Benchmarking asset allocation strategies in the presence of liability constraints

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Abstract

Portfolio managers are in general evaluated relative to a benchmark and adapt their allocation strategies to account for the benchmark performance. A major issue for portfolio managers of liability driven institutions is that no benchmark is given to them, although they face mid-term objectives with short term constraints. No performance attribution methodology may then be used to serve as a reference. Assessing the performance of the asset manager as an agent, represents a major stake for the institution as a principal delegating a mandate of asset management. We propose an optimal asset allocation approach taking into account liability constraints to build a benchmark. This benchmark will be used to compare the ex-post effective performance of the asset manager to the effective performance of the ex-ante optimal dynamic asset allocation.

1 Introduction

A major issue for the asset managers of liability driven institutions (LDI, e.g. private institutions such as pension funds or insurance companies, and public pension funds) is to assess the performance of their investments taking into account the constraints laid down by the financial department of the institution. Indeed, for intermediate to big institutions, the asset management is parted from the financial department: together the financial department, including the asset and liabilities management (ALM), builds a framework of admissible strategies and delegate the investments to the asset management branch through an asset management mandate. The investment framework must take into account various factors that structure the company balancesheet such as liability constraints, solvency targets, accounting constraints and risk limits. Generally, most of the constraints are embedded into the strategic asset allocation defined by the institutions and conveyed to the asset manager: liability constraints and solvency targets can be boiled down in a plain table that displays the range of admissible cash flow for each asset class. Thus, the asset manager has the mandate to maximise the financial performance of her investments while limiting them to the macro-structure defined by the institution.

A vast literature on liability driven investments has grown considerably for fifteen years addressing various issues, particularly how to best manage the assets in order either to increase the net asset value or to contain the solvency risk, taking the liabilities into account. Boulier et al. (2011) [5] focus on defined-contribution plans where a guarantee is given on the benefits and show

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that the optimal composition can be divided into three different parts (a loan equal to the present value of the contributions, a contingent claim and a hedging fund); they then define dynamic asset allocations which evolve with the stochastic interest rate and the contributor's savings account. Martellini et al. (2012) [25], dividing the allocation into a risky and a safe block, study a wide range of dynamic risk-controlled strategies (using for example a floor as a function of the liabilities, possibly a performance cap in addition to the floor, or corporate bonds in the liability-hedging portfolio) and they show that it is an efficient way to improve the surplus while controlling the induced increase in risk. Deguest et al. (2013) [11] also evidence that dynamic insurance strategies, notably including equities, enable LDI to reconcile short term constraints with long term performance seeking, adjusting the risk budget with the market conditions. Baldeaux and Platen (2013) [1] build an investment strategy that minimizes the cost of hedging of the liabilities replicating given claims. As for Han and Hung (2012) [22], they highlight an optimal asset allocation for a defined-contribution pension plan hedged against a stochastic inflation using inflation-indexed bonds. Nevertheless, this literature focuses on the actions that an LDI can undertake to manage its balance sheet, in particular the financial performance and the solvency, but do not consider the actions that the asset manager of the LDI can engage within the constraints.

In terms of performance attribution, much of the studies have been done on an unconstrained universe - generally on mutual funds - from single-factor models to multi-factor models (see Lehman and Modest (1987) [23] for a comparison between both models): Ferson and Schadt (1996) [17], Ferson and Warther (1996) [18] and Christopherson et al. (1998) [10] used multifactor benchmarks constructed on publicly available information. Basso and Funari (2005) [2] build a multiple-indicator model (generalized DEA performance index) to evaluate the performance of mutual funds. Empirical work has also been done, such as Brooks and Porter (2012) [6] that examines the performance of almost 2000 actively managed equity funds from 1994 to 2005. Nonetheless, few papers address performance attribution issue in a constrained environment. Bertrand (2008) [3] compares the information ratio under the tracking-error constraint to the same ratio without this constraint and evidences the distortion of the performance measure as soon as one introduces a risk constraint.

To our knowledge, it is relatively rare to use performance attribution methodology, taking into account the framework imposed by a financial department to an asset manager, as a reference. Yet, assessing the performance of the asset manager as an agent, represents a major stake for the institution as a principal delegating a mandate of asset management.

In this paper, we address the issue of performance attribution for a constrained asset management. For this purpose, we use an optimal asset allocation approach to build a benchmark used for the performance measure. The aim is to compare the ex-post effective performance of the asset manager to the effective performance of the ex-ante optimal dynamic asset allocation (called the *benchmark*) determined within the range of constraints that the asset manager faces, mainly the strategic asset allocation provided by the mandator (the client of the asset manager). The optimisation is carried out under the real world probability measure corresponding to either the asset manager forecast or the market consensus forecast. Therefore, if the realised performance of the asset manager is greater than the benchmark one, it means that the asset manager has created a value that the quantitative portfolio construction was not been able to do. On the contrary, if the realised performance of the asset manager is lower than the benchmark one, the asset manager should have chosen the quantitative portfolio construction.

The aim of the asset manager is to maximize an expected utility on the final wealth, under a certain probabilistic constraint which outperforms the benchmark. Different levels of constraints are discussed in hierarchical order: the strongest one in the almost sure sense and the weaker one in probability. We also consider, in the spirit of Föllmer and Leukert (2000) [19], the expected shortfall constraint where a loss function is concerned. We are in particular interested in the impact of credit risk on the asset allocation. The asset portfolio contains, besides the cash account and the stock, a family of credit sensitive bonds, which are characterized by different credit ratings and associated default intensities. We adopt a typical reduced-form model framework by using the affine model for the intensity processes. In such a setting, it is important to specify the risk premia of each asset and choose a suitable risk-neutral probability measure. As stated by Duffie and Singleton (2003) [14, Section 5.2], modelling default-risk premia is conceptually more challenging than modelling interest rate risk. We fix, following Duffie (2002) [12], some equivalent probability measure $\mathbb{Q}$ based on discounting at the short rate r, which is also described by an affine model, and provide suitable conditions for calculating the change of probability measure (between $\mathbb{Q}$ and the real world probability $\mathbb{P}$) and the associated change of intensity. As we have explained, the optimization is done under $\mathbb{P}$ with the performance constraint. A related strain of literature deals with the portfolio insurance problem, see for example Grossman and Vila (1989) [20], Gundel and Weber [21], Boyle and Tian (2007) [8] and El Karoui et al. (2005) [15]. Compared to these previous works, the contribution of this paper is to include the credit risk into the optimization problem and to use the performance attribution as a stochastic benchmark, which reflects the requirements of asset managers. From the point of view of technical concerns, the possibility of default event leads to a random weight in the objective utility function. To solve this weighted utility maximization problem, we use a duality method to obtain optimal allocation strategy, where different Lagrange multipliers are used to deal with performance constraints.

In this paper, we propose a methodology to build a benchmark that encompasses the constraints provided to the asset manager by the LDI. We show that, in a market with credit and default risks, this benchmark always exists under certain consitions, for three different constraint specifications. Therefore, one can realize a classic performance attribution against the defined benchmark in order to assess the out- or under-performance of the asset manager.

The remaining parts of the paper are organized as follows. In Section 2, the state variables of the financial economy are introduced, the assets of the financial market are characterized, and the liability driven benchmark is defined. In Section 3, the optimization problem with it several constraints based on the liability driven benchmark is presented and solutions are provided. A numerical application is given in Section 4 that characterizes the dynamic asset allocation in the case of the almost surely constraint. Section 5 summarizes the conclusion of this research.

2 Model setup and dynamics of asset prices

In this section, we introduce a general continuous-time model for our asset allocation problem. We let the financial market be described by a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions. We assume that the market is arbitrage-free and we denote by $\mathbb{Q}$ a risk-neutral measure equivalent to $\mathbb{P}$.

For any $\boldsymbol{a} = (a_1, \cdots, a_n)'$ and $\boldsymbol{b} = (b_1, \cdots, b_n)'$, $\boldsymbol{a} * \boldsymbol{b}$ denotes $(a_1 b_1, \cdots, a_n b_n)'$, and $\boldsymbol{a} \cdot \boldsymbol{b}$ denotes the inner product $a_1 b_1 + \cdots + a_n b_n$.

2.1 State variables and zero-coupon bonds

Regarding the asset side, interest risk, credit risk and equity risk appear as the three most relevant risk factors for our asset manager. The "classical" and main approach of the dynamic asset allocation literature studies a representative agent dynamically allocating her wealth into several asset categories, usually consisting of a (default risk-free) bond, and an equity (or equity index).

This approach fails to incorporate corporate and sovereign bond markets that have been developing very fast during the last decades and provide distinct risk-return profile to investors. Incorporating defaultable bond in the analysis of asset allocation not only recognizes current reality of the financial world, but also contributes to the literature of asset allocation by explicitly investigating the impacts of credit risk on investors' decisions. The increase of investments in credit bonds (corporate or financial bonds) has been boosted by the low rates environment in Europe as well as in the US since the beginning of the crisis. The collapse of the interest rates has been explained by the post-crisis slowdown in the world economic growth, the low inflation environment that illustrates a sluggish recovery notably in Europe, an accomodating monetary policy in Europe, in the UK, in Japan and in the US until the 2015 tapering and, lastly, a drop of the global risk premia on the market. Thus, on 31 December 2015, the 10 year Bund traded at 0.63%, the 10 year Euribor Swap traded at 1% and the 10 year T Bond traded at 2.27% (even though it traded at 1.64% on 30 January 2015). In this respect, investors have been searching for yield and thus have diversified their investments towards more risky asset classes, in particular towards lower-rated corporate bonds (often increasing their high-yield exposure) and even longer maturities.

