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Optimal investment with counterparty risk: a default-density model approach

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Abstract We consider a financial market with a stock exposed to a counterparty risk inducing a jump in the price, and which can still be traded after this default time. The jump represents a loss or gain of the asset value at the default of the counterparty. We use a default-density modelling approach, and address in this incomplete market context the problem of expected utility maximization from terminal wealth. We show how this problem can be suitably decomposed in two optimization problems in a default-free framework: an after-default utility maximization and a global before-default optimization problem involving the former one. These two optimization problems are solved explicitly, respectively, by duality and dynamic programming approaches, and provide a detailed description of the optimal strategy. We give some numerical results illustrating the impact of counterparty risk and the loss or gain given default on optimal trading strategies, in particular with respect to the Merton portfolio selection problem. For example, this explains how an investor can take advantage of a large loss of the asset value at default in extreme situations as observed during the financial crisis.

Keywords Counterparty risk · Contagious loss or gain · Density of default time · Optimal investment · Duality · Dynamic programming · Backward stochastic differential equation (BSDE)

Mathematics Subject Classification (2000) 60J75 · 91B28 · 93E20

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1 Introduction

In a financial market, the default of one firm has usually important influences on the other ones. This has been shown clearly by several recent default events during the credit crisis. The impact of a counterparty default may arise in various contexts. In terms of credit spreads, one observes in general a positive "jump" of the default intensity, called the contagious jump and investigated first by Jarrow and Yu [9]. For the credit derivative CDS, Brigo and Capponi [4] have considered the case where not only the underlying credit name, but also the transaction counterparty (buyer or seller of CDS) may default. In terms of asset (or stock) values for a firm, the default of a counterparty will in general induce a drop, or sometimes a rise, of its value process. The drop corresponds to a contagious loss when the asset is positively correlated with the counterparty, while the rise often represents a negative correlation situation (the asset of one firm in a duopoly competition for example). In this paper, we analyze the impact of this risk on the optimal investment problem. More precisely, we consider an agent who invests in a risky asset exposed to a counterparty risk, and we are interested in the optimal trading strategy and performance, i.e., the value function, when taking into account the possibility of default of a counterparty, together with the instantaneous loss or gain of the asset at the default time.

The global market information containing default is modeled by the progressive enlargement of a reference filtration, denoted by $\mathbb{F}$, representing the default-free information. The default time τ is in general a totally inaccessible stopping time with respect to the enlarged filtration $\mathbb{G}$, but is not an $\mathbb{F}$-stopping time. We shall work with a density hypothesis on the conditional law of default given $\mathbb{F}$. This hypothesis has been introduced by Jacod [8] in an initial enlargement of filtrations framework, and has been adopted recently by El Karoui et al. [6] in the progressive enlargement setting for credit risk analysis. The density approach is particularly suited for studying what goes on after the default, i.e., on $\{\tau \le t\}$. For the before-default analysis on $\{\tau > t\}$, there exists an explicit relationship between the density approach and the widely used intensity approach.

The market model considered in the $\mathbb{G}$-filtration is incomplete due to the jump induced by the default time. The general optimal investment problem in an incomplete market has been studied by Kramkov and Schachermayer [12] via duality methods. Concerning the default risk, this problem has been treated in Blanchet-Scalliet et al. [2], Bouchard and Pham [3], and Collin-Dufresne and Hugonnier [5], where the agent can no longer invest in the stock once the default occurs; see also Sircar and Zariphopoulou [15] for utility indifference pricing of multiname credit derivatives. Optimal investment problems with a risky asset subject to a jump induced by a counterparty default (and which can still be traded after the default, as in our context) were recently studied by Lim and Quenez [13] for a utility maximization criterion, and by Ankirchner et al. [1] for utility indifference pricing with exponential utility. Both these papers used a direct BSDE approach in the $\mathbb{G}$-filtration for studying the corresponding stochastic control problem, in the spirit of Morlais [14] for market models with jumps. The solution to their problem is then characterized through a BSDE with jumps. In this paper, we provide an alternative approach, which makes use of the specific feature of the single jump induced by the counterparty default. A natural idea is to separate the initial problem into a problem after the default and a problem before the default. We show how this can be achieved successfully by relying on the density hypothesis on the default time, and we derive a suitable decomposition of the initial optimization problem into an after-default one and a global before-default one. The key feature is that both problems are reduced to a market setting in the reference filtration $\mathbb{F}$, and the solution of the latter problem depends on the solution of the former. These two optimization problems in complete markets are solved by duality and dynamic programming approaches, and the main advantage is to give a better insight and more explicit results than the incomplete market framework. The interesting feature of our decomposition is to provide a nice description of the optimal strategy switching at the default time τ. Moreover, the fairly explicit solution (for the CRRA utility function) makes clear the roles in the investment strategy played by the default time τ and the loss or gain given default, as shown by some numerical examples.

