\int_0^t\int_0^{r_{s-}} N(ds,du)$ and r_t as

$$r_t = r^*(t) + \int_0^t\int_0^{r_{s-}} e^{-a(t-s)}N(ds,du).$$

In this form, the self-exciting feature can be observed as follows: the frequency of jumps grows with the process itself due to the presence of the integral with respect to the variable u. Moreover, when r^* takes a particular form, r is a branching process which belongs to the family of affine processes in finance (see Duffie et al. [11]).

Let us now come back to the integral representation (2.2) of the α-CIR model. We let $\sigma = 0$ and $\mu(d\zeta) = \delta_1(dz)$; then the (non-compensated) Poisson measure

$N(ds, du, d\zeta)$ reduces to a random measure on $\mathbb{R}_+^2$ with intensity $ds\,du$, denoted by $N(ds, du)$. Hence r can be rewritten as

$$r_t = r_0 + abt - \int_0^t (a + \sigma_Z) r_s \, ds + \sigma_Z \int_0^t \int_0^{r_{s-}} N(ds, du).$$

We note that r is the intensity of the Hawkes process $\int_0^t \int_0^{r_{s-}} N(ds, du)$ by using the equivalent form

$$r_t = r_0 e^{-(a+\sigma_Z)t} + \frac{ab}{a+\sigma_Z}(1 - e^{(a+\sigma_Z)t}) + \int_0^t \int_0^{r_{s-}} e^{-(a+\sigma_Z)(t-s)} N(ds, du). \quad (2.5)$$

As a consequence, α-CIR integral type processes, and in particular the α-CIR processes, can be seen as marked Hawkes processes influenced by a Brownian noise.

Furthermore consider a sequence of processes $(r_t^{(n)})_{t\geq 0}$ defined by (2.5) with parameters $(a/n, nb, \sigma_Z)$. Note that as $n \to \infty$, we have

$$(r_{nt}^{(n)}/n) \xrightarrow{\mathcal{L}} (Y_t) \quad \text{in } D(\mathbb{R}_+),$$

where Y follows a CIR model given by $Y_t = \int_0^t a(b - Y_s)\, ds + \sigma_Z \int_0^t \int_0^{Y_s} W(ds, du)$ and $D(\mathbb{R}_+)$ denotes the space of càdlàg functions with the Skorokhod topology. Therefore, a sequence of rescaled Hawkes processes converges weakly to the CIR process; see Jaisson and Rosenbaum [25] for more details, notably on the convergence of the nearly unstable Hawkes process with general kernel, after suitable rescaling, to a CIR process.

3 CBI characterization

In this section, we first recall some important results on CBI processes from Dawson and Li [10] and their link to affine term structure models from Filipović [18]. We then use the CBI characterization to deduce the positivity condition and the limit distribution properties of the α-CIR model.

3.1 Recall on CBI processes and affine property

The CBI processes have been introduced by Kawazu and Watanabe [27]. A Markov process X with state space $\mathbb{R}_+$ is called a *continuous state branching process with immigration*, characterized by a branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$, if its characteristic representation is given, for $p \geq 0$, by

$$\mathbb{E}_x[e^{-pX_t}] = \exp\left(-xv(t, p) - \int_0^t \Phi\big(v(s, p)\big)\, ds \right), \quad (3.1)$$

where the function $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies the differential equation

$$\frac{\partial v(t, p)}{\partial t} = -\Psi\big(v(t, p)\big), \qquad v(0, p) = p, \quad (3.2)$$

and Ψ and Φ are functions of the variable $q \geq 0$ given by

$$\Psi(q) = \beta q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du),$$

$$\Phi(q) = \gamma q + \int_0^\infty (1 - e^{-qu})\nu(du),$$

with $\sigma, \gamma \geq 0$ and $\beta \in \mathbb{R}$. In addition, π and ν are two Lévy measures such that $\int_0^\infty (u \wedge u^2)\pi(du) < \infty$ and $\int_0^\infty (1 \wedge u)\nu(du) < \infty$.

The α-CIR model (2.1) is a CBI process by considering its integral representation.

Proposition 3.1 (*Dawson and Li* [10, *Theorem* 3.1].) *The α-CIR integral type process r defined in* (2.2) *is a CBI process with the branching mechanism Ψ given by*

$$\Psi(q) = aq + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-q\sigma_Z\zeta} - 1 + q\sigma_Z\zeta)\mu(d\zeta) \tag{3.3}$$

and the immigration rate $\Phi(q) = abq$.

As a consequence, with the Lévy measure (2.3), the short rate r defined in (2.1) is a CBI process with the branching mechanism given by

$$\Psi_\alpha(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha \tag{3.4}$$

and the immigration rate given by $\Phi(q) = abq$.

Remark 3.2 The branching property can be interpreted in the following pathwise sense as in [10, Theorem 3.2]. Let $r_0^{(i)} \in \mathbb{R}_+$ and $b^{(i)} \in \mathbb{R}$, $i \in \{1, 2\}$, be such that $r_0 = r_0^{(1)} + r_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$. Then there exist independent processes $r^{(i)}$ in the families α-CIR$(a, b^{(i)}, \sigma, \sigma_Z, \alpha)$ with initial values $r_0^{(i)}$ such that $r = r^{(1)} + r^{(2)}$.

The link between CBI processes and affine term structure models has been established in [18] (see also [19]). We recall the joint Laplace transform of a CBI process and its integrated process, which has an exponential-affine structure depending on an auxiliary function which solves a first order nonlinear ordinary differential equation (ODE). This result will be useful for the bond pricing in the next section.

Proposition 3.3 (*Filipović* [18, *Theorem* 5.3]) *Let X be a CBI* (Ψ, Φ) *process given by* (3.1) *with $X_0 = x$. For nonnegative real numbers ξ and θ, we have*

$$\mathbb{E}_x\big[e^{-\xi X_t - \theta\int_0^t X_s\,ds}\big] = \exp\bigg(-xv(t,\xi,\theta) - \int_0^t \Phi\big(v(s,\xi,\theta)\big)\,ds\bigg), \tag{3.5}$$

where $v(t,\xi,\theta)$ is the unique solution of

$$\frac{\partial v(t,\xi,\theta)}{\partial t} = -\Psi\big(v(t,\xi,\theta)\big) + \theta, \qquad v(0,\xi,\theta) = \xi.$$

3.2 Positivity and limit distribution of the α-CIR model

In the rest of this section, we use the CBI characterization to deduce some properties of the α-CIR model. First, we show that the usual condition of inaccessibility of the point 0 is preserved when we extend the CIR to an α-CIR model.