In what follows, we introduce a vector of state variables $\mathbf{\Phi}_t = (r_t, \varphi_t^1, \ldots, \varphi_t^n)'$ that will be used to incorporate both stochastic interest and credit risk in our financial market, based on a general affine model framework (see Duffie [12]). The first component is the short-term interest. The other components are components of the intensities of default risk. We assume that there are $n+1$ rating categories $R_0, R_1, \ldots, R_n$ corresponding to increasing levels of default risk. The rating class R_0 must be associated with default-free bonds. The other ratings are driven by default intensities $\lambda_t^1, \ldots, \lambda_t^n$ such that, for any $i = 1, \ldots, n$,

$$\lambda_t^i = \varphi_t^1 + \cdots + \varphi_t^i. \tag{2.1}$$

Hence φ_t^i represents the credit spread between the two rating categories R_{i-1} and R_i. We also define $\lambda_t^0 = 0$.

We assume that $(\mathbf{\Phi}_t)_{t\geq 0}$ is an affine diffusion process with the following dynamics under $\mathbb{Q}$:

$$d\mathbf{\Phi}_t = (K_0 + K_1\mathbf{\Phi}_t)dt + \sigma(\mathbf{\Phi}_t)d\mathbf{V}_t \tag{2.2}$$

where $K_0 \in \mathbb{R}^{n+1}$, $K_1 \in \mathbb{R}^{(n+1)\times(n+1)}$, $\mathbf{V} = (W_t^{r,\mathbb{Q}}, V_t^1, \ldots, V_t^n)'$ is a $(n+1)$-dimensional standard Brownian motion, and $\sigma(\boldsymbol{\varphi})$ is a $(n+1)\times(n+1)$ matrix such that

$$\big(\sigma(\boldsymbol{\varphi})\sigma(\boldsymbol{\varphi})'\big)_{ij} = H_{0ij} + H_{1ij}\cdot\boldsymbol{\varphi},$$

where

$$H_{0ij} \in \mathbb{R},\ H_{1ij} \in \mathbb{R}^{n+1} \text{ for } (H_0, H_1) \in \mathbb{R}^{(n+1)\times(n+1)} \times \mathbb{R}^{(n+1)\times(n+1)\times(n+1)}$$

(see e.g. Duffie and Singletton (2003) [14, Appendix A.4]).

The price of a default-free zero-coupon bond in rating class R_0 with maturity T is given by

$$B_0(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T r_s ds\Big)\,|\mathcal{F}_t\Big]. \tag{2.3}$$

For any rating category R_i, $i = 1, \ldots, n$, the pre-default price of a zero-coupon bond with maturity T is given by

$$B_i(t,T) = \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T (r_s + \lambda_s^i)ds\Big)\,|\mathcal{F}_t\Big]. \tag{2.4}$$

for any $t < \tau_i$ where τ_i denotes the default time of the bond of rating R_i. If the bond is of zero-recovery rate, then $B_i(t,T) = 0$ for any $t \geq \tau_i$. In our setting, we will discuss the recovery at default more in detail in the next section. Note that, for any $i = 0, \dots, n$, $r_t + \lambda_t^i = \boldsymbol{h}^i \cdot \boldsymbol{\Phi}_t$ where $\boldsymbol{h}^i$ is a $(n+1)$-dimensional vector such that $h_k^i = 1$ for $k = 1, \dots, i+1$ and $h_k^i = 0$ for $k = i+1, \dots, n+1$. Then, using Duffie (2002) [12], the pre-default zero-coupon bond price can be expressed as

$$B_i(t,T) = \exp(-A_i(T-t) \cdot \boldsymbol{\Phi}_t + C_i(T-t)) \tag{2.5}$$

where $A_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n+1}$ and $C_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ are solutions of the following system of Riccati equations:

$$\frac{dA_i(\varphi)}{d\varphi} = h_i + K_1' A_i(\varphi) - \frac{1}{2} A_i(\varphi)' H_1 A_i(\varphi), \tag{2.6}$$

$$\frac{dC_i(\varphi)}{d\varphi} = \quad - K_0' A_i(\varphi) + \frac{1}{2} A_i(\varphi)' H_0 A_i(\varphi) \tag{2.7}$$

for all $\boldsymbol{\varphi} \geq 0$, with the initial condition $A_i(0) = 0_{\mathbb{R}^{n+1}}$ and $C_i(0) = 0$, and $A_i(\varphi)' H_1 A_i(\varphi)$ denotes $\sum_{j,k} A_{i,j}(\varphi) A_{i,k}(\varphi) H_{1jk}$, $A_{i,j}$ being the j^{th} component of A_i. Note that the last $n+1-i$ components of A_i are equal to 0.

By observing that the process defined by $\exp(-\int_0^t (r_s + \lambda_s^i) ds) B_i(t,T)$ is a $(\mathbb{Q}, \mathbb{F})$-martingale and by applying Itô lemma on (2.5), the pre-default dynamics of B_i under $\mathbb{Q}$ writes

$$\frac{dB_i(t,T)}{B_i(t,T)} = \boldsymbol{h}^i \cdot \boldsymbol{\Phi}_t dt - A_i(T-t) \cdot \sigma(\boldsymbol{\Phi}_t) d\mathbf{V}_t. \tag{2.8}$$

Our specification enables to introduce default risk and it is very important in practise for an asset manager of a LDI since every year various event of default occur. For example in 2015 in Europe, there were nine defaulted bonds, of which, prior to default, six were rated CCC (four Greek banks and two British corporates) and three were rated CC (one Ukrainian bank and a Belgian and a British corporate). In contrast, in 2014, six defaults were identified with a better rating one year prior to default (four of which were rated single B). Finally, in 2013, sixteen defaults occurred in Europe involving a wide range of companies and countries. It should be noted that a default might be either a missed interest or a missed principal payment.

2.2 The financial market

We assume that the asset-manager can invest in a market composed of different asset classes: one risk-free asset (cash account), a default-free zero-coupon bond, n corporate or sovereign default-sensitive bonds and a stock.

The price of the risk-free asset S^0 is given by

$$S_t^0 = \exp\Big(\int_0^t r_s ds\Big), \quad S_0^0 = 1. \tag{2.9}$$

We assume that the asset manager may invest in bonds with stable duration over time (constant maturity rolling bonds) as in Boulier et al. (2001) [5] and Han and Hung (2012) [22]: in each rating category $R_0, R_1, \dots, R_n$, the market trades a zero-coupon bond with constant term-to-maturities over time T_i, $i = 0, \dots, n$, and price $B_i(t) = B_i(t, t+T_i)$. Note that we only consider one maturity for each rating category, it will be justified further. The hypothesis of constant duration bonds is not so strong for an asset manager of a LDI (and even more appropriate) since, when the annual redemptions are close to the annual subscriptions (i.e. for a flat business evolution), the liability cash flows profile remain quite unchanging. As a consequence, the shape of the asset cash flows remain pretty unchanging as well over time since they are regularly readjusted. Therefore, one can use constant duration bonds to proxy the asset and the liability cash flows of a flat business evolution of an institutional investor. Their risk neutral dynamics are given by

$$\frac{dB_i(t)}{B_i(t)} = \boldsymbol{h}^i \cdot \boldsymbol{\Phi}_t dt - A_i(T_i) \cdot \sigma(\boldsymbol{\Phi}_t) d\mathbf{V}_t. \tag{2.10}$$

We further assume that the risk-neutral dynamics of the stock price is given by

$$\frac{dS_t}{S_t} = r_t dt + \sigma_S(\rho dW_t^{r,\mathbb{Q}} + \sqrt{1-\rho^2} dW_t^{S,\mathbb{Q}}) \tag{2.11}$$

where σ_S is the stock volatility and $W^{S,\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$, that is independent of $W^{r,\mathbb{Q}}$ but is such that for $i = 1, \ldots, n$, $d\left\langle V^i, W^{S,\mathbb{Q}}\right\rangle_t = \rho^{i,S} dt$ where $\rho^{i,S} \in (-1,1)$. The parameter ρ captures the degree of correlation between the short term interest risk and the stock price, the parameters ρ_i the correlation between the default intensities and the stock price.

Given the two-dimensional Brownian motion $W^{\mathbb{P}} = (W^{r,\mathbb{P}}, W^{S,\mathbb{P}})'$ and a process $\alpha_t = (\alpha_t^r, -\alpha_t^S)$, the risk-neutral probability $\mathbb{Q}$ is defined, following [Duffie (2002) [12], Section 4], by the change of probability measure with the following Radon-Nikodym derivative

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \alpha_s dW_s^{\mathbb{P}} - \frac{1}{2}\int_0^t \|\alpha_s\|^2 ds\right). \tag{2.12}$$

Note that under this change of measures, $W^{\mathbb{P}}$ and $W^{\mathbb{Q}}$ are related through $dW_t^{r,\mathbb{Q}} = dW_t^{r,\mathbb{P}} - \alpha_t^r dt$ and $dW_t^{S,\mathbb{Q}} = dW_t^{S,\mathbb{P}} + \alpha_t^S dt$. The process α^r is the risk premium associated with interest-rate risk whereas α^S corresponds to the market price of the stock market risk. If we require that the dynamics of r remains in the same family after the equivalent change of probability measure, a possible choice of the interest-rate risk premium would be a constant $\alpha_t^r = \alpha^r$ under the Vasicek model or $\alpha_t^r = \alpha^r\sqrt{r_t}$ under the CIR model (see below). At a first stage, we can consider that the risk premium associated with the stock market α^S is constant, i.e., $\alpha_t^S = \alpha^S$.

By (2.12) and Girsanov's theorem, we have under probability $\mathbb{P}$ that

$$\mathbf{V}^{\mathbb{P}} = (W^{r,\mathbb{P}}, V^{1,\mathbb{P}}, \ldots, V^{n,\mathbb{P}})' \text{ where } V_t^{i,\mathbb{P}} = V_t^i - \int_0^t \alpha_u^S d\left\langle V^i, W^{S,\mathbb{Q}}\right\rangle_u \tag{2.13}$$

is a $(n+1)$-dimensional standard Brownian motion.