The article is organized as follows. In Sect. 2, we present the model and the investment problem, and introduce the default density hypothesis. We then explain in Sect. 3 how to decompose the optimal investment problem into the before-default and after-default ones. We solve these two optimization problems in Sect. 4, by using the duality approach for the after-default one and the dynamic programming approach for the global before-default one. We examine in more detail the popular case of a CRRA utility function and finally, numerical results illustrate the impact of counterparty risk on optimal trading strategies, in particular with respect to the classical Merton portfolio selection problem.

2 The contagion risk model

We consider a financial market model with a riskless bond assumed for simplicity equal to one, and a stock subject to a counterparty risk. The dynamics of the risky asset are affected by another firm, the counterparty, which may default at some random time, inducing consequently a jump in the asset price. However, this stock still exists and can be traded after the default of the counterparty.

• *Market information and density hypothesis.* Let us fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with a Brownian motion $W = (W_t)_{t\in[0,T]}$ over a finite horizon $T < \infty$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ the natural filtration of W. We are given a nonnegative and finite random variable τ, representing the default time, on $(\Omega, \mathcal{G}, \mathbb{P})$. Before the default time τ, the filtration $\mathbb{F}$ represents the information accessible to the investors. When the default occurs, the investors observe it and add this new information to the reference filtration $\mathbb{F}$. We then introduce $D_t = 1_{\tau\le t}$, $0 \le t \le T$, the filtration $\mathbb{D} = (\mathcal{D}_t)_{t\in[0,T]}$ generated by this jump process, and call $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ the progressively enlarged filtration $\mathbb{F} \vee \mathbb{D}$, representing the structure of the global information available for the investors over $[0, T]$.

In the sequel, we shall make the following *standing assumption*, called *density hypothesis*, on the default time of the counterparty. For any $t \in [0, T]$, the conditional distribution of τ given $\mathcal{F}_t$ admits a density with respect to Lebesgue measure, i.e., there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable positive functions $(\omega, \theta) \mapsto \alpha_t(\theta)$ such that

$$\textbf{(DH)} \quad \mathbb{P}[\tau \in d\theta | \mathcal{F}_t] = \alpha_t(\theta)\, d\theta, \quad t \in [0, T].$$

We note that for any $\theta \geq 0$, the process $\{\alpha_t(\theta), 0 \leq t \leq T\}$ is a $(\mathbb{P}, \mathbb{F})$-martingale.

Remark 2.1 The progressively enlarged filtration $\mathbb{G}$ is the smallest filtration containing $\mathbb{F}$ which makes τ a stopping time. We recall from [11], Lemma 4.4, the decomposition of any $\mathbb{G}$-predictable and adapted process. Let φ be a $\mathbb{G}$-predictable (resp. adapted) process. Then there exist an $\mathbb{F}$-adapted process $\varphi^{\mathbb{F}}$ and a family of processes $\{\varphi_t^d(\theta), \theta \leq t \leq T, \theta \in [0, T]\}$, where $\varphi_t^d(\theta)$ is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ for all $t \in [0, T]$, such that

$$\varphi_t = \varphi_t^{\mathbb{F}} 1_{t \leq \tau} + \varphi_t^d(\tau) 1_{t > \tau}, \quad 0 \leq t \leq T,$$
$$\big(\text{resp. } \varphi_t^{\mathbb{F}} 1_{t < \tau} + \varphi_t^d(\tau) 1_{t \geq \tau}, \quad 0 \leq t \leq T\big).$$