Proposition 3.4 *For the α-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process with $\alpha \in (1, 2)$, the point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0.*

Proof We apply the result of Duhalde et al. [13, Theorem 2] for CBI processes to obtain that 0 is an inaccessible boundary point for an α-CIR integral type process if and only if

$$\int_\theta^\infty \frac{dz}{\Psi(z)} \exp\left(\int_\theta^z \frac{\Phi(x)}{\Psi(x)}\,dx\right) = \infty$$

for some positive constant θ, where Ψ is given by (3.3) and $\Phi(q) = abq$. We now focus on the α-CIR process. Let $\Psi^*(q) = aq + \sigma^2 q^2/2$ be the branching mechanism of the classical CIR process viewed as a CBI process. We have $\Psi_\alpha \geq \Psi^*$, where Ψ_α is the branching mechanism of the α-CIR process, given in (3.4). Therefore

$$\int_\theta^\infty \frac{dz}{\Psi_\alpha(z)} \exp\left(\int_\theta^z \frac{\Phi(x)}{\Psi_\alpha(x)}\,dx\right) \leq \int_\theta^\infty \frac{dz}{\Psi^*(z)} \exp\left(\int_\theta^z \frac{\Phi(x)}{\Psi^*(x)}\,dx\right).$$

In particular, if 0 is an inaccessible boundary for the α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, then the inequality $2ab \geq \sigma^2$ holds, thanks to the classical inaccessibility criterion for the CIR process.

Conversely, if the inequality $2ab \geq \sigma^2$ holds, then one has

$$\frac{\Phi(x)}{\Psi_\alpha(x)} \geq \frac{1}{x}\left(1 + O(x^{\alpha-2})\right) \quad \text{as } x \to \infty.$$

So there exists some constant $C > 0$ (depending on θ) such that

$$\int_\theta^z \frac{\Phi(x)}{\Psi_\alpha(x)}\,dx \geq \log(z/\theta) - C.$$

Hence

$$\int_\theta^\infty \frac{dz}{\Psi_\alpha(z)} \exp\left(\int_\theta^z \frac{\Phi(x)}{\Psi_\alpha(x)}\,dx\right) \geqslant \frac{1}{e^C\theta} \int_\theta^\infty \frac{z}{\Psi_\alpha(z)}\,dz = \infty.$$

□

Remark 3.5 The result of Proposition 3.4 is not true when $\alpha = 2$. In this case, the α-CIR model reduces to a classic CIR model, but with a modified volatility term. Therefore for the α-CIR$(a, b, \sigma, \sigma_Z, 2)$ process, the point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2 + 2\sigma_Z^2$. We note that when the α-CIR process contains the jump part, i.e., when $\alpha < 2$, the parameter σ_Z does not intervene in the boundary condition.

The next result shows that the α-CIR process converges to the standard CIR model as α tends to 2.

Proposition 3.6 *Let $r^{(\alpha)} = (r_t^{(\alpha)})_{t\geq 0}$ denote the α-CIR process with parameters $(a, b, \sigma, \sigma_Z, \alpha)$. Then as $\alpha \to 2$, $r^{(\alpha)}$ converges in distribution on $D(\mathbb{R}_+)$ to the CIR process $r^{(2)}$.*

Proof The generator of a CBI process X is the operator $\mathcal{L}$ acting on $C_0^2(\mathbb{R}_+)$ given by

$$\mathcal{L}f(x) = \frac{\sigma^2}{2}xf''(x) + (\gamma - \beta x)f'(x) + x\int_0^\infty \big(f(x+u) - f(x) - uf'(x)\big)\pi(du) + \int_0^\infty \big(f(x+u) - f(x)\big)\nu(du). \tag{3.6}$$

Let $P^{(\alpha)}$ be the transition semigroup of the CBI process $r^{(\alpha)}$ and $A^{(\alpha)}$ its generator. Denote $e_p(x) = e^{-px}$ for $p > 0$ and $x \geq 0$. Then by (3.6),

$$\begin{aligned} A^{(\alpha)}e_p(x) &= -e_p(x)\big(x\Psi_\alpha(p) + \Phi(p)\big) \\ &= -e^{-px}\left(x\left(ap + \frac{\sigma^2}{2}p^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}p^\alpha\right) + abp\right). \end{aligned}$$

We have

$$\lim_{\alpha\to 2}\sup_{x\in\mathbb{R}_+} |A^{(\alpha)}e_p(x) - A^{(2)}e_p(x)| = 0.$$

Denote by D_1 the linear hull of $\{e_p : p > 0\}$. Then D_1 is an algebra which strongly separates the points of $\mathbb{R}_+$. Let $C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ vanishing at infinity. By the Stone–Weierstrass theorem, D_1 is dense in $C_0(\mathbb{R}_+)$. Since D_1 is invariant under $P^{(2)}$ by (3.1), it is a core of $A^{(2)}$ by Ethier and Kurtz [17, Proposition 3.3]. Then using [17, Corollary 8.7], we have the weak convergence of the processes as α tends to 2. □

Finally, we characterize the ergodic distribution of the α-CIR process. Note that the first part of the following result was also shown in Keller-Ressel and Steiner [29, Theorem 3.16] (see also [28]).

Proposition 3.7 *The α-CIR integral type process defined in (2.2) has a limit distribution, whose Laplace transform is given by*

$$\mathbb{E}[e^{-pr_\infty}] = \exp\left(-\int_0^p \frac{\Phi(q)}{\Psi(q)}dq\right), \quad p \geq 0. \tag{3.7}$$

Moreover, the process is exponentially ergodic, namely

$$\|\mathbb{P}[r_t \in \cdot\,] - \mathbb{P}[r_\infty \in \cdot\,]\| \leqslant C\rho^t$$

for some positive constants C and $\rho < 1$, where $\|\cdot\|$ denotes the total variation norm.

Proof The branching mechanism Ψ is bounded from below by $aq + \frac{1}{2}\sigma^2 q^2$. Hence one has

$$\int_0^1 \frac{\Phi(q)}{\Psi(q)}\,dq \leqslant \int_0^1 \frac{abq}{aq+\frac{1}{2}\sigma^2 q^2}\,dq < \infty.$$

By [31, Theorem 3.20], we obtain that the process r defined in (2.2) has a limit distribution, whose Laplace transform is given by $\exp(-\int_0^\infty \Phi(v(t,p))\,dt)$, where the function v is defined in (3.2). A change of variables $q = v(t,p)$ in the above formula leads to (3.7). The last assertion follows from [32, Theorem 2.5]. □

4 Application to bond pricing

In this section, we apply the α-CIR model to interest rate modeling and pricing. Since the α-CIR model admits the CBI properties, we give a closed-form expression of the bond price thanks to the related affine term structure property in [18]. Moreover, we focus on the behaviors of bond prices and bond yields. In particular, we analyze the decreasing property of the bond prices with respect to the parameter α.

4.1 Zero-coupon bond pricing

We begin by making precise the equivalent probability measures. The following proposition shows that the short interest rate r given by the α-CIR model remains in the class of integral type processes under an equivalent change of probability.