Using (2.13) under the equivalent change of measure, the pre-default dynamics of B_i under $\mathbb{P}$ is given by

$$\frac{dB_i(t)}{B_i(t)} = \left(r_t + \lambda_t^i - A_i(T_i) \cdot \sigma(\boldsymbol{\Phi}_t)\alpha_t^Z\right) dt - A_i(T_i) \cdot \sigma(\boldsymbol{\Phi}_t) d\mathbf{V}_t^{\mathbb{P}} \tag{2.14}$$

where $\alpha_t^Z = (-\alpha_t^r, \rho^{1,S}\alpha_t^S, \cdots, \rho^{n,S}\alpha_t^S)'$, and the dynamics of S under $\mathbb{P}$ is given by

$$\frac{dS_t}{S_t} = (r_t + \sigma_S(-\rho\alpha_t^r + \sqrt{1-\rho^2}\alpha_t^S))dt + \sigma_S(\rho dW_t^{r,\mathbb{P}} + \sqrt{1-\rho^2} dW_t^{S,\mathbb{P}}). \tag{2.15}$$

We give two examples where the zero-coupon bonds have explicit dynamics.

Example 1 *In the multivariate Vasicek model, $K_1 = -aId_{n+1}$ with $a > 0$ and $\sigma(\boldsymbol{\Phi}_t)$ is a constant diagonal matrix. As a consequence, the components of $\boldsymbol{\Phi}_t$ are independent univariate Vasicek processes with the following dynamics under $\mathbb{Q}$:*

$$dr_t = a(b - r_t)dt + \sigma_r dW^{r,\mathbb{Q}} \tag{2.16}$$

$$d\varphi_t^i = a(b_i - \varphi_t^i)dt + \sigma_i dV_t^i, \; i = 1, \ldots, n. \tag{2.17}$$

Nevertheless, the Vasicek model has a number of disadvantages: the interest rate can be negative (see CIR, exponential Vasicek, Black-Derman-Toy and Black-Karasinski models for amendments to address this issue), it is a one factor model with constant coefficients hence the interest rates are perfectly correlated and the pricing of fixed income products are flawed. The dynamics of the zero-coupons under $\mathbb{P}$ are then the following ones:

$$\frac{dB_0(t)}{B_0(t)} = (r_t + \phi(T_0)\sigma_r\alpha^r)dt - \phi(T_0)\sigma_r dW_t^{r,\mathbb{P}} \tag{2.18}$$

$$\frac{dB_i(t)}{B_i(t)} = \left(r_t + \lambda_t^i + \phi(T_i)\Big(\sigma_r\alpha^r - \sum_{k=1}^{i}\sigma_k\rho^{k,S}\alpha_t^S\Big)\right)dt - \phi(T_i)\left(\sigma_r dW_t^{r,\mathbb{P}} + \sigma_1 dV_t^{1,\mathbb{P}} + \cdots + \sigma_i dV_t^{i,\mathbb{P}}\right), \tag{2.19}$$

where $\phi(x) = \frac{1-e^{-ax}}{a}$.

Example 2 *In the multivariate CIR model, $K_1 = -aId_{n+1}$ with $a > 0$ and $\sigma(\mathbf{\Phi}_t)$ is a diagonal matrix proportional to $\sqrt{\mathbf{\Phi}_t}$. As a consequence, the components of $\mathbf{\Phi}_t$ are independent univariate CIR processes with the following dynamics under $\mathbb{Q}$:*

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW^{r,\mathbb{Q}} \tag{2.20}$$

$$d\varphi_t^i = a(b_i - \varphi_t^i)dt + \sigma\sqrt{\varphi_t^i}dV_t^i, \; i = 1, \ldots, n \tag{2.21}$$

The CIR model prevents the rates from being negative. Bu it should be noted that it is also a one factor model. Roggers (1995) [26], Duffie and Kan (1996) [13] propose multifactor models to improve it. Moreover the coefficient are neither time varying (see Maghsoodi (1996) [24] and Brigo and Mercurio (2001) [7] for amendments to address this point) nor stochastic (see Chen (1996) [9]). The dynamics of the zero-coupons under $\mathbb{P}$ are the following ones:

$$\frac{dB_0(t)}{B_0(t)} = (r_t + \psi(T_0)\sigma\alpha^r r_t)dt - \psi(T_0)\sigma\sqrt{r_t}dW_t^{r,\mathbb{P}} \tag{2.22}$$

$$\begin{aligned}\frac{dB_i(t)}{B_i(t)} &= \left(r_t + \lambda_t^i + \psi(T_i)\sigma\Big(\alpha^r r_t - \sum_{k=1}^{i}\rho^{k,S}\sqrt{\varphi_t^k}\alpha_t^S\Big)\right)dt \\ &\quad - \sigma\psi(T_i)\left(\sqrt{r_t}dW_t^{r,\mathbb{P}} + \sqrt{\varphi_t^1}dV_t^{1,\mathbb{P}} + \cdots + \sqrt{\varphi_t^i}dV_t^{i,\mathbb{P}}\right)\end{aligned} \tag{2.23}$$

where $\psi(x) = \frac{2(1-e^{-hx})}{h+a+(h-a)e^{-hx}}$ *and* $h = \sqrt{a^2 + 2\sigma^2}$.

2.3 The liability driven benchmark

The strategic asset allocation (SAA) is provided by the financial department of the LDI. It is generally defined by the ALM together with the financial department in order to both steer the solvency of the institution and to reflect a financial forecast on the asset classes. Hence, the LDI proposes a framework of admissible strategies consistent with the liability constraints, and delegates the investments to the asset manager. Thus, for instance, an LDI with a strong positive forecast on equities must overweight its equity allocation while making sure that the estimated prospective solvency of the institution will meet the requirements defined by the executive management. This asset allocation is generally defined as a breakdown per asset classes of the annual cash flows: it is a flow SAA rather than a stock SAA. It may either be defined with some leeway per asset class, thus the asset manager has the possibility of generating performance via a marginal allocation effect and a selection effect (bond and stock picking), or be strictly specified so that the only outperformance source for the asset manager is the selection effect. For example, the SAA with some leeway can be represented as follows:

Asset classes	Rating	Target	Minimum	Maximum
Government bonds	AAA	24%	21%	27%
	AA	22%	19%	25%
	A	1%	0%	2%
	BBB	1%	0%	1%
	Total	**1%**	**0%**	**1%**
Corporate bonds	AAA	1%	0%	4%
	AA	14%	11%	17%
	A	15%	12%	18%
	BBB	1%	0%	4%
	Total	**31%**	**28%**	**34%**
Equities		**9%**	**6%**	**12%**
Money market funds		**12%**	**9%**	**15%**
Total		**100%**		

Table 1. Example of a Strategic Asset Allocation

When the SAA is built with upper and lower bounds on each asset class, one may take the average target of each asset class provided that the sum of each target amounts to 100%. Otherwise, one may normalize the asset allocation.

In the absence of a benchmark, the SAA may be considered as a naive benchmark for the investment process. For this reason, we suggest to choose the SAA as a "*liability driven benchmark*" that will be denoted by $(H_t)_{t\geq 0}$. Therefore, the SAA is a natural candidate to determine the constrained optimal strategy (see hereafter) and will be designated as the benchmark.

3 The optimization problem with constraints

The asset manager invests the wealth in a portfolio which is composed of the financial assets described in the previous section : the cash (risk-free asset), $n+1$ zero-coupon bonds of different credit qualities and one stock. Therefore, the total number of risky assets in the portfolio is $N = n+2$. The objective is to maximize the expected institution's utility of the final wealth under certain asset-liability performance constraint and over the period $[0,T]$.

We denote by $\widetilde{\boldsymbol{S}} = (B_0, B_1, \ldots, B_n, S)'$ the vector of prices of the risky financial assets in the portfolio which satisfies under $\mathbb{P}$ the SDE

$$d\widetilde{\boldsymbol{S}}_t = \widetilde{\boldsymbol{S}}_t * (\widetilde{\boldsymbol{\mu}}_t dt + \Sigma_t d\boldsymbol{W}_t) \tag{3.24}$$

where $\widetilde{\boldsymbol{\mu}}$ and Σ are adapted processes valued in $\mathbb{R}^N$ and $\mathbb{R}^{N\times N}$ respectively, and $\boldsymbol{W} = \left(W^{r,\mathbb{P}}, (\boldsymbol{V}_t^{\mathbb{P}})', W^{S,\mathbb{P}}\right)'$ is a N-dimensional Brownian motion. Note that Σ_t^{-1} is well defined for each t since we have chosen a unique zero-coupon bond for each rating. The actualized price vector of risky assets is then given by $\boldsymbol{S} = (S^0)^{-1}\widetilde{\boldsymbol{S}}$, where S^0 denotes the value of the risk-free asset, which is given in (2.9). It satisfies the SDE

$$d\boldsymbol{S}_t = \boldsymbol{S}_t * (\boldsymbol{\mu}_t dt + \Sigma_t d\boldsymbol{W}_t),$$

where $\boldsymbol{\mu}_t = \widetilde{\boldsymbol{\mu}}_t - r_t(1, ...1)' := (\mu_t^1 - r_t, \cdots, \mu_t^N - r_t)$.