Remark 2.2 The density hypothesis on a random time is usual in the theory of initial enlargement of filtrations, and was introduced by Jacod [8]. The **(DH)** hypothesis was recently adopted by El Karoui et al. [6] in the progressive enlargement of filtrations for credit risk modeling. Notice that in the particular case where the family of densities satisfies $\alpha_T(t) = \alpha_t(t)$ for all $0 \leq t \leq T$, we have $\mathbb{P}[\tau > t|\mathcal{F}_t] = \mathbb{P}[\tau > t|\mathcal{F}_T]$. This corresponds to the so-called immersion hypothesis (or **H**-hypothesis), which is a familiar condition in credit risk analysis, and means equivalently that any square-integrable $\mathbb{F}$-martingale is a square-integrable $\mathbb{G}$-martingale. The **H**-hypothesis appears natural for the analysis on before-default events when $t < \tau$, but is actually restrictive when it concerns after-default events on $\{t \geq \tau\}$; see [6] for a more detailed discussion. By considering here the whole family $\{\alpha_t(\theta), t \in [0, T], \theta \in \mathbb{R}_+\}$, we obtain additional information for the analysis of after-default events, which is crucial for our purpose.

Let us also mention that the classical intensity of default can be expressed in an explicit way by means of the density (see [6]). Indeed, the $(\mathbb{P}, \mathbb{G})$-predictable compensator of $D_t = 1_{\tau \leq t}$ is given by $\int_0^{t \wedge \tau} \alpha_\theta(\theta)/G_\theta \, d\theta$, where $G_t = \mathbb{P}[\tau > t|\mathcal{F}_t]$ is the conditional survival probability. In other words, the process $M_t = D_t - \int_0^{t\wedge\tau} \alpha_\theta(\theta)/G_\theta \, d\theta$ is a $(\mathbb{P}, \mathbb{G})$-martingale. Thus, by observing from the martingale property of $\{\alpha_t(\theta), 0 \leq t \leq T\}$ that $G_t = \int_t^\infty \alpha_t(\theta)\, d\theta = \int_t^\infty \mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]\, d\theta$, we completely recover the intensity process $\lambda_t^{\mathbb{G}} = 1_{t \leq \tau} \alpha_t(t)/G_t$ from the knowledge of the process $\{\alpha_t(t), t \geq 0\}$. However, given the intensity $\lambda^{\mathbb{G}}$, we can only obtain some part of the density family, namely $\alpha_t(\theta)$ for $\theta \geq t$.

• *Asset price model under counterparty risk.* We consider a risky asset subject to counterparty risk, and with $\mathbb{G}$-adapted price process S given by

$$S_t = S_t^{\mathbb{F}} 1_{t < \tau} + S_t^d(\tau) 1_{t \geq \tau}, \quad 0 \leq t \leq T, \tag{2.1}$$

where $S^{\mathbb{F}}$ is an $\mathbb{F}$-adapted process evolving according to

$$dS_t^{\mathbb{F}} = S_t^{\mathbb{F}}\big(\mu_t^{\mathbb{F}}\,dt + \sigma_t^{\mathbb{F}}\,dW_t\big), \quad S_0^{\mathbb{F}} = S_{0-}, \quad 0 \le t \le T, \tag{2.2}$$

and $\{S_t^d(\theta), \theta \le t \le T,\ \theta \in [0,T]\}$ is a measurable (in θ) family of $\mathbb{F}$-adapted processes governed by

$$dS_t^d(\theta) = S_t^d(\theta)\big(\mu_t^d(\theta)\,dt + \sigma_t^d(\theta)\,dW_t\big), \quad \theta < t \le T, \tag{2.3}$$