Proposition 4.1 *Fix $T^\star$ large enough. Let r be an α-CIR$(a,b,\sigma,\sigma_Z,\alpha)$ process as in (2.1) under the probability measure $\mathbb{P}$ and assume that the filtration $\mathbb{F}$ is generated by the random fields W and $\widetilde{N}$. Fix $\eta \in \mathbb{R}$ and $\theta \in \mathbb{R}_+$ and define, for $t \in [0,T^\star]$,*

$$U_t := \eta \int_0^t \int_0^{r_s} W(ds,du) + \int_0^t \int_0^{r_{s-}} \int_0^\infty (e^{-\theta\zeta} - 1)\widetilde{N}(ds,du,d\zeta).$$

Then the Doléans-Dade exponential $\mathcal{E}(U)$ is a martingale on $[0,T^\star]$ and the probability measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_{T^\star}} = \mathcal{E}(U)_{T^\star}$$

is equivalent to $\mathbb{P}$. Moreover, r is under $\mathbb{Q}$ an α-CIR integral type process as in (2.2) with the parameters $(a',b',\sigma',\sigma'_Z,\mu'_\alpha)$, where

$$a' = a - \sigma\eta - \frac{\alpha\sigma_Z}{\cos(\pi\alpha/2)}\theta^{\alpha-1}, \qquad b' = ab/a', \qquad \sigma' = \sigma, \qquad \sigma'_Z = \sigma_Z$$

and

$$\mu'_\alpha(d\zeta) = -\frac{1_{\{\zeta>0\}}e^{-\theta\zeta}}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}\,d\zeta.$$

Proof The pair (r, U) is a time-homogeneous affine process (cf. [9, Theorem 6.2]). The Doléans-Dade exponential $\mathcal{E}(U)$ is a true martingale by checking that the conditions in [26, Corollary 3.2] are satisfied; so it defines an equivalent probability measure $\mathbb{Q}$. Note that $Y = \mathcal{E}(U)$ is the unique strong solution of the equation $dY_t = Y_{t-}\, dU_t$. Then for any function $f \in C^2(\mathbb{R}_+)$, the process

$$Y_t f(r_t) - \int_0^t Y_s f'(r_s)\Big(ab - \Big(a - \sigma\eta - \sigma_Z \int_0^\infty \zeta(e^{-\theta\zeta} - 1)\mu_\alpha(d\zeta)\Big)r_s\Big)\, ds$$
$$- \frac{\sigma^2}{2}\int_0^t Y_s f''(r_s) r_s\, ds$$
$$- \int_0^t Y_s r_s\, ds \int_0^\infty \big(f(r_{s-} + \sigma_Z\zeta) - f(r_s) - f'(r_{s-})\sigma_Z\zeta\big)e^{-\theta\zeta}\mu_\alpha(d\zeta), \quad t \geq 0$$

is a local martingale, which implies that r is under $\mathbb{Q}$ an α-CIR integral type process with parameters $(a', b', \sigma', \sigma'_Z, \mu'_\alpha)$. □

Remark 4.2 Usually we choose η and θ such that $a' > 0$. When $\theta = 0$, μ'_α coincides with μ_α given in (2.3), so that an α-CIR process will remain in the same class under an equivalent change of probability. When $\theta > 0$, the α-CIR process becomes an α-CIR integral type process driven by a tempered stable process under the change of probability measure. In this case, Proposition 3.3, which is on general CBI processes, still allows to compute the bond prices.

In the following, we give the zero-coupon price as a consequence of Proposition 3.3. The short rate r is supposed to follow the α-CIR model of parameter $(a, b, \sigma, \sigma_Z, \alpha)$ under the equivalent risk-neutral probability $\mathbb{Q}$. Recall that the value of a zero-coupon bond of maturity T at time $t \leq T$ is given by

$$B(t, T) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s\, ds\Big) \,\Big|\, \mathcal{F}_t\Big].$$

Proposition 4.3 *Let the short rate r be given by the α-CIR model (2.1) under the probability measure $\mathbb{Q}$. Then the zero-coupon bond price is given by*

$$B(t, T) = \exp\Big(-r_t v(T - t) - ab\int_0^{T-t} v(s)\, ds\Big), \tag{4.1}$$

where $v(s)$ is the unique solution of the equation

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi_\alpha\big(v(t)\big), \quad v(0) = 0, \tag{4.2}$$

with $\Psi_\alpha(q) = aq + \frac{\sigma^2}{2}q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)}q^\alpha$ as in (3.4). Moreover, we have

$$v(t) = f^{-1}(t), \quad \textit{where } f(t) = \int_0^t \frac{dx}{1 - \Psi_\alpha(x)}.$$

Proof Applying (3.5) with $\xi = 0$ and $\theta = 1$, we have

$$\mathbb{E}^{\mathbb{Q}}\big[e^{-\int_t^T r_s\,ds}\big|\mathcal{F}_t\big] = \exp\left(-r_t v(T-t) - ab\int_0^{T-t} v(s)\,ds\right),$$

where $v(t)$ is the unique solution of (4.2) with Ψ_α given in (3.4). Since $\Psi_\alpha(\cdot)$ is a nonnegative, increasing and convex function, the equation $\Psi_\alpha(x) = 1$ has a unique positive solution denoted by x_0. For $0 \le x < x_0$, $1 - \Psi_\alpha(x) > 0$. Note that $f(u)$ is strictly increasing in $u \in [0, x_0)$ and $f(u) \to \infty$ as $u \to x_0$. It follows from (4.2) that

$$\int_0^{v(t)} \frac{dv}{1 - \Psi_\alpha(v)} = t.$$

Let t tend to infinity on both sides of the above equality. Then $v(t) \to x_0$ as $t \to \infty$ and $v(t) < x_0$ for any $t \ge 0$. Also by (4.2), $v(t)$ is strictly increasing. So one has $v(t) = f^{-1}(t)$. □

Remark 4.4 The bond price can also be obtained directly by using the generalized Riccati equation as in Duffie et al. [11] and Keller-Ressel and Steiner [29]. Denote $A(x) := -ab\int_0^x v(s)\,ds$ and $B(x) := -v(x)$. Then these satisfy the generalized Riccati equations

$$\begin{cases} \partial_x A(x) = F(B(x)), & A(0) = 0, \\ \partial_x B(x) = R(B(x)) - 1, & B(0) = 0, \end{cases}$$

where the functions F and R are given by $F(u) = abu$ and $R(u) = \Psi_\alpha(-u)$. The bond price is then given as $B(t, T) = \exp(A(T-t) + r_t B(T-t))$. We also consider the quasi-mean-reversion which is defined as the solution of $R(-1/\lambda) = 1$ (cf. [29, Definition 3.2]). From the proof of Proposition 4.3, we see that in the α-CIR model, the quasi-mean-reversion parameter is given by $\lambda = 1/x_0$, where x_0 is the unique positive solution of the equation $\Psi_\alpha(x) = 1$.

4.2 Behaviors of bond price and bond yield

We now focus on the properties of the bond prices obtained in Proposition 4.3 and the corresponding bond yield curves.