Let $\boldsymbol{\pi}$ be a predictable process valued in $\mathbb{R}^N$ which represents the quantity of assets invested in the risky assets. The total actualized wealth of the portfolio is denoted by X and satisfies the following SDE

$$dX_t = \boldsymbol{\pi}_t \cdot d\boldsymbol{S}_t, \quad X_0 = x_0,$$

where x_0 is the initial endowed wealth. Note that the quantity invested in the risk free asset is equal to

$$\pi_t^0 = X_t - \boldsymbol{\pi}_t \cdot \boldsymbol{S}_t$$

since the actualized price of the cash is 1. If $\boldsymbol{\alpha}$ denotes the proportion of the wealth invested in the assets, then

$$\frac{dX_t}{X_t} = \boldsymbol{\alpha}_t \cdot \frac{d\boldsymbol{S}_t}{\boldsymbol{S}_t}, \quad X_0 = x_0.$$

Hence we have

$$\boldsymbol{\alpha}_t = (\alpha_t^1, \cdots, \alpha_t^N)' = \frac{\boldsymbol{\pi}_t * \boldsymbol{S}_t}{X_t}.$$

Accordingly, the proportion invested in the risk-free asset is

$$\alpha_t^0 = 1 - (\alpha_t^1 + \cdots + \alpha_t^N).$$

In practice, the asset manager chooses in her portfolio the bonds of investment grade which have relatively low probability to default. Let $\tau = \min\{\tau_1, \ldots, \tau_n\}$, where τ_i denotes the default time of the i^{th} defaultable bond B_i (corresponding to the credit rating class R_i). So τ represents the first default time among the risky bonds. We consider the portfolio value at the horizon time T with the two cases whether a default event occurs before T or not, i.e. $\tau > T$ or $\tau \le T$. If a default occurs, a recovery rule will be applied to calculate the portfolio value given default, according to the practice of asset manager (which we will explain more in detail in the sequel). Hence the terminal value of the portfolio which takes into consideration the default risk admits a regime switching and is given by

$$Y_T = \mathbb{I}_{\{\tau > T\}} X_T + \sum_{i=1}^n \mathbb{I}_{\{\tau = \tau_i \le T\}} C^i Y$$

where C^i is a random variable taking values in $[0, 1]$ and representing the recovery rate of the portfolio if the first default corresponds to the i^{th} underlying firm and occurs before the maturity date. The random variable Y is a reference value for the portfolio. For example, it can be the face value of the portfolio at the initial time (recovery of face value), or the value of a comparative portfolio which is less sensitive to the default risk (e.g. recovery of treasury value). Following the cases, the description of the recovery procedure can be different for the bonds, according to the provision politics of the asset manager. We assume in addition that there is no simultaneous defaults, that is, $\tau_i \neq \tau_j$ almost surely for $i \neq j$ and that the recovery rate C^i is independent of the other random variables. This specification is a decent representation of the reality on financial markets where each asset B_i can default and, in this case, there is a recovery notional. Moreover, the restriction regarding the fact that no simultaneous defaults can occur is not an obstacle since we can reduce the time period to overcome this issue.

As explained in Section 2.3, the asset manager considers the liability driven benchmark (H_t) as a reference of the liability allocation. The asset manager is required that the optimal strategy outperforms the liability driven benchmark in some probability sense.

Let $U : \mathbb{R}_+ \to \mathbb{R}$ be the institution's utility function which is assumed to be strictly concave, of class C^1 on $(0, +\infty)$ and to satisfy the Inada condition, i.e., $\lim_{x\to 0+} U'(x) = +\infty$ and $\lim_{x\to+\infty} U'(x) = 0$. The asset manager's optimization problem can be described as

$$\text{(P)} \qquad \max_{\boldsymbol{\alpha}\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}[U(Y_T)] \text{ such that } Y_T^{\alpha} \text{ outperforms } H_T \text{ in some probabilistic sense}$$

where $\mathcal{A}$ denotes the set of all admissible strategies satisfying

$$\int_0^T |\boldsymbol{\alpha}_t \cdot \boldsymbol{\mu}_t| dt + \int_0^T |\Sigma_t \boldsymbol{\alpha}_t|^2 dt < \infty.$$

The objective function is equal to

$$\mathbb{E}_{\mathbb{P}}[U(Y_T)] = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{\tau>T\}}U(X_T) + \sum_{i=1}^{n} \mathbb{I}_{\{\tau=\tau_i\leq T\}}U(C^iY)] = \mathbb{E}_{\mathbb{P}}\Big[G_T U(X_T) + \sum_{i=1}^{n} P_i U(C^iY)\Big]$$

where $G_T = \mathbb{P}(\tau > T|\mathcal{F}_T)$ is the conditional survival probability of the first default and can be viewed as a random weight of the optimization problem, and $P_i = \mathbb{P}(\tau = \tau_i \leq T|\mathcal{F}_T)$ is the marginal probability of the first default time. With the recovery rule described above, we see that the investment strategy concerns only the set $\{\tau > T\}$.

The conditional survival and default probabilities can be computed explicitly with a tractable model of correlation such as the doubly stochastic model where the default correlation comes from the covariation in default intensities. More precisely, we assume that the H-hypothesis holds for the first default τ and the filtration $\mathbb{F}$, that is, any $\mathbb{F}$-martingale remains a martingale with respect to the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ where $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\mathbb{I}_{\{\tau\leq s\}}, s \leq t)$. This hypothesis is a standard one in the credit risk analysis (see e.g. [16]). In this case, the conditional survival probability $G = (G_t, t \geq 0)$, which is the Azéma supermartingale of τ, is given by $G_t = \exp(-\int_0^t \lambda_s ds)$ where λ is the intensity of τ and satisfies the relation $\lambda_{t\wedge\tau} = \sum_{i=1}^n \lambda^i_{t\wedge\tau}$ for any $t \geq 0$. In a similar way, the probabilities P_i can also be calculated, see [14, Section 10.6.3] for more details.

In the following, we make precise the optimization problem with different ALM constraints. First of all, we consider a portfolio insurance problem under the almost surely constraint. The problem (P) can be described as follows:

$$\text{(P1)} \qquad \max_{\boldsymbol{\alpha}\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}[G_T U(X_T)] \text{ such that } X_T^{\alpha} \geq H_T \text{ almost surely.}$$

The optimization problem (P1) imposes a European type constraint where the optimal strategy outperforms a stochastic threshold with probability 1. Without the constraint, the problem reduces to a standard optimal investment allocation problem. This problem also generalizes the insurance optimization problem introduced in Deguest et al. (2013) [11] for an investor who only faces uncertainty in interest rates and the equity risk premium, but is imposed the maximum performance constraint.

Instead of considering the almost surely constraint in (P1), one may release the constraint by using a confidential level of probability p which is strictly smaller than 1, then the problem becomes

$$\text{(P2)} \qquad \max_{\boldsymbol{\alpha}\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}[G_T U(X_T)] \text{ such that } \mathbb{P}(X_T^{\alpha} \geq H_T) \geq p, \ p \in (0, 1) \qquad (3.25)$$

When $p = 1$, then the problem (P2) coincides with (P1).

We can also introduce risk tolerance to the optimization problem. Following Föllmer and Leukert (2000) [19], we consider that a bound is imposed on the expected shortfall, weighted by a loss function $l : \mathbb{R} \to \mathbb{R}$, which is assumed to be convex and decreasing. The optimization problem with such constraint can be described as

$$\text{(P3)} \qquad \max_{\boldsymbol{\alpha}\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}[G_T U(X_T)] \ \text{ such that } \ \mathbb{E}_{\mathbb{P}}[G_T l(X_T^{\alpha} - H_T)] \leq L \tag{3.26}$$

A typical example is the "call" function $l(x) = (-x)^+$ which takes consideration of the positive part of the loss. The parameter L represents the risk tolerance for the loss, which is fixed by the asset manager.

3.1 Solution under the almost surely constraint

In this subsection, we focus on the optimization problem under the almost surely constraint. We introduce an auxiliary probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{P}$, under which the price process $\boldsymbol{S}$ is a local martingale over $[0, T]$. Denote by $Z = (Z_t, t \geq 0)$ the Radon-Nikodym derivative of $\widetilde{\mathbb{Q}}$ with respect to $\mathbb{P}$, which is a $\mathbb{P}$-martingale, i.e.,

$$Z_t = \exp\left(-\int_0^t (\Sigma_s^{-1}\boldsymbol{\mu}_s)\cdot d\boldsymbol{W}_s - \frac{1}{2}\int_0^t \|\Sigma_s^{-1}\boldsymbol{\mu}_s\|^2 ds\right).$$

Note that $\widetilde{\mathbb{Q}}$ is different from $\mathbb{Q}$, which is constructed in (2.12). In fact, the measure $\widetilde{\mathbb{Q}}$ is used as a technical tool for the optimization and can be viewed as a "risk-neutral probability" with respect to the ambient filtration $\mathbb{F}$.