$$S_\theta^d(\theta) = S_{\theta-}^{\mathbb{F}}(1-\gamma_\theta). \tag{2.4}$$

The asset price process S is càdlàg, and may jump at time 0 if there is a default at $\tau = 0$. Here, we denote by S_{0-} the initial value of the asset. The coefficients $\mu^{\mathbb{F}}$, $\sigma^{\mathbb{F}}$ are $\mathbb{F}$-adapted processes, $(\omega,\theta) \mapsto \mu_t^d(\theta)$, $\sigma_t^d(\theta)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions for all $t \in [0,T]$, and γ is an $\mathbb{F}$-adapted process. We assume that $\sigma_t^{\mathbb{F}} > 0$, $0 \le t \le T$, $\sigma_t^d(\theta) > 0$, $\theta < t \le T$, $\theta \in [0,T]$, and for all $\theta \in [0,T]$ the integrability conditions

$$\int_0^T \left|\frac{\mu_t^{\mathbb{F}}}{\sigma_t^{\mathbb{F}}}\right|^2 dt + \int_\theta^T \left|\frac{\mu_t^d(\theta)}{\sigma_t^d(\theta)}\right|^2 dt + \int_0^T \left|\sigma_t^{\mathbb{F}}\right|^2 dt + \int_\theta^T \left|\sigma_t^d(\theta)\right|^2 dt < \infty \quad \text{a.s.} \tag{2.5}$$

and

$$-\infty < \gamma_t < 1, \quad 0 \le t \le T \quad \text{a.s.}, \tag{2.6}$$

which ensure that the dynamics (2.2), (2.3) are well defined and the stock price remains (strictly) positive over $[0,T]$ (if the initial stock price S_{0-} is > 0) and locally bounded.

The interpretation of the contagion risk model for the asset price S is the following. The process $S^{\mathbb{F}}$ represents the asset price before the default, and there is a jump of the stock price at the default time of the counterparty, represented by the process γ, which may take positive or negative values, corresponding to a proportional loss or gain on the stock price. After the default at time, $\tau = \theta$, $S^d(\theta)$ represents the asset price process, where there is a change of regime in the coefficients depending on the default time. One typical situation can be as follows: In the case of a downward (resp. upward) jump in the asset price at the default time $\tau = \theta$, the rate of return $\mu^d(\theta)$ should be smaller (resp. greater) than the rate of return $\mu^{\mathbb{F}}$ before the default, and this gap should increase when the default occurs early, i.e., $\mu^d(\theta)$ is increasing (resp. decreasing) in θ with $\mu^d(\theta) <$ (resp. $>$) $\mu^{\mathbb{F}}$. For example, we may choose in the case of a downward jump, i.e., $\gamma > 0$, a rate of return coefficient in the form $\mu^d(\theta) = \mu^{\mathbb{F}}\theta/T$, for $\theta \in [0,T]$. On the other hand, we also expect that the volatility $\sigma^d(\theta)$ after the default is greater than the volatility $\sigma^{\mathbb{F}}$ before the default, and this gap should also increase when the default occurs early. An example of such a volatility coefficient is $\sigma^d(\theta) = \sigma^{\mathbb{F}}(2-\theta/T)$.

Remark 2.3 Under the **(DH)** hypothesis, a $(\mathbb{P},\mathbb{F})$-Brownian motion W is a $\mathbb{G}$-semimartingale and admits an explicit decomposition in terms of the density α given by (see [10], Sect. 1 and [6], Proposition 5.9)

$$W_t = \hat{W}_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle W, G\rangle_s}{G_s} + \int_{\tau\wedge t}^{t} \frac{d\langle W_s, \alpha_s(\tau)\rangle}{\alpha_s(\tau)} =: \hat{W}_t^{\mathbb{G}} + A_t, \quad 0 \le t \le T, \tag{2.7}$$

where $\hat{W}^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-Brownian motion, and A is a finite variation $\mathbb{G}$-adapted process. Denoting by μ and σ the $\mathbb{G}$-adapted processes $\mu_t = \mu_t^{\mathbb{F}} 1_{t<\tau} + \mu_t^d(\tau) 1_{t\ge\tau}$ and $\sigma_t = \sigma_t^{\mathbb{F}} 1_{t<\tau} + \sigma_t^d(\tau) 1_{t\ge\tau}$, we see from (2.1)–(2.3) that the dynamics of the stock price process S can be written as

$$dS_t = S_{t-}(\mu_t\, dt + \sigma_t\, dW_t - \gamma_t\, dD_t), \quad 0 \le t \le T. \tag{2.8}$$