Proposition 4.5 *The function v is increasing with respect to $\alpha \in (1, 2]$. In particular, the bond price $B(0, T)$ is decreasing with respect to α.*

Proof We write the function v as $v(t, \alpha)$ to emphasize the dependence on the parameter α. Since $1 - \Psi_\alpha(u)$ is a decreasing concave function of u and $\Psi_\alpha(0) = 0$, there is a unique positive solution, denoted by $v^*(\alpha)$, to the equation $1 - \Psi_\alpha(u) = 0$. It is not hard to see that $0 \le v(s, \alpha) < v^*(\alpha)$ and $\lim_{s\to\infty} v(s, \alpha) = v^*(\alpha)$. Moreover, from the relation $1 - \Psi_\alpha(v^*(\alpha)) = 0$, we obtain that $(\sigma_Z v^*(\alpha))^\alpha \le -\cos(\pi\alpha/2) \le 1$ and hence $\sigma_Z v^*(\alpha) \le 1$.

For any $t \in \mathbb{R}_+$, one has

$$t = \int_0^{v(t,\alpha)} \frac{dx}{1 - \Psi_\alpha(x)}.$$

Taking the derivative with respect to α, we obtain

$$\frac{1}{1 - \Psi_\alpha(v(t,\alpha))} \frac{\partial v}{\partial \alpha}(t,\alpha) + \int_0^{v(t,\alpha)} \frac{1}{(1 - \Psi_\alpha(x))^2} \frac{\partial \Psi_\alpha}{\partial \alpha}(x)\, dx = 0.$$

Note that by (3.4),

$$\frac{\partial \Psi_\alpha}{\partial \alpha}(x) = -\frac{\sin(\pi\alpha/2)}{\cos^2(\pi\alpha/2)} \frac{\pi}{2} (\sigma_Z x)^\alpha - \frac{(\sigma_Z x)^\alpha}{\cos(\pi\alpha/2)} \ln(\sigma_Z x) \le 0$$

on $x \in (0, v^*(\alpha)]$, since $\sigma_Z v^*(\alpha) \le 1$ and $\cos(\pi\alpha/2) < 0$. Therefore we obtain $\partial v/\partial\alpha \ge 0$, so that the function v is increasing with respect to α. In particular, the bond price $B(0,T)$ is a decreasing function of α. □

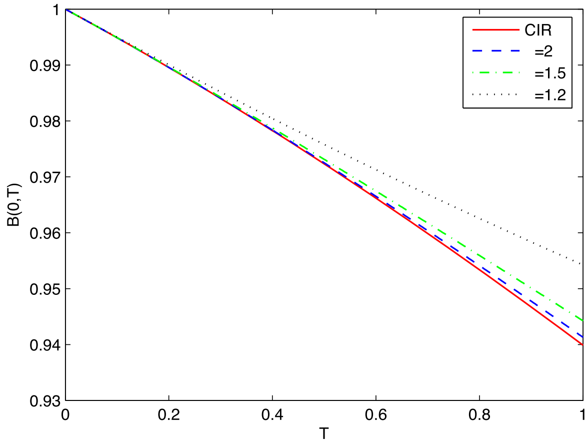
Proposition 4.5 seems to be surprising at first sight since the parameter α is an inverse measure of the heaviness of the distribution tails—the closer α to 1, the more likely the large jumps appear (see also Sect. 5). In addition, in the α-CIR model, α coincides with the so-called generalized Blumenthal–Getoor index, which is defined as $\inf\{\beta > 0 : \sum_{0<s\le T} \Delta r_s^\beta < \infty \text{ a.s.}\}$ with $\Delta r_s := r_s - r_{s-}$ and T a time horizon (see e.g. Aït-Sahalia and Jacod [2]) and is often used to measure the activity of the small jumps in a semimartingale. When $\mu_\alpha(du)$ is defined by (2.3), this index is reduced to $\inf\{\beta > 0 : \int_0^T r_s\, ds \int_0^1 u^\beta \mu_\alpha(du) < \infty \text{ a.s.}\}$ and thus is equal to α. The index $\alpha \in (1,2)$ shows that the jumps are of infinite variation. The explanation of Proposition 4.5 is based on the self-exciting property discussed in Sect. 2.2. For the compensated α-stable Lévy process Z in the α-CIR model (2.1), a smaller α is related to a deeper (negative) compensation and hence a stronger mean-reversion. Then as the interest rate becomes low because of the mean-reversion effect, the self-exciting property will imply a decreasing frequency of jumps and enforce the tendency of low interest rates.

Figure 2 plots the bond prices $B(0,T)$ given by (4.1) in Proposition 4.3. Besides the three values of α of 2, 1.5 and 1.2, we also consider the bond price in the classic CIR model (when $\sigma_Z = 0$). We observe, as already shown in Proposition 4.5, that for a fixed maturity, the bond prices are decreasing with respect to the value of α, with the lowest price in the CIR model. This observation means that compared to the standard CIR model, the α-CIR model with $\alpha \in (1,2]$ allows to better describe the low interest rate behavior from the point of view of bond pricing.

Remark 4.6 Extensions of the CIR model with jumps have already been considered in the literature. For example, the following model for the risk-neutral short interest rate is introduced in Duffie and Gârleanu [12] and then discussed in [18, 29]: they consider

$$dr_t^J = a(b - r_t^J)\, dt + \sigma\sqrt{r_t^J}\, dB_t + dJ_t, \qquad r_0^J = r_0,$$

Fig. 2 Bond prices $B(0,T)$ as function of the maturity T with different values of α: blue line for $\alpha=2$, green line for $\alpha=1.5$, black line for $\alpha=1.2$ and in comparison with the CIR model in red



where $(J_t)_{t\geq 0}$ is a compound Poisson process with intensity $c>0$ and exponentially distributed jumps of mean $m>0$. This model can provide positive jumps besides the standard CIR model. But the bond prices obtained are in general smaller than those in the CIR model, which means that it is difficult to reconcile the jumps with low interest rates.

In a similar way as in [12], we consider a CIR model with jumps where the short rate $r^L=(r_t^L)_{t\geq 0}$ satisfies

$$dr_t^L = a(b-r_t^L)\,dt + \sigma\sqrt{r_t^L}\,dB_t + \sigma_Z\,dZ'_t, \qquad r_0^L = r_0, \tag{4.3}$$

with $Z'=(Z'_t)_{t\geq 0}$ being a Lévy process whose big jump behavior is similar to Z and whose small jumps behave like an $\alpha-1$ Lévy subordinator (to ensure that the short rate takes positive values and (4.3) is well defined). More precisely, Z' is given by

$$\mathbb{E}[e^{-qZ'_t}] = \exp\left(-t\int_0^\infty (1-e^{-q\zeta})\mu'(d\zeta)\right)$$

and the Lévy measure $\mu'(d\zeta)$ is given by

$$\mu'(d\zeta) = -\frac{1_{\{\zeta>0\}}(1-e^{-\zeta})\,d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1<\alpha<2. \tag{4.4}$$

The short rate r^L in (4.3) is still a CBI process and admits the integral form

$$r_t^L = r_0 + \int_0^t a(b-r_s^L)\,ds + \sigma\int_0^t \sqrt{r_s^L}\,dB_t + \sigma_Z\int_0^t\int_0^{r_0}\int_0^\infty \zeta N(ds,du,d\zeta), \tag{4.5}$$

where $N(ds,du,d\zeta)$ is a (non-compensated) Poisson random measure with intensity $ds\,du\,\mu'(d\zeta)$. In this model, the bond prices are no longer decreasing with respect to the parameter α. Moreover, similarly as in a usual CIR model with jumps, the bond prices are all lower than the CIR bond prices.