Proposition 1 *Let I be the inverse function of U'. If the equation*

$$\mathbb{E}_{\mathbb{P}}[Z_T \max(I(\widehat{y}Z_T/G_T), H_T)] = x_0$$

has a solution $\widehat{y}$, then

1. *the optimal utility expectation for the problem (P1) is given by*

$$\max_{\boldsymbol{\alpha}\in\mathcal{A},\, X_T^{\alpha}\geq H_T} \mathbb{E}_{\mathbb{P}}[G_T U(X_T^{\alpha})] = \mathbb{E}_{\mathbb{P}}\big[G_T U\big(\max(I(\widehat{y}Z_T/G_T), H_T)\big)\big];$$

2. *if $\boldsymbol{\alpha}^* \in \mathcal{A}$ satisfies $X_T^{\alpha^*} = \max(I(\widehat{y}Z_T/G_T), H_T)$, then it is an optimal investment strategy to the problem (P1);*

3. *let $(Y, \boldsymbol{\xi})$ be the solution to the following backward stochastic differential equation (BSDE)*

$$Y_t = \max(I(\widehat{y}Z_T/G_T), H_T) - \int_t^T (\boldsymbol{\xi}_t \cdot \Sigma_s^{-1}\boldsymbol{\mu}_s) ds - \int_t^T \boldsymbol{\xi}_s \cdot d\boldsymbol{W}_s,$$

then the optimal wealth is given by $X^{\alpha^*} = Y$ and an optimal investment strategy can be written as

$$\boldsymbol{\alpha}^* = \frac{\left(\Sigma^{-1}\right)'\boldsymbol{\xi}}{Y}. \tag{3.27}$$

Proof: For any $\lambda > 0$, let

$$\widetilde{U}(\lambda, y) = \sup_{x\geqslant\lambda}(U(x) - xy). \tag{3.28}$$

The supremum in (3.28) is attained at $x = \max(I(y), \lambda)$. Therefore one has

$$\widetilde{U}(\lambda, y) = U(\max(I(y), \lambda)) - y \max(I(y), \lambda).$$

Since $\boldsymbol{S}$ is a local martingale under $\widetilde{\mathbb{Q}}$, for any investment strategy $\boldsymbol{\alpha} \in \mathcal{A}$, the value process X^α is a non-negative $\widetilde{\mathbb{Q}}$-local martingale and hence a supermartingale. Then the process ZX^α is a $\mathbb{P}$-supermartingale. Let $\boldsymbol{\alpha}$ be an investment strategy such that $X_T^\alpha \geqslant H_T$. By (3.28), for any $y > 0$ one has

$$U(X_T^\alpha) - (yZ_T/G_T)X_T^\alpha \leqslant \widetilde{U}(H_T, yZ_T/G_T).$$

Hence

$$\mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)] \leqslant y\mathbb{E}_\mathbb{P}[Z_T X_T^\alpha] + \mathbb{E}_\mathbb{P}[G_T \widetilde{U}(H_T, yZ_T/G_T)].$$

Since ZX^α is a $\mathbb{P}$-supermartingale, one has $\mathbb{E}_\mathbb{P}[Z_T X_T^\alpha] \leq X_0^\alpha = x_0$. Therefore

$$\mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)] \leqslant yx_0 + \mathbb{E}_\mathbb{P}[G_T U(\max(I(yZ_T/G_T), H_T))] - y\mathbb{E}_\mathbb{P}[Z_T \max(I(yZ_T/G_T), H_T)].$$

If $\widehat{y}$ is a solution to the equation

$$\mathbb{E}_\mathbb{P}[Z_T \max(I(\widehat{y}Z_T/G_T), H_T)] = x_0,$$

then one has

$$\mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)] \leqslant \mathbb{E}_\mathbb{P}[G_T U\big(\max(I(\widehat{y}Z_T/G_T), H_T)\big)].$$

If moreover $\boldsymbol{\alpha}^*$ is an investment strategy such that $X_T^{\alpha^*} = \max(I(\widehat{y}Z_T/G_T), H_T)$, then it is an optimal investment strategy since one has $X_T^{\alpha^*} \geqslant H_T$ and $\mathbb{E}_\mathbb{P}[G_T U(X_T^{\alpha^*})] \geqslant \mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)]$ for any $\boldsymbol{\alpha} \in \mathcal{A}$ such that $X_T^\alpha \geqslant H_T$. Since

$$\mathbb{E}_{\widetilde{\mathbb{Q}}}[\max(I(\widehat{y}Z_T/G_T), H_T)] = \mathbb{E}_\mathbb{P}[Z_T \max(I(\widehat{y}Z_T/G_T), H_T)] = x_0, \tag{3.29}$$

then by the martingale representation theorem, there always exists $\boldsymbol{\alpha}^* \in \mathcal{A}$ which satisfies the relation $X_T^{\alpha^*} = \max(I(\widehat{y}Z_T/G_T), H_T)$.

Finally, if we take $\boldsymbol{\alpha}^*$ as in (3.27), then the process Y verifies the stochastic differential equation

$$dY_t = (\boldsymbol{\xi}_t \cdot \Sigma_t^{-1}\boldsymbol{\mu}_t)dt + \boldsymbol{\xi}_t \cdot d\mathbf{W}_t = Y_t\boldsymbol{\alpha}_t^* \cdot (\boldsymbol{\mu}_t dt + \Sigma_t d\mathbf{W}_t) = Y_t\boldsymbol{\alpha}_t^* \cdot \frac{d\mathbf{S}_t}{\mathbf{S}_t}.$$

Moreover, by (3.29) we obtain that $Y_0 = x_0$. Therefore one has $\max(I(\widehat{y}Z_T/G_T), H_T) = Y_T = X_T^{\alpha^*}$. The proposition is thus proved. □

Example 3 *Let us consider the CRRA utility function*

$$U(x) = \frac{x^p}{p}, \quad p < 1, p \neq 0, x > 0.$$

In this case $I(y) = y^{1/(p-1)}$. By Proposition 1, we search for the solution $\widehat{y}$ to the equation

$$\mathbb{E}_\mathbb{P}\left[Z_T \max\left((\widehat{y}Z_T/G_T)^{1/p-1}, H_T\right)\right] = x_0.$$

An optimal investment strategy $\boldsymbol{\alpha}^*$ is determined by the relation

$$X_T^{\alpha^*} = \max\left((\widehat{y}Z_T/G_T)^{1/(p-1)}, H_T\right).$$

Example 4 *Let us consider the logarithmic utility function*

$$U(x) = \log x, \quad x > 0.$$

We have

$$X_T^{\alpha^*} = \max\left(G_T/\widehat{y}Z_T, H_T\right)$$

such that

$$\mathbb{E}_{\mathbb{P}}\left[(1/\widehat{y} - Z_T H_T/G_T)^+\right] + \mathbb{E}_{\mathbb{P}}\left[Z_T H_T/G_T\right] = x_0.$$

3.2 Solution under the probability constraint

For the optimization problem (P2), we note that the probabilistic constraint is equivalent to $\mathbb{P}(X_T^\alpha < H_T) \leq 1 - p$. We introduce, for any $(y, \lambda, h) \in \mathbb{R}_+^2 \times \mathbb{R}$, the Lagrange multiplier as

$$\widetilde{U}(y, \lambda, h) = \sup_{x>0}\{U(x) - xy - \lambda(\mathbb{I}_{\{x<h\}} - (1-p))\}.$$

As before, let $I(\cdot)$ be the inverse function of U'. Then the supremum defining the function $\widetilde{U}$ is attainted at the point $J(y, \lambda, h) \in \mathbb{R}_+$ with

$$J(y, \lambda, h) = \begin{cases} I(y) & \text{if } I(y) \geq h \text{ or if } I(y) < h \text{ and } U(I(y)) - yI(y) - \lambda > U(h) - yh \\ h & \text{if } I(y) < h \text{ and } U(I(y)) - yI(y) - \lambda \leq U(h) - yh. \end{cases}$$

Proposition 2 *Assume that the system of equations on* (y, λ)

$$\begin{cases} \mathbb{E}_{\mathbb{P}}[Z_T J(yZ_T/G_T, \lambda/G_T, H_T)] = x_0 \\ \mathbb{P}(J(yZ_T/G_T, \lambda/G_T, H_T) < H_T) = 1 - p, \end{cases}$$

has a solution $(\widehat{y}, \widehat{\lambda})$, *then the optimal value in the problem (P2) is* $\mathbb{E}_{\mathbb{P}}[U(J(\widehat{y}Z_T, \widehat{\lambda}, H_T))]$. *Moreover, if* $\boldsymbol{\alpha}^* \in \mathcal{A}$ *satisfies* $X_T^{\alpha^*} = J(\widehat{y}Z_T, \widehat{\lambda}, H_T)$, *then it is an optimal strategy of the problem (P2). More precisely, let* $(Y, \boldsymbol{\xi})$ *be the solution to the BSDE*

$$Y_t = J(\widehat{y}Z_T, \widehat{\lambda}, H_T) - \int_t^T (\boldsymbol{\xi}_s \cdot \Sigma_s^{-1}\boldsymbol{\mu}_s)ds - \int_t^T \boldsymbol{\xi}_s \cdot d\mathbf{W}_s. \tag{3.30}$$

Then an optimal investment strategy is given by $\boldsymbol{\alpha}^* = (\Sigma^{-1})'\boldsymbol{\xi}/Y$.

Since the proof of this proposition shares a large number of arguments with the proof of Proposition 1, it is given in the Appendix.

Example 5 *For the CRRA utility function, we have*

$$J(y, \lambda, h) = \begin{cases} y^{1/(p-1)}, & \text{if } y \leq h^{p-1}, \\ y^{1/(p-1)}, & \text{if } y > h^{p-1} \text{ and } (1/p - 1)y^{p/(p-1)} - \lambda > h^p/p - yh, \\ h, & \text{otherwise.} \end{cases}$$

For the logarithmic utility function,

$$J(y, \lambda, h) = \begin{cases} 1/y, & \text{if } y \leq h^{-1}, \\ 1/y, & \text{if } y > h^{-1} \text{ and } yh - \log(yh) > \lambda + 1, \\ h, & \text{otherwise.} \end{cases}$$

3.3 Solution under the expected shortfall constraint

We now consider the problem with the expected shortfall constraint. Let $v(\cdot,\cdot)$ be a measurable function on $\mathbb{R}_+ \times \mathbb{R}$, which is convex and descreasing on its first coordinate. We assume in addition that, for any $h \in \mathbb{R}$, the function $\partial_x v(x,h)$ on x is bounded, and

$$\lim_{x\to+\infty} \partial_x v(x,h) = 0.$$

Corresponding to the constraint in (P3), we let $v(x,h) = l(x-h) - L/\mathbb{P}(\tau > T)$, where L is the threshold. With the above notation, (P3) is equivalent to the following optimization problem

$$\max_{\substack{\alpha\in\mathcal{A} \\ \mathbb{E}_{\mathbb{P}}[G_T v(X_T^\alpha, H_T)]\le 0}} \mathbb{E}_{\mathbb{P}}[G_T U(X_T^\alpha)].$$

For fixed $\lambda > 0$ and $h \in \mathbb{R}$, the function

$$(x>0) \longmapsto U'(x) - y - \lambda\partial_x v(x,h)$$

is strictly decreasing. Moreover, by the hypothesis on the functions U and v, one has

$$\lim_{x\to 0+} U'(x) - \lambda\partial_x v(x,h) = +\infty, \quad \lim_{x\to+\infty} U'(x) - \lambda\partial_x v(x,h) = 0.$$

We let

$$J(y,\lambda,h) := \inf\{x > 0 \,|\, U'(x) - \lambda\partial_x(x,h) \le y\}.$$

Then the function $(x > 0) \mapsto U(x) - xy - \lambda v(x,h)$ attains its maximal value at the point $x = J(y,\lambda,h)$, which we denote by $\widetilde{U}(y,\lambda,h)$.