Moreover, by the Itô martingale representation theorem for the Brownian filtration $\mathbb{F}$, the finite variation part A in (2.7) is written in the form $A = \int_0^t a_s ds$ for some $\mathbb{G}$-adapted process $a = (a_t)_{t\in[0,T]}$. Let us then define the $\mathbb{G}$-adapted process

$$\beta_t = \frac{\mu_t + \sigma_t a_t - \gamma_t \lambda_t^{\mathbb{G}}}{\sigma_t}, \quad 0 \le t \le T,$$

and consider the Doléans–Dade exponential local martingale $Z_t^{\mathbb{G}} = \mathcal{E}(-\int \beta\, d\hat{W}^{\mathbb{G}})_t$, $0 \le t \le T$. By assuming that $Z^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-martingale (which is satisfied, e.g., under the Novikov criterion $\mathbb{E}[\exp(\int_0^T \frac{1}{2}|\beta_t|^2\, dt)] < \infty$), this defines a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ with Radon–Nikodým density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^{\mathbb{G}} = \exp\left(-\int_0^T \beta_t\, d\hat{W}_t^{\mathbb{G}} - \frac{1}{2}\int_0^T |\beta_t|^2\, dt\right),$$

under which, by Girsanov's theorem, $\overline{W}^{\mathbb{G}} = \hat{W}^{\mathbb{G}} + \int \beta\, dt$ is a $(\mathbb{Q}, \mathbb{G})$-Brownian motion and M is a $(\mathbb{Q}, \mathbb{G})$-martingale, so that the dynamics of S follows the $(\mathbb{Q}, \mathbb{G})$-local martingale

$$dS_t = S_{t-}\big(\sigma_t\, d\overline{W}_t^{\mathbb{G}} - \gamma_t\, dM_t\big).$$

We thus have the "no-arbitrage" condition

$$\mathcal{M}(\mathbb{G}) := \big\{\mathbb{Q} \sim \mathbb{P} \text{ on } (\Omega, \mathcal{G}_T) : S \text{ is a } (\mathbb{Q}, \mathbb{G})\text{-local martingale}\big\} \neq \emptyset. \tag{2.9}$$

• *Portfolio and wealth processes.* Consider now an investor who can trade continuously in this financial market by holding a positive wealth at any time. This is mathematically quantified by a $\mathbb{G}$-predictable process $\pi = (\pi_t)_{t\in[0,T]}$, called trading strategy and representing the proportion of wealth invested in the stock. By decomposing the $\mathbb{G}$-predictable process π in the form $\pi_t = \pi_t^{\mathbb{F}} 1_{t\le\tau} + \pi_t^d(\tau) 1_{t>\tau}$, $0 \le t \le T$, where $\pi^{\mathbb{F}}$ is $\mathbb{F}$-adapted, representing the proportion of wealth invested before the default, and $\pi_t^d(\theta)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, representing the proportion of wealth invested after the default at time $\tau = \theta$, the $\mathbb{G}$-adapted wealth process is given by

$$X_t = X_t^{\mathbb{F}} 1_{t<\tau} + X_t^d(\tau) 1_{t\ge\tau}, \quad 0 \le t \le T, \tag{2.10}$$

where $X^{\mathbb{F}}$ is the wealth process in the default-free market, governed by

$$dX_t^{\mathbb{F}} = X_t^{\mathbb{F}} \pi_t^{\mathbb{F}} \frac{dS_t^{\mathbb{F}}}{S_t^{\mathbb{F}}} = X_t^{\mathbb{F}} \pi_t^{\mathbb{F}} \big(\mu_t^{\mathbb{F}}\, dt + \sigma_t^{\mathbb{F}}\, dW_t\big), \quad X_0^{\mathbb{F}} = X_{0-}, \quad 0 \le t \le T, \tag{2.11}$$

and $X^d(\theta)$ is the wealth process after the default at time $\tau = \theta$, governed by

$$\begin{aligned} dX_t^d(\theta) &= X_t^d(\theta)\pi_t^d(\theta)\frac{dS_t^d(\theta)}{S_t^d(\theta)} \\ &= X_t^d(\theta)\pi_t^d(\theta)\big(\mu_t^d(\theta)\, dt + \sigma_t^d(\theta)\, dW_t\big), \quad \theta < t \le T, \\ X_\theta^d(\theta) &= X_{\theta-}^{\mathbb{F}}\big(1 - \pi_\theta^{\mathbb{F}}\gamma_\theta\big). \end{aligned} \tag{2.12}$$