We are interested in the behavior of the bond yields following [29]. From Proposition 4.3, the zero-coupon yield $Y(t,\theta)$ is given by $Y(t,0) := r_t$ and

$$Y(t,\theta) := -\frac{1}{\theta}\log B(t,t+\theta) = r_t\frac{v(\theta)}{\theta} + \frac{ab\int_0^\theta v(s)\,ds}{\theta}, \quad \theta > 0.$$

The long-term yield, which is the asymptotic level of the yield curve when $\theta \to \infty$, is given by $b_{\text{asym}} = abx_0$, where x_0 is the unique positive solution of $\Psi_\alpha(x) = 1$ as in the proof of Proposition 4.3. This result corresponds to the equality $b_{\text{asymp}} = -F(-1/\lambda)$ in [29, Theorem 3.7] (see Remark 4.4).

We can also have a closer look at the bond yield shapes. Let $b_{\text{norm}} := ab/\Psi'_\alpha(x_0)$ and $b_{\text{inv}} = b$, and note that $b_{\text{norm}} < b_{\text{inv}}$. From [29, Theorem 3.9], we verify that the yield curve $Y(t,\theta)$ is normal (i.e., strictly increasing with respect to θ) when $r_t \le b_{\text{norm}}$; humped (i.e., has one local maximum and no minimum) when $b_{\text{norm}} < r_t < b_{\text{inv}}$; and inverse (strictly decreasing with respect to θ) when $r_t \ge b_{\text{inv}}$.

5 Analysis of jumps

This section is focused on the jump part of the short interest rate r. In particular, we are interested in the large jumps which capture significant changes in the interest rate dynamics.

5.1 Behavior of large jumps

Let us fix a jump threshold $y = \sigma_Z\overline{y} > 0$. In this subsection, we study the following two quantities: the number of large jumps whose jump sizes are larger than y, and the first large jump time. For this purpose, we separate the large and small jumps and use the non-compensated version of the Poisson random measure in the integral form (2.2). The small jumps with infinite activity can be approximated by a second Brownian motion, for instance, in the spirit of Asmussen and Rosiński [4]. Then the α-CIR process can be written in the form

$$\begin{aligned} r_t = r_0 &+ \int_0^t a\left(b - \frac{\sigma_Z r_s \Theta(\alpha,y)}{a} - r_s\right)ds + \sigma\int_0^t\int_0^{r_s} W(ds,du) \\ &+ \sigma_Z\int_0^t\int_0^{r_{s-}}\int_0^{\overline{y}} \zeta\widetilde{N}(ds,du,d\zeta) \\ &+ \sigma_Z\int_0^t\int_0^{r_{s-}}\int_{\overline{y}}^\infty \zeta N(ds,du,d\zeta), \end{aligned}$$

where

$$\Theta(\alpha,y) = -\frac{1}{\cos(\pi\alpha/2)\Gamma(-\alpha)}\int_{\overline{y}}^\infty \frac{d\zeta}{\zeta^\alpha} = \frac{2}{\pi}\alpha\Gamma(\alpha-1)\sin\left(\frac{\pi\alpha}{2}\right)\left(\frac{y}{\sigma_Z}\right)^{1-\alpha}$$

and N is the (non-compensated) Poisson random measure corresponding to $\widetilde{N}$. Let

$$\widetilde{a}(\alpha, y) = a + \sigma_Z \Theta(\alpha, y), \qquad \widetilde{b}(\alpha, y) = \frac{ab}{a + \sigma_Z \Theta(\alpha, y)}. \tag{5.1}$$

We introduce the auxiliary process which represents the truncated interest rate r except for the jumps larger than y as

$$\begin{aligned}\widehat{r}_t^{(y)} = r_0 + \int_0^t \widetilde{a}(\alpha, y)\big(\widetilde{b}(\alpha, y) - \widehat{r}_s^{(y)}\big)\,ds + \sigma \int_0^t \int_0^{\widehat{r}_s^{(y)}} W(ds, du) \\ + \sigma_Z \int_0^t \int_0^{\widehat{r}_{s-}^{(y)}} \int_0^{\overline{y}} \zeta \widetilde{N}(ds, du, d\zeta).\end{aligned} \tag{5.2}$$

For any jump threshold $y > 0$, the process $\widehat{r}^{(y)}$ coincides with r up to the first large jump time $\tau_y := \inf\{t > 0 : \Delta r_t > y\}$. The process $\widehat{r}^{(y)}$ is a CBI process with the branching mechanism given by

$$\Psi_\alpha^{(y)}(q) := \left(a + \sigma_Z^\alpha \int_y^\infty \zeta \mu_\alpha(d\zeta)\right) q + \frac{1}{2}\sigma^2 q^2 + \sigma_Z^\alpha \int_0^y (e^{-q\zeta} - 1 + q\zeta)\mu_\alpha(d\zeta) \tag{5.3}$$

and the immigration rate given by $\Phi(q) = \widetilde{a}(\alpha, y)\widetilde{b}(\alpha, y)q = abq$.

Let J_t^y denote the number of jumps of r with jump size larger than y in $[0, t]$, i.e.,

$$J_t^y := \sum_{0<s\le t} 1_{\{\Delta r_s > y\}}.$$

Using the integral representation (2.2), we have

$$J_t^y = \int_0^t \int_0^{r_{s-}} \int_{y/\sigma_Z}^\infty N(ds, du, d\zeta) = \int_0^t \int_0^{r_{s-}} \int_{\overline{y}}^\infty N(ds, du, d\zeta).$$

Since $\mu_\alpha((0, \infty)) = \infty$, we have $\lim_{y\to 0} J_t^y = \infty$ a.s. In the following, we show that the Laplace transform of this counting process is exponential-affine, where the exponent coefficient satisfies a nonlinear ODE.

Proposition 5.1 *Let r be an α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process with initial value $r_0 \ge 0$. Then for any $p \ge 0$ and $t \ge 0$,*

$$\mathbb{E}[e^{-pJ_t^y}] = \exp\left(-\ell(p, y, t)r_0 - ab\int_0^t \ell(p, y, s)\,ds\right) \tag{5.4}$$

where $\ell(p, y, t)$ is the unique solution of the equation

$$\frac{\partial \ell(p, y, t)}{\partial t} = \sigma_Z^\alpha \int_y^\infty (1 - e^{-p-\ell(p,y,t)\zeta})\mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}\big(\ell(p, y, t)\big) \tag{5.5}$$

with initial condition $\ell(p, y, 0) = 0$ and $\Psi_\alpha^{(y)}$ given by (5.3).