Proposition 3 *Assume that the system of equations*

$$\begin{cases} \mathbb{E}_{\mathbb{P}}[Z_T J(yZ_T/G_T, \lambda, H_T)] = x_0 \\ \mathbb{E}_{\mathbb{P}}[v(J(yZ_T/G_T,\lambda,H_T),H_T)] = 0 \end{cases}$$

on (y,λ) admits a solution $(\widehat{y},\widehat{\lambda})$. Then the optimal value of the optimization problem (P3) is $\mathbb{E}_{\mathbb{P}}[J(\widehat{y}Z_T,\widehat{\lambda},H_T)]$. Moreover, if $\boldsymbol{\alpha}^*$ satisfies $X_T^{\alpha^*} = J(\widehat{y}Z_T,\widehat{\lambda},H_T)$, then it is an optimal strategy of the problem (P3). Finally, let $(Y,\boldsymbol{\xi})$ be the solution to the BSDE

$$Y_t = J(\widehat{y}Z_T,\widehat{\lambda},H_T) - \int_t^T (\boldsymbol{\xi}_t\cdot\Sigma_s^{-1}\boldsymbol{\mu}_s)ds - \int_t^T \boldsymbol{\xi}_s\cdot d\mathbf{W}_s. \tag{3.31}$$

Then $\boldsymbol{\alpha}^* = (\Sigma^{-1})'\boldsymbol{\xi}/Y$ is an optimal investment strategy.

Since the proof of this proposition shares a large number of arguments with the proof of Proposition 1, it is given in the Appendix.

Example 6 *Let $U(x) = x^p/p$ be the CRRA utility function, $p < 1$, $p \neq 0$. Let $v(x,h) = (h-x)_+ - L$, where $L > 0$ is a constant. We have*

$$U'(x) - y - \lambda\partial_x v(x,h) = x^{p-1} - y + \lambda\mathbb{I}_{\{x\leqslant h\}}.$$

Hence

$$J(y,\lambda,h) = \begin{cases} (y-\lambda)^{1/(p-1)}, & \text{if } \lambda < y \le \lambda + h^{p-1}, \\ y^{1/(p-1)}, & \text{if } y > \lambda + h^{p-1}, \\ h, & \text{if } y \leqslant \lambda. \end{cases}$$

Therefore

$$v(J(y,\lambda,h),h) = (h-(y-\lambda)^{1/(p-1)})\mathbb{I}_{\lambda<y\le\lambda+h^{p-1}} - L.$$

Hence the equation becomes

$$\begin{cases} \mathbb{E}_{\mathbb{P}}\Big[Z_T(\widehat{y}Z_T/G_T-\widehat{\lambda})^{1/(p-1)}\mathbb{I}_{\{0<\widehat{y}Z_T/G_T-\widehat{\lambda}\le H_T^{p-1}\}}\Big] + \mathbb{E}_{\mathbb{P}}\Big[Z_T(\widehat{y}Z_T/G_T)^{1/(p-1)}\mathbb{I}_{\{\widehat{y}Z_T/G_T>\widehat{\lambda}+H_T^{p-1}\}}\Big] \\ \quad +\mathbb{E}_{\mathbb{P}}\Big[Z_TH_T\mathbb{I}_{\{\widehat{y}Z_T\le\widehat{\lambda}G_T\}}\Big] = x_0, \\ \mathbb{E}_{\mathbb{P}}\Big[G_T(H_T-(\widehat{y}Z_T/G_T-\widehat{\lambda})^{1/(p-1)})_+\mathbb{I}_{\{\widehat{\lambda}G_T<\widehat{y}Z_T\}}\Big] = L. \end{cases}$$

4 Numerical application

In this section, we give a numerical example to show the optimal strategy derived under the almost surely constraint (Problem (P1)). We consider the case where the financial assets are: the cash, the default-free zero-coupon, a corporate bond and one stock. The state variables have the following dynamics under $\mathbb{Q}$:

$$\begin{pmatrix} dr_t \\ d\varphi_t^1 \end{pmatrix} = \begin{pmatrix} a(b-r_t) \\ a_1(b_1-\varphi_t^1) \end{pmatrix} dt + \begin{pmatrix} \sigma\sqrt{r_t} & 0 \\ 0 & \sigma_1\sqrt{\varphi_t^1} \end{pmatrix} \begin{pmatrix} dW^{r,\mathbb{Q}} \\ dV_t^1 \end{pmatrix}.$$

The dynamics of assets under $\mathbb{P}$ are the following ones:

$$\begin{aligned} \frac{dB_0(t)}{B_0(t)} &= (r_t + \psi_{a,\sigma}(T_0)\sigma\alpha^r r_t)dt - \psi_{a,\sigma}(T_0)\sigma\sqrt{r_t}dW_t^{r,\mathbb{P}} \\ \frac{dB_1(t)}{B_1(t)} &= \left(r_t + \lambda_t^i + \psi_{a,\sigma}(T_1)\sigma\alpha^r r_t - \rho^{1,S}\psi_{a_1,\sigma_1}(T_1)\sqrt{\varphi_t^1}\alpha^S\right) dt \\ &\quad - \sigma\psi_{a,\sigma}(T_1)\sqrt{r_t}dW_t^{r,\mathbb{P}} - \sigma_1\psi_{a_1,\sigma_1}(T_1)\sqrt{\varphi_t^1}dV_t^{1,\mathbb{P}} \\ \frac{dS_t}{S_t} &= (r_t + \sigma_S(-\rho\alpha^r\sqrt{r_t} + \sqrt{1-\rho^2}\alpha^S))dt + \sigma_S(\rho dW_t^{r,\mathbb{P}} + \sqrt{1-\rho^2}dW_t^{S,\mathbb{P}}) \end{aligned}$$

where $\psi_{a,\sigma}(x) = \frac{2(1-e^{-hx})}{h+a+(h-a)e^{-hx}}$ and $h = \sqrt{a^2+2\sigma^2}$.

Table 2 lists the parameters characterizing the financial market. These parameters are consistent with the French financial market in 2015.

The institution's utility function is the logarithmic function. We compare our optimal allocation for the optimization problem (P1) with the optimal allocation for the following problem: $\max_{\boldsymbol{\alpha}\in\mathcal{A}} \mathbb{E}_{\mathbb{P}}[\log(X_T)]$ for which the solution is given by $\boldsymbol{\beta}_t^* = (\Sigma_t\Sigma_t')^{-1}\boldsymbol{\mu}_t$. The main differences between both problems are that the almost surely constraint and the default risk (G_T) are ignored.

The driven liability is chosen as

$$H_t = q_0S_t^0 + q_1B_0(t) + q_2B_1(t) + q_3S_t$$

with $q_0 = 0.1/S_0^0$, $q_1 = 0.4/B_1(0)$, $q_1 = 0.35/B_1(0)$ and $q_3 = 0.15/S_0$ such that $H_0 = 1$.

Short term interest		Default intensity	
a	0.6	a_1	1.1
b	0.01	b_1	0.005
σ	0.1	σ_1	0.1
α^r	0.1		
ZC bond B_0		ZC bond B_1	
T_0	1	T_1	1
Stock S		Initial value	
σ_S	0.2	r_0	0.01
α^S	0.7	φ_0^1	0.005
ρ	0	H_0, S_0	1
$\rho^{1,S}$	-0.5	x_0	1.02

Table 2. Values of model parameters

The optimal value of the portfolio is given by

$$Y_t = X_t^{\alpha^*} = \mathbb{E}_{\mathbb{Q}}\left[\max(I(\widehat{y}Z_T/G_T), H_T)|\mathcal{F}_t\right]$$

where $\mathcal{F}_t = \sigma\left((W_u^{r,\mathbb{Q}})_{u\in[0,t]}, (V_t^1)_{u\in[0,t]}, (W_t^{S,\mathbb{Q}})_{u\in[0,t]}\right)$, and the optimal proportions satisfy $\boldsymbol{\alpha}_t^* = (\Sigma_t^{-1})'\boldsymbol{\xi}_t/X_t^{\alpha^*}$ with

$$\begin{aligned}
\xi_{1,t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}_{\mathbb{Q}}\left[X_{t+\Delta t}^{\alpha^*}(W_{t+\Delta t}^{r,\mathbb{Q}} - W_t^{r,\mathbb{Q}})|\mathcal{F}_t\right]\\
\xi_{2,t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}_{\mathbb{Q}}\left[X_{t+\Delta t}^{\alpha^*}(V_{t+\Delta t}^{1} - V_t^{1})|\mathcal{F}_t\right]\\
\xi_{3,t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}_{\mathbb{Q}}\left[X_{t+\Delta t}^{\alpha^*}(W_{t+\Delta t}^{S,\mathbb{Q}} - W_t^{S,\mathbb{Q}})|\mathcal{F}_t\right].
\end{aligned}$$

We computed all the conditional expectations over $N = 10^5$ paths and chose $\Delta t = 0.001$. Figure 1 gives four examples of paths of B_0, B_1, S, H and 11 values of $Y_{t_i} = X_{t_i}^{\alpha^*}$ and $\boldsymbol{\alpha}_{t_i}^*/\boldsymbol{\beta}_{t_i}^*$ for $t_i = iT/10$ for these paths.