We say that a trading strategy π (identified with the pair $(\pi^{\mathbb{F}}, \pi^d)$) is admissible, and we write $\pi \in \mathcal{A}$, if for all $\theta \in [0, T]$,

$$\int_0^T \big|\pi_t^{\mathbb{F}} \sigma_t^{\mathbb{F}}\big|^2\, dt + \int_\theta^T \big|\pi_t^d(\theta)\sigma_t^d(\theta)\big|^2\, dt < \infty \quad \text{and} \quad \pi_\theta^{\mathbb{F}}\gamma_\theta < 1 \quad \text{a.s.}$$

This means that the dynamics of the wealth process is well defined with a strictly positive wealth at any time (if we start from a positive initial capital $X_{0-} > 0$).

3 Decomposition of the utility maximization problem

We are given a utility function U defined on $(0, \infty)$, strictly increasing, strictly concave and C^1 on $(0, \infty)$, and satisfying the Inada conditions $U'(0+) = \infty$, $U'(\infty) = 0$. The performance of an admissible trading strategy $\pi \in \mathcal{A}$, associated with a wealth process X in (2.10) and starting at time 0 from $X_{0-} > 0$, is measured over the finite horizon T by

$$J_0(\pi) = \mathbb{E}\big[U(X_T)\big],$$

and the optimal investment problem is formulated as

$$V_0 = \sup_{\pi \in \mathcal{A}} J_0(\pi). \tag{3.1}$$

Problem (3.1) is a maximization problem of expected utility from terminal wealth in an incomplete market due to the jump of the risky asset. This optimization problem can be studied by convex duality methods. Actually, under the condition that

$$V_0 < \infty, \tag{3.2}$$

which is satisfied under (2.9) once

$$\mathbb{E}\bigg[\tilde{U}\bigg(y\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg)\bigg] < \infty \quad \text{for some } y > 0,$$

where $\tilde{U}(y) = \sup_{x>0}[U(x) - xy]$, and under the so-called condition of reasonable asymptotic elasticity,

$$AE(U) := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1,$$

we know from the general results of Kramkov and Schachermayer [12] that there exists a solution to (3.1). We also have a dual characterization of the solution, but this does not lead to explicit results due to the incompleteness of the market, i.e., the infinite cardinality of $\mathcal{M}(\mathbb{G})$. One can also deal directly with problem (3.1) by dynamic programming methods in the $\mathbb{G}$-filtration, as done recently in Lim and Quenez [13], where the authors consider a similar model as in (2.8) by assuming that W is a $\mathbb{G}$-Brownian motion, i.e., $W = \hat{W}^{\mathbb{G}}$ (see Remark 2.3). This means that the **H**-hypothesis is in fact satisfied implicitly. We also mention a recent related paper by Ankirchner et al. [1], who considered an indifference pricing problem for the exponential utility function in a market with a risky asset subject to a single jump, and adopted as in Lim and Quenez [13] a BSDE approach for solving this stochastic control problem. In both papers, the authors studied the problem globally in the $\mathbb{G}$-filtration, which leads to a derivation of the solution in terms of a BSDE with jumps. This method does not really use the particular feature of the single jump at the default, and the results obtained are rather similar to those derived in market models with jumps, as in Morlais [14], for which it is in general difficult to obtain an explicit characterization of the solution.