Proof Denote

$$F(q) := \sigma_Z^\alpha \int_y^\infty \mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}(q), \tag{5.6}$$

which is a decreasing concave function, and $G(q) := \sigma_Z^\alpha \int_y^\infty e^{-p-q\zeta}\mu_\alpha(d\zeta)$, which is a decreasing convex function of q. Since $p \geq 0$, one has $F(0) \geq G(0)$. Moreover, for q large enough, $F(q) < 0 < G(q)$. Thus there is a unique positive solution, denoted by $\ell^* > 0$, to the equation

$$F(q) - G(q) = \sigma_Z^\alpha \int_y^\infty (1 - e^{-p-q\zeta})\mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}(q) = 0.$$

One has $F(q) - G(q) > 0$ when $0 \leq q < \ell^*$, and $F(q) - G(q) < 0$ when $q > \ell^*$. Moreover, $\Gamma(\ell) := \int_0^\ell \frac{1}{F(q)-G(q)}\,dq$ is an increasing function from $[0, \ell^*)$ to $[0, \infty)$, and its inverse function $\ell(p, y, \cdot) : [0, \infty) \to [0, \ell^*)$ exists. We then have, for any $t \geq 0$,

$$\int_0^t \frac{1}{F(\ell(p, y, s)) - G(\ell(p, y, s))}\, d\ell(p, y, s) = t,$$

which implies (5.5). Since $F(q) - G(q)$ is locally Lipschitz, the uniqueness follows.

The pair (J^y, r) is an affine Markov process taking values in $\mathbb{N}_0 \times \mathbb{R}_+$, where $\mathbb{N}_0 := \{0, 1, \ldots\}$. By Duffie et al. [11, Theorem 2.7], the generator of (J^y, r) acting on a function $f(x, n, t)$ is given by

$$\begin{aligned}
&\mathcal{A}f(x, n, t) \\
&= \frac{\partial f}{\partial t}(x, n, t) + a(b - x)\frac{\partial f}{\partial x}(x, n, t) + \frac{1}{2}\sigma^2 x \frac{\partial^2 f}{\partial x^2}(x, n, t) \\
&\quad + \sigma_Z^\alpha x \int_0^y \left(f(x + \zeta, n, t) - f(x, n, t) - \zeta \frac{\partial f}{\partial x}(x, n, t)\right) \mu_\alpha(d\zeta) \\
&\quad + \sigma_Z^\alpha x \int_y^\infty \left(f(x + \zeta, n + 1, t) - f(x, n, t) - \zeta \frac{\partial f}{\partial x}(x, n, t)\right) \mu_\alpha(d\zeta),
\end{aligned}$$

where $f(x, n, t)$ is differentiable with respect to t and twice differentiable with respect to x, and the measure $\mu_\alpha(d\zeta)$ is defined by (2.3). Let p and θ be nonnegative numbers and $T \geq 0$ a time horizon. Then it follows again from [11, Theorem 2.7] that

$$\mathbb{E}[e^{-pJ_T^y - \theta r_T} | \mathcal{F}_t] = \exp\left(-C_1(t)J_t^y - C_2(t)r_t - ab\int_0^{T-t} C_2(s)\, ds \right),$$

where

$$\begin{cases}
C_1(t) = p, \quad t \geq 0, \\
C_2'(t) = \Psi_\alpha^{(y)}(C_2(t)) + \sigma_Z^\alpha \int_y^\infty (e^{-C_2(t)\zeta - C_1(t)} - 1)\mu_\alpha(d\zeta)\ C_2(0) = \theta.
\end{cases}$$

We fix p, y and let $\ell(p, y, t) = C_2(t)$. Then the special case $\theta = 0$ leads to (5.4). □

Now we consider the first time when the jump size of the short rate r is larger than $y=\sigma_Z\overline{y}$, i.e.,

$$\tau_y = \inf\{t>0 : \Delta r_t > y\}. \tag{5.7}$$

We show that this random time also exhibits an exponential-affine cumulative distribution function. The following result gives its distribution function as a consequence of the above proposition.

Corollary 5.2 *For any $t \geq 0$, we have*

$$\mathbb{P}[\tau_y > t] = \exp\left(-\ell(y,t)r_0 - ab\int_0^t \ell(y,s)\,ds\right), \tag{5.8}$$

where $\ell(y,t)$ is the unique solution of the ODE

$$\frac{d\ell}{dt}(y,t) = \sigma_Z^\alpha \int_y^\infty \mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}\big(\ell(y,t)\big) \tag{5.9}$$

with initial condition $\ell(y,0)=0$ and $\Psi_\alpha^{(y)}$ given by (5.3).

Proof We have

$$\mathbb{P}[\tau_y > t] = \mathbb{P}[J_t^y = 0] = \lim_{p\to\infty} \mathbb{E}\big[e^{-pJ_t^y}\big]. \tag{5.10}$$

By Proposition 5.1, it suffices to prove that the limit function of $\ell(p,y,t)$ when $p\to\infty$ is the unique solution to (5.9). For any $q\geq 0$,

$$\sigma_Z^\alpha \int_y^\infty (1-e^{-p-q\zeta})\mu_\alpha(d\zeta) - \Psi^{(y)}(q) \leq \sigma_Z^\alpha \int_y^\infty \mu_\alpha(d\zeta) - aq.$$

By (5.5) in Proposition 5.1, we obtain

$$\ell(p,y,t) \leq \frac{\sigma_Z^\alpha}{a}(1-e^{-at}) \int_y^\infty \mu_\alpha(d\zeta).$$

Moreover, $\ell(p,x,t)$ is increasing with respect to p. So $\ell(y,t) := \lim_{p\to\infty}\ell(p,y,t)$ exists. Again by (5.5),

$$\ell(p,y,t) = \int_0^t \left(\sigma_Z^\alpha \int_y^\infty (1-e^{-p-\ell(p,y,s)\zeta})\mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}\big(\ell(p,y,s)\big)\right) ds.$$

Since $\Psi_\alpha^{(y)}(q)$ is locally Lipschitz and $e^{-p-\ell(p,y,s)\zeta} \leq e^{-p}$, taking limit as $p\to\infty$ on both sides of the above equation gives

$$\ell(y,t) = \int_0^t \left(\sigma_Z^\alpha \int_y^\infty \mu(d\zeta) - \Psi_\alpha^{(y)}\big(\ell(y,s)\big)\right) ds,$$

which implies that the function $\ell(y,t)$ is the unique solution to (5.9). We then conclude the proof by (5.10) and the monotone convergence theorem. □

Remark 5.3 Corollary 5.2 has the alternative form

$$\mathbb{P}[\tau_y > t] = \mathbb{E}\Big[\exp\Big(-\sigma_Z^\alpha\Big(\int_y^\infty \mu_\alpha(d\zeta)\Big)\Big(\int_0^t \widehat{r}_s^{(y)}\,ds\Big)\Big)\Big], \tag{5.11}$$

where $\widehat{r}^{(y)}$ is defined by (5.2). It shows that the distribution of the first jump time τ_y can also be given by using the Laplace transform of the integrated auxiliary process $\widehat{r}^{(y)}$, evaluated on $\sigma_Z^\alpha \int_y^\infty \mu_\alpha(d\zeta)$, that is, the mass of the Lévy measure whose jump size is larger than y. The proof of (5.11) is based on the fact that $\widehat{r}^{(y)}$ is a CBI process. More precisely, for any $\theta > 0$, we have

$$\mathbb{E}\big[e^{-\theta\int_0^t \widehat{r}_s^{(y)}\,ds}\big] = \exp\Big(\widehat{\ell}(\theta,t)r_0 - ab\int_0^t \widehat{\ell}(\theta,s)\,ds\Big),$$

where $\widehat{\ell}(\theta,t)$ is the unique solution of

$$\frac{d\widehat{\ell}(\theta,t)}{dt} = \theta - \Psi_\alpha^{(y)}\big(\widehat{\ell}(\theta,t)\big)$$

with $\widehat{\ell}(\theta,0) = \theta$. Then (5.8) can be rewritten in the form (5.11). When $b = 0$, (5.11) recovers a result of He and Li [24, Theorem 3.2].