For paths 1, 2, 4, $X_T^{\alpha^*} = H_T$, while for path 3, $X_T^{\alpha^*} > H_T$. Actually, approximately only 11% of paths have a terminal value of the optimal portfolio which is larger than the value of the liability driven benchmark. It can be observed that the higher the value of the optimal portfolio from the liability driven benchmark, the closer the values of $\boldsymbol{\alpha}^*$ to $\boldsymbol{\beta}^*$.

Since the stock upside potential (i.e. the stock expected risk premium) is high compared to the other classes upside, one can notice that it is a strong outperformance lever for the benchmark. Thus the ratio $\boldsymbol{\alpha}^*/\boldsymbol{\beta}^*$ is highly correlated with the values of S. Hence the allocations that overweight equities (higher ratio $\boldsymbol{\alpha}^*/\boldsymbol{\beta}^*$) are set up when the equity price increases. Conversely the optimal strategy typically implies a reduction to equity allocation when a drop of equity prices has led to a substantial diminution of the price of the portfolio.

Moreover, when the stock expected risk premia and the volatilities of the asset classes are low, the ratio $\boldsymbol{\alpha}^*/\boldsymbol{\beta}^*$ should remain far from 1 and the terminal wealth of the different constrained asset allocations are pretty similar. In the current financial environment where equities and fixed income volatilities and the equity risk premium are high, the outcomes of all admissible strategies are very different and the choice of the benchmark thus has significant implications.

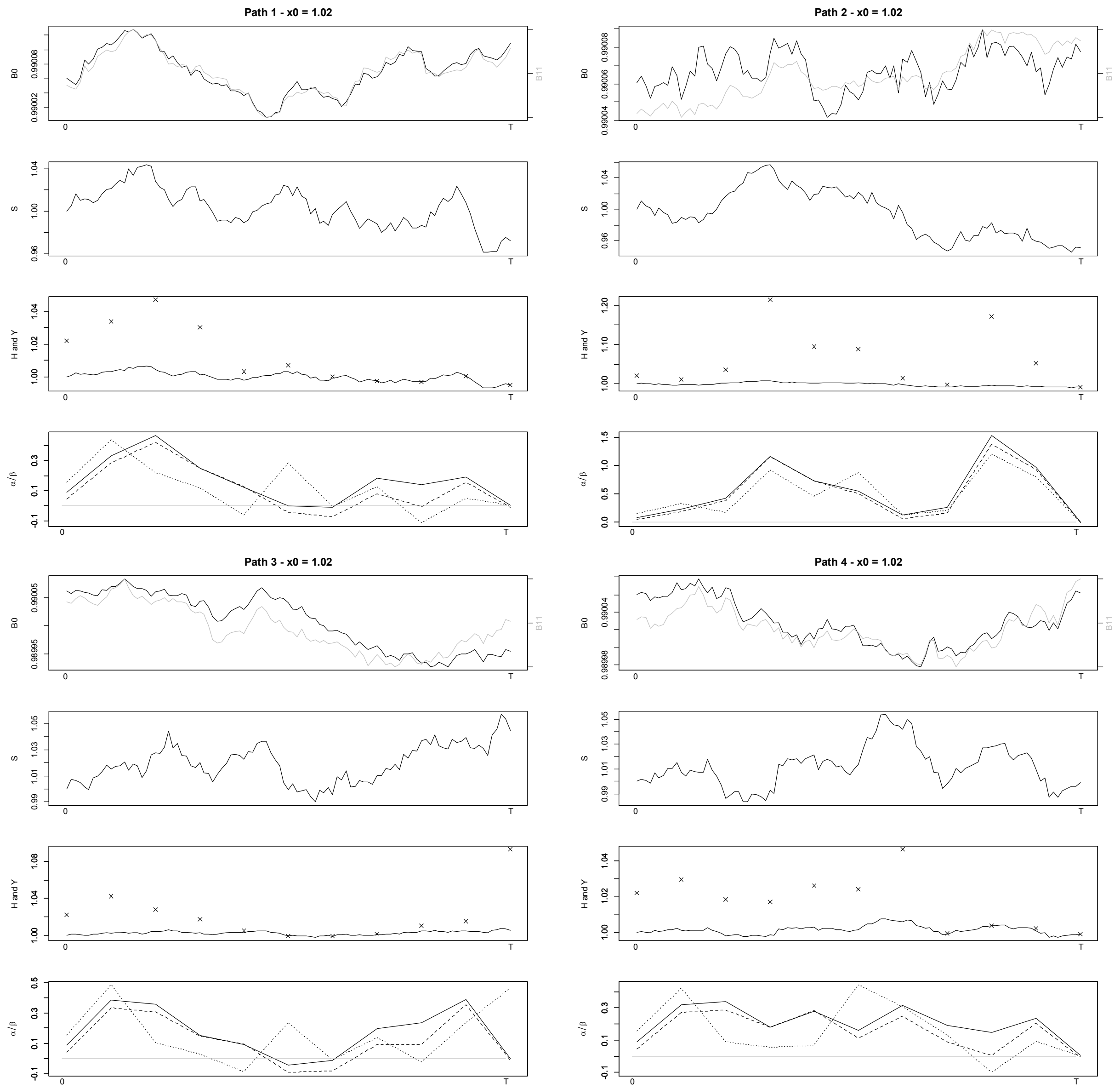

Figure 1: Four paths of the stock prices B_0 (black), B_1 (gray), S (black), the values of optimal allocation portfolio Y (×), of the liability driven benchmark H (black) and of the ratio $\boldsymbol{\alpha}^*/\boldsymbol{\beta}^*$ (— $\alpha^*_{1,t}$, - - - - $\alpha^*_{2,t}$, · · · · $\alpha^*_{3,t}$,)

5 Conclusions

In this paper we propose an original performance attribution methodology for a liability driven institution that wishes to measure the financial performance of its delegated asset management. More generally we offer a performance attribution methodology with credit and default risks under allocation constraints.

The methodology is the following: (i) define a naïve benchmark, generally the libabillity driven benchmark (H_t), (ii) define the strength of the constraint on the terminal wealth (almost surely, in probability or an expected shortfall constraint), (iii) determine the optimal asset allocation under the ex-ante probability (called *the benchmark*), (iv) run a classic performance attribution against the benchmark to deduce the out- or under-performance, the selection effect and the possible allocation effect.

We show that this benchmark always exists under certain conditions, for three different constraint specifications, and we characterize it. On the numerical simulation, we evidence that, under a specified probability consistent with the French financial market in 2015, there exist 11% of admissible paths among which one can extract the optimal one.

This methodology implicitly suggests that either the asset manager choses the ex-ante optimal constrained-allocation or she deviates from it in the hopes of creating alpha. Hence, the asset manager is evaluated against the benchmark we have defined hereinabove.

There are some avenues for further research, notably broadening the constraints under which the asset manager invests. In practise, in addition to the allocation constraints, the framework imposed to the asset manager may include accounting constraints (upper and lower limit for the realised gain or loss that have an impact on the profit and loss account) and/or risk constraints (no short sale).

References

[1] Baldeaux, J. and Platen, E., Liability driven investlents under a benchmark based approach. *Quantitative Finance Research Centre - Research paper* 325, 2013.

[2] Basso, A. and Funari, S., A generalized performanca attribution technique for mutual funds. *Central European Journal of Operations Research*, 13:65 84, 2005.

[3] Bertrand, P., Risk-adjusted performance attribution and portfolio optimisations under trackingerror constraints. *Journal of Asset Management*, 10:75 88, 2008.

[4] Binay, M., Performance attribution of us institutonal investors. *Financial Management*, Summer 2005:127 152, 2005.

[5] Boulier, J.-F., Huang, S. and Taillard, G., Optimal management under stochastic interest rates: the case of a protected defined contribution pension fund. *Insurance: Mathematics and Economics*, 28: 173-189, 2001.

[6] Brooks, L.D. Brooks and Porter, G., Mutual fund performance attribution: 1994-2005. *Financial Services Review*, 21:259 273, 2012.

[7] Brigo, D. and Mercurio, F., A deterministic-shift extension of analytically tractable and time-homogeneous short rate models. *Finance and Stochastics*, 5: 369–388, 2001.

[8] Boyle, P. and Tian, W. Portfolio management with constraints. *Mathematical Finance*, 17: 319-343, 2007.

[9] Chen, L., Stochastic Mean and Stochastic Volatility - A Three-Factor Model of the Term Structure of Interest Rates and Its Application to the Pricing of Interest Rate Derivatives. *Financial Markets, Institutions, and Instruments*, 5: 1–88, 1996.

[10] Christopherson, J.A., Ferson, W. and Glassman, D.A., Conditionning manager akoha on economic information: Another look ate the persistence of performance. *Review of Financial Studies*, 11 (1): 111 142, 1998.

[11] Deguest, R., Martellini, L. and Milhau, V., Hedging versus insurance: long-horizon investing with short-term constraints. Preprint EDHEC, 2013.

[12] Duffie, D., Credit Risk Modeling with Affine Processes. Cattedra Galileana Lectures, Scuola Normale, Pisa, April, 2002, *Journal of Banking and Finance*, 29: 2751-2802, 2005.

[13] Duffie, D. and Kan, R., A Yield-Factor Model of Interest Rates. *Mathematical finance*, 6: 379-406, 1995.

[14] Duffie, D. and Singleton, K.J., Credit Risk: pricing, measurement and management. Princeton University Press, 2003.

[15] El Karoui, N., Jeanblanc, M. and Lacoste, V., Optimal portfolio management with American capital guarantee, *Journal of Economic Dynamics and Control*, 29: 449-468, 2005.

[16] Elliott, R. J., Jeanblanc, M. and Yor, M., On models of default risk. *Mathematical Finance*, 10(2):179195, 2000.

[17] Ferson, W. and Schadt, R., Measuring fund strategy and performance in changing economic conditions. *Journal of Finance*, 51 (2):425 461, 1996.

[18] Ferson, W. and Warther, V., Evaluating fund peformance in a dynamic market. *Financial Analysts Journal*, 52 (6):20 28, 1996.

[19] Föllmer, H. and Leukert P., Efficient hedging: cost vs. shortfall risk. *Finance and Stochastics*, 4: 117-146, 2000.

[20] Grossman, S. and Vila, J.-L., Portfolio insurance in complete markets: a note. *Journal of Business*, 62: 473-476, 1989.

[21] Gundel, A. and Weber, S., Robust utility maximization with limited downside risk in incomplete markets, *Stochastic Processes and Their Applications*, 117: 1663-1688, 2007.

[22] Han, N.-W. and Hung, M.-W., Optimal asset allocation for DC pension plans under inflation. *Insurance: Mathematics and Economics*, 51: 172-181, 2012.

[23] Lehman, B. and Modest, D., Mutual fund performance evaluation: A comparision of benchmarks and benchmark comparisons. *Journal of Finance*, 42 (2):233 265, 1987.

[24] Maghsoodi, Y., Solution of the extended CIR Term Structure and Bond Option Valuation. *Mathematical Finance*, 6: 89–109, 1996.

[25] Martellini, L., Milhau, V. and Tarelli, A., Dynamic investment strategies for corporate pension funds in the presence of sponsor risk. Preprint EDHEC, March, 2012.

[26] Rogers, L.C.G., Which model for term-structure of interest rates should one use?. *Mathematical Finance*, 65: 93-116, 1995.

6 Appendix. Proofs of the main results

6.1 Proof of Proposition 2

Let $\boldsymbol{\alpha} \in \mathcal{A}$ be an investment strategy which satisfies the constraint $\mathbb{P}(X_T^\alpha < H_T) \le 1 - p$. For any $y > 0$ and $\lambda > 0$, we have

$$\begin{aligned} U(X_T^\alpha) &\le \widetilde{U}(yZ_T/G_T, \lambda/G_T, H_T) + (yZ_T/G_T)X_T^\alpha + (\lambda/G_T)(1_{\{X_T < H_T\}} - (1-p)) \\ &= U(J(yZ_T/G_T, \lambda/G_T, H_T)) - (yZ_T/G_T)J(yZ_T/G_T, \lambda/G_T, H_T) \\ &\quad - (\lambda/G_T)(1_{\{J(yZ_T/G_T,\lambda/G_T,H_T)<H_T\}} - (1-p)) \\ &\quad + (yZ_T/G_T)X_T^\alpha + (\lambda/G_T)(1_{\{X_T<H_T\}} - (1-p)). \end{aligned}$$

We multiply the both sides of the equality by G_T and then take the expectation with respect to $\mathbb{P}$. By using the facts that ZX^α is a $\mathbb{P}$-martingale and that $\mathbb{P}(X_T^\alpha < H_T) \le 1 - p$, we obtain

$$\begin{aligned} \mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)] &\le \mathbb{E}_\mathbb{P}[G_T U(J(yZ_T/G_T, \lambda/G_T, H_T))] - y\mathbb{E}_\mathbb{P}[Z_T J(yZ_T/G_T, \lambda/G_T, H_T)] \\ &\quad - \lambda(\mathbb{P}(J(yZ_T/G_T, \lambda/G_T, H_T) < H_T) - (1-p)) + yx_0. \end{aligned}$$

Therefore, if $(\widehat{y}, \widehat{\lambda})$ is a solution to the equation

$$\begin{cases} \mathbb{E}_\mathbb{P}[Z_T J(\widehat{y}Z_T/G_T, \widehat{\lambda}/G_T, H_T)] = x_0 \\ \mathbb{P}(J(\widehat{y}Z_T/G_T, \widehat{\lambda}/G_T, H_T) < H_T) = 1-p, \end{cases} \tag{6.32}$$

then one has

$$\mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)] \leqslant \mathbb{E}_\mathbb{P}[G_T U(J(\widehat{y}Z_T/G_T, \widehat{\lambda}/G_T, H_T))].$$

If moreover $\boldsymbol{\alpha}^* \in \mathcal{A}$ is an investment strategy such that $X_T^{\alpha^*} = J(\widehat{y}Z_T/G_T, \widehat{\lambda}/G_T, H_T)$, then it is an optimal strategy since one has $\mathbb{P}(X_T^{\alpha^*} < H_T) = 1-p$ and $\mathbb{E}_\mathbb{P}[X_T^\alpha] \le \mathbb{E}_\mathbb{P}[X_T^{\alpha^*}]$ for any $\boldsymbol{\alpha} \in \mathcal{A}$ such that $\mathbb{P}(X_T^\alpha < H_T) \le 1-p$. By the martingale representation theorem, such optimal strategy always exists, provided that the parameters $(\widehat{y}, \widehat{\lambda})$ are well defined. Finally, similar as in Proposition 1, the process Y verifies the equation

$$dY_t = (\boldsymbol{\xi}_t \cdot \Sigma_t^{-1}\boldsymbol{\mu}_t)d_t + \boldsymbol{\xi}_t \cdot d\boldsymbol{W}_t = Y_t \boldsymbol{\alpha}_t^* \cdot \frac{d\mathbf{S}_t}{\mathbf{S}_t}$$

with $Y_0 = x_0$. Therefore one has $J(\widehat{y}Z_T/G_T, \widehat{\lambda}/G_T, H_T) = Y_T = X_T^{\alpha^*}$, which gives the terminal condition of (3.30).

6.2 Proof of Proposition 3

Let $\boldsymbol{\alpha}$ be an investment strategy. For any $y > 0$ and any $\lambda > 0$ we have

$$\begin{aligned} U(X_T^\alpha) &\le \widetilde{U}(yZ_T/G_T, \lambda, H_T) + (yZ_T/G_T)X_T^\alpha + \lambda v(X_T^\alpha, H_T) \\ &= U(J(yZ_T/G_T, \lambda, H_T)) - (yZ_T/G_T)J(yZ_T/G_T, \lambda, H_T) \\ &\quad - \lambda v(J(yZ_T/G_T, \lambda, H_T), H_T) + (yZ_T/G_T)X_T^\alpha + \lambda v(X_T^\alpha, H_T). \end{aligned}$$

Multiply both sides of the equality by G_T and then take the expectation. By using the facts that ZX^α is a $\mathbb{P}$-martingale and that $\mathbb{E}_\mathbb{P}[G_T v(X_T^\alpha, H_T)] \le 0$, we obtain

$$\begin{aligned} \mathbb{E}_\mathbb{P}[G_T U(X_T^\alpha)] &\le \mathbb{E}_\mathbb{P}[G_T U(J(yZ_T/G_T, \lambda, H_T))] - y\mathbb{E}_\mathbb{P}[Z_T J(yZ_T/G_T, \lambda, H_T)] \\ &\quad - \lambda \mathbb{E}_\mathbb{P}[G_T v(J(yZ_T/G_T, \lambda, H_T), H_T)] + yx_0. \end{aligned}$$

Therefore, if $(\widehat{y}, \widehat{\lambda})$ is the solution to the equation

$$\begin{cases} \mathbb{E}_{\mathbb{P}}[Z_T J(\widehat{y} Z_T/G_T, \widehat{\lambda}, H_T)] = x_0, \\ \mathbb{E}_{\mathbb{P}}[G_T v(J(\widehat{y} Z_T/G_T, \widehat{\lambda}, H_T), H_T)] = 0. \end{cases} \tag{6.33}$$

Then

$$\mathbb{E}_{\mathbb{P}}[G_T U(X_T^{\alpha})] \leqslant \mathbb{E}_{\mathbb{P}}[G_T U(J(\widehat{y} Z_T, \widehat{\lambda}, H_T))].$$

If there exists an strategy $\boldsymbol{\alpha}^*$ such that $X_T^{\boldsymbol{\alpha}^*} = J(\widehat{y} Z_T/G_T, \widehat{\lambda}, H_T)$, then it is an optimal strategy since one has $\mathbb{E}_{\mathbb{P}}[G_T v(X_T^{\boldsymbol{\alpha}^*}, \widehat{\lambda}, H_T)] = 0$ and $\mathbb{E}_{\mathbb{P}}[X_T^{\boldsymbol{\alpha}}] \leq \mathbb{E}_{\mathbb{P}}[X_T^{\boldsymbol{\alpha}^*}]$ for any $\boldsymbol{\alpha} \in \mathcal{A}$ such that $\mathbb{E}_{\mathbb{P}}[G_T v(X_T^{\boldsymbol{\alpha}}, \widehat{\lambda}, H_T)] \leq 0$. By the martingale representation theorem, such optimal strategy always exists, provided that the parameters $(\widehat{y}, \widehat{\lambda})$ are well defined. Finally, we note that the terminal condition of the BSDE (3.31) is given by $J(\widehat{y} Z_T/G_T, \widehat{\lambda}, H_T) = Y_T = X_T^{\boldsymbol{\alpha}^*}$. The proposition is thus proved.

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