Proposition 5.4 *We have* $\mathbb{P}[\tau_y < \infty] = 1$. *Furthermore,*

$$\mathbb{E}[\tau_y] = \int_0^{\ell_y^*} \frac{1}{F(u)} \exp\Big(-ur_0 - \int_0^u \frac{abs}{F(s)}\,ds\Big)\,du < \infty, \tag{5.12}$$

where ℓ_y^* *is the unique solution of the equation* $F(q) = 0$ (*with the function* F *defined by* (5.6)), *which identifies with* $\lim_{t\to\infty} \ell(y,t)$, *where the function* $\ell(y,t)$ *is given by* (5.9).

Proof We note as in the proof of Proposition 5.1 that F is a decreasing concave function and $F(0) > 0$. Hence the equation $F(q) = 0$ admits a unique positive solution, denoted as $\ell_y^* > 0$. One has $F(q) > 0$ when $q \in [0, \ell_y^*)$. By (5.9),

$$\int_0^{\ell(y,t)} \frac{1}{F(q)}\,dq = t, \tag{5.13}$$

which implies that $0 \le \ell(y,t) < \ell_y^*$ for any $t \ge 0$. Then $\ell(y,t)$ is strictly increasing in t. Letting t tend to ∞ in (5.13), we deduce that $\lim_{t\to\infty} \ell(y,t) = \ell_y^* > 0$. Then $\int_0^\infty \ell(y,s)\,ds = \infty$. Hence by Corollary 5.2, $\mathbb{P}[\tau_y = \infty] = 0$.

For the expectation, note that $\mathbb{E}[\tau_y] = \int_0^\infty \mathbb{P}[\tau_y > t]\,dt$. Then by Corollary 5.2,

$$\begin{aligned}\mathbb{E}[\tau_y] &= \int_0^\infty \exp\Big(-\ell(y,t)r_0 - ab\int_0^t \ell(y,s)\,ds\Big)\,dt \\ &= \int_0^{\ell_y^*} \frac{1}{F(u)} \exp\Big(-ur_0 - \int_0^u \frac{abs}{F(s)}\,ds\Big)\,du,\end{aligned}$$

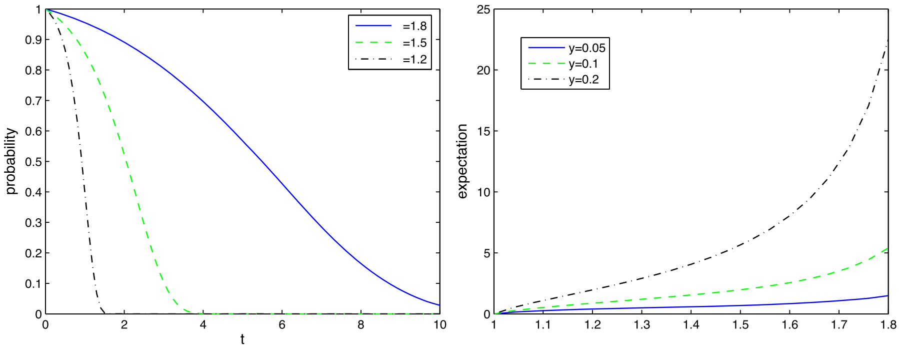


Fig. 3 (*Left*) Probability $\mathbb{P}[\tau_y > t]$ with different values of α: blue line for $\alpha = 1.8$, green line for $\alpha = 1.5$, black line for $\alpha = 1.2$. (*Right*) Expectation of the first jump time τ_y of the short rate r whose jump size is larger than y with different values of y: blue line for $y = 0.05$, green line for $y = 0.1$, black line for $y = 0.2$

where the second equality follows from (5.9). Since F is decreasing, $F'(\ell_x^*) < 0$ and by concavity,

$$\begin{aligned}\frac{1}{F(u)} \exp\left(-ur_0 - \int_0^u \frac{abs}{F(s)}\, ds\right) \\ \sim \frac{c}{F'(\ell_y^*)(u - \ell_y^*)} \exp\left(-ur_0 - \int_0^u \frac{abs}{F'(\ell_y^*)(s - \ell_y^*)}\, ds\right)\end{aligned}$$

for some constant $c > 0$, where $A(u) \sim B(u)$ means that the quotient $B(u)/A(u)$ tends to 1 as $u \to \ell_y^*$. Then $\mathbb{E}[\tau_y] < \infty$ follows from

$$\int_0^{\ell_y^*} \frac{1}{F'(\ell_y^*)(u - \ell_y^*)} \exp\left(-ur_0 - \int_0^u \frac{abs}{F'(\ell_y^*)(s - \ell_y^*)}\, ds\right) du < \infty. \qquad \square$$

We illustrate in Fig. 3 the behaviors of the first large jump time τ_y where the short rate process exceeds y. The parameters are $a = 0.1$, $b = 0.1$, $\sigma = 0.1$, $\sigma_Z = 0.1$, $r_0 = 0.2$ and $y = 0.1$. The first graph shows the probability $\mathbb{P}[\tau_y > t]$, given by (5.8) in Corollary 5.2, as a function of t for different values of α. We see that this probability converges to 0 very quickly for smaller values of α, and much more slowly for larger values of α. In particular, when α is equal to 2, the convergence time will tend to infinity for a CIR process. The second graph illustrates the expectation of τ_y as a function of α, which is given by (5.12) in Proposition 5.4. The expected jump time is increasing with α. Both graphs show that for a smaller α, the first large jump is likely to occur sooner.

5.2 Jump behavior of the locally equivalent model and comparison

In this subsection, we consider the behaviors of the first large jump in the model (4.3) and make a comparison with the α-CIR model. By comparing (4.5) and (2.2), we note that the difference lies in the integral interval in the jump term which is fixed at the initial value r_0 in (4.5), while adapted to the current level of the interest rate r_t in (2.2).

Proposition 5.5 *Let* $\tau_y^L := \inf\{t > 0 : \Delta r_t^L > y\}$ *denote the first time when the jump size of the locally equivalent process* r^L *is larger than* y. *Then*

$$\mathbb{P}[\tau_y^L > t] = \exp\Big(-r_0\mu'\big((y,\infty)\big)t\Big), \tag{5.14}$$

where $\mu'(\zeta)$ *is given by* (4.4). *Moreover, we have the asymptotic tail probability of maximal jump when* y *goes to* ∞ *as*

$$\mathcal{M}_L(t,y) := \mathbb{P}\Big[\sup_{0<s\le t}\Delta r_s^L > y\Big] \sim C_\alpha r_0 t\, y^{-\alpha},$$

where $C_\alpha := \frac{2}{\pi}\Gamma(\alpha)\sin(\pi\alpha/2)$.

Proof By (4.5), we have

$$\mathbb{P}[\tau_y^L > t] = \mathbb{P}\bigg[\int_0^t\int_0^{r_0}\int_{\overline{y}}^{\infty} N(ds,du,d\zeta) = 0\bigg],$$

where $\overline{y} = y/\sigma_Z$. Then (5.14) is obtained by a direct integration. The asymptotic tail is a consequence of the equality $\mathbb{P}[\sup_{0<s\le t}\Delta r_s^L > y] = 1 - \mathbb{P}[\tau_y^L < t]$ and the fact that $\mu'((y,\infty)) \sim C_\alpha y^{-\alpha}$ as y goes to ∞. □

In a similar way, for the α-CIR process, we have the following result.

Proposition 5.6 *The distribution function of the first large jump* τ_y *of the* α*-CIR process* r *defined in* (5.7) *satisfies the inequality*

$$\mathbb{P}[\tau_y \le t] \le C_\alpha\, y^{-\alpha}\bigg(\widetilde{b}(\alpha,y)t + \frac{r_0 - \widetilde{b}(\alpha,y)}{\widetilde{a}(\alpha,y)}(1 - e^{-\widetilde{a}(\alpha,y)t})\bigg),$$

where $\widetilde{a}(\alpha,y)$ *and* $\widetilde{b}(\alpha,y)$ *are given by* (5.1). *Moreover, as* y *goes to* ∞, *we have*

$$\mathcal{M}_r(t,y) := \mathbb{P}\Big[\sup_{0<s\le t}\Delta r_s > y\Big] \sim C_\alpha\bigg(bt + \frac{r_0 - b}{a}(1 - e^{-at})\bigg)y^{-\alpha}.$$

Proof For the α-CIR process, applying (5.11), we have

$$\mathbb{P}[\tau_y > t] = \mathbb{E}\bigg[\exp\bigg(-C_\alpha y^{-\alpha}\int_0^t \widehat{r}_s^{(y)}\,ds\bigg)\bigg]. \tag{5.15}$$

Note that $\mathbb{E}[\widehat{r}_t^{(y)}] = \widetilde{b}(\alpha, y)(1 - e^{-\widetilde{a}(\alpha,y)t}) + r_0 e^{-\widetilde{a}(\alpha,y)t}$. Thus by (5.15), we obtain the inequality by convexity. For the asymptotic tail, by (5.9), we have

$$\ell(y,t) = \sigma_Z^\alpha \int_y^\infty \mu_\alpha(d\zeta) \int_0^\infty e^{-a(t-s)}\, ds - \sigma_Z^\alpha \int_y^\infty \zeta\, \mu_\alpha(d\zeta) \int_0^t e^{-a(t-s)} \ell(y,s)\, ds$$
$$- \frac{\sigma^2}{2} \int_0^t e^{-a(t-s)} \ell^2(y,s)\, ds - \int_0^t e^{-a(t-s)} \overline{\Psi}_\alpha^{(y)}\big(\ell(y,s)\big)\, ds, \tag{5.16}$$

where $\overline{\Psi}_\alpha^{(y)}(q) = \sigma_Z^\alpha \int_0^y (e^{-q\zeta} - 1 + q\zeta)\mu_\alpha(d\zeta)$. This also shows that

$$\ell(y,t) \le -\frac{\sigma_Z^\alpha}{a\cos(\pi\alpha/2)\alpha\Gamma(-\alpha)}(1 - e^{-at})y^{-\alpha} = C_\alpha \frac{\sigma_Z^\alpha}{a}(1 - e^{-at})y^{-\alpha} \tag{5.17}$$

since $-(\alpha\cos(\pi\alpha/2)\Gamma(-\alpha))^{-1} = C_\alpha$. By (5.16), we also have

$$y^\alpha \ell(y,t) = -\frac{\sigma_Z^\alpha}{\alpha\cos(\pi\alpha/2)\Gamma(-\alpha)} \int_0^t e^{-a(t-s)}\, ds$$
$$+ \frac{\sigma_Z^\alpha}{(\alpha - 1)\cos(\pi\alpha/2)\Gamma(-\alpha)} y^{-1} \int_0^t e^{-a(t-s)} \ell(y,s)\, ds$$
$$- \frac{\sigma^2}{2} \int_0^t e^{-a(t-s)} \ell^2(y,s) y^\alpha\, ds - \int_0^t e^{-a(t-s)} \overline{\Psi}_\alpha^{(1)}\big(y\,\ell(y,s)\big)\, ds.$$

This implies, by combining with (5.17), that as $y \to \infty$,

$$y^\alpha \ell(y,t) \longrightarrow -\frac{\sigma_Z^\alpha}{\alpha\cos(\pi\alpha/2)\Gamma(-\alpha)} \int_0^t e^{-a(t-s)}\, ds = C_\alpha \sigma_Z^\alpha \frac{1 - e^{-at}}{a}. \tag{5.18}$$

Furthermore, this convergence is locally uniform in t. By Corollary 5.2,

$$\mathbb{P}\Big[\sup_{0<s\le t} \Delta r_s > y\Big] = \mathbb{P}[\tau_y \le t] = 1 - e^{-\ell(y,t)r_0 - ab\int_0^t \ell(y,s)\, ds}$$
$$\sim \ell(y,t)r_0 + ab \int_0^t \ell(y,s)\, ds.$$

We have the tail of the jump of r by (5.18). □

Remark 5.7 Comparing $\mathcal{M}_L$ and $\mathcal{M}_r$, we have that when t goes to 0, the two asymptotic tail probabilities coincide, whereas when t is large enough, $\mathcal{M}_r$ is approximately proportional to the long term interest rate b.

Remark 5.8 We have noted that for $0 < t < \tau_y$, $r_t = \widehat{r}_t^{(y)}$. Then for any T,

$$\sup_{0\le t\le T} \left| E\Big[\exp\Big(-\int_0^t r_s\, ds\Big)\Big] - E\Big[\exp\Big(-\int_0^t \widehat{r}_s^{(y)}\, ds\Big)\Big] \right|$$
$$\le 2\mathbb{P}[\tau_y \le T] = \mathbb{P}\Big[\sup_{0<s\le T} \Delta r_s > y\Big].$$

By Proposition 5.5, one has $\mathbb{P}[\sup_{0<s\leq T} \Delta r_s > y] \sim C(T)y^{-\alpha}$, where $C(T)$ is a constant depending on T. This means that as $y \to \infty$, r can be approximated by $\widehat{r}^{(y)}$ with rate $y^{-\alpha}$. In the approximation sense, we see the role of the big jumps which leads to the additional negative drift term shown in (5.2) and forces the interest rate to remain at a low level as α decreases to 1.

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