We provide here an alternative approach by fully making use of the specific feature of the single jump of the stock induced by the default time. While it is intuitively natural to separate the initial problem into an after- and before-default optimization problem, we show how this can be derived rigorously by means of the density hypothesis on the default time. The key result of this approach is to reduce the initial incomplete market problem formulated in the $\mathbb{G}$-filtration into two portfolio optimization problems with respect to the reference filtration $\mathbb{F}$, hence in a complete market, and simpler to solve: the after-default and before-default maximization problems, the solution to the latter depending on the former. Notice that our approach does not require the **H**-hypothesis, as usually assumed in the direct global method in the $\mathbb{G}$-filtration. Moreover, this gives a better understanding of the optimal strategy and allows us to derive explicit results in some particular cases of interest.

The derivation starts as follows. Let $\pi = (\pi^{\mathbb{F}}, \pi^d) \in \mathcal{A}$ and X the associated wealth given in (2.10). Then, under the density hypothesis **(DH)** and by the law of iterated conditional expectations, the performance measure may be written as

$$\begin{aligned} J_0(\pi) &= \mathbb{E}\big[\mathbb{E}\big[U(X_T)\big|\mathcal{F}_T\big]\big] \\ &= \mathbb{E}\big[U\big(X_T^{\mathbb{F}}\big)\mathbb{P}[\tau > T|\mathcal{F}_T] + \mathbb{E}\big[U\big(X_T^d(\tau)\big)1_{\tau\le T}\big|\mathcal{F}_T\big]\big] \\ &= \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T U\big(X_T^d(\theta)\big)\alpha_T(\theta)\,d\theta\bigg], \end{aligned} \tag{3.3}$$

where $G_T = \mathbb{P}[\tau > T|\mathcal{F}_T] = \int_T^\infty \alpha_T(\theta)\,d\theta$.

Let us introduce the value process of the "after-default" optimization problem,

$$V_\theta^d(x) = \operatorname*{ess\,sup}_{\pi^d(\theta)\in\mathcal{A}_d(\theta)} J_\theta^d\big(x, \pi^d(\theta)\big), \quad (\theta, x) \in [0,T]\times(0,\infty), \tag{3.4}$$

$$J_\theta^d\big(x, \pi^d(\theta)\big) = \mathbb{E}\big[U\big(X_T^{d,x}(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big],$$

where $\mathcal{A}_d(\theta)$ is the set of $(\mathcal{F}_t)_{\theta<t\le T}$-adapted processes $\{\pi_t^d(\theta), \theta < t \le T\}$ satisfying $\int_\theta^T |\pi_t^d(\theta)\sigma_t^d(\theta)|^2\,dt < \infty$ a.s. and $\{X_t^{d,x}(\theta), \theta \le t \le T\}$ is the solution to (2.12) controlled by $\pi^d(\theta) \in \mathcal{A}_d(\theta)$, starting from x at time θ. Thus, V^d is the value process of an optimal investment problem in a market model after default. Notice that the coefficients (μ^d, σ^d) of the model depend on the initial time θ when the maximization is performed, and the utility function in the criterion is weighted by $\alpha_T(\theta)$. We shall see in the next section how to deal with these peculiarities for solving (3.4) and proving the existence and characterization of an optimal strategy.

The main result of this section is to show that the original problem (3.1) can be split into the above after-default optimization problem, and a global optimization problem in a before-default market.

Theorem 3.1 *Assume that* $V_\theta^d(x) < \infty$ *a.s. for all* $(\theta, x) \in [0,T]\times(0,\infty)$. *Then we have*

$$V_0 = \sup_{\pi^{\mathbb{F}}\in\mathcal{A}_{\mathbb{F}}} \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T V_\theta^d\big(X_\theta^{\mathbb{F}}\big(1-\pi_\theta^{\mathbb{F}}\gamma_\theta\big)\big)\,d\theta\bigg], \tag{3.5}$$

where $\mathcal{A}_{\mathbb{F}}$ *is the set of* $\mathbb{F}$*-adapted processes* $\pi^{\mathbb{F}}$ *such that* $\int_0^T |\pi_t^{\mathbb{F}}\sigma_t^{\mathbb{F}}|^2 dt < \infty$ *and* $\pi_t^{\mathbb{F}}\gamma_t < 1$ *a.s. for any* $t \in [0,T]$.

Proof Given $\pi = (\pi^{\mathbb{F}}, \pi^d) \in \mathcal{A}$, we have the relation (3.3) for $J_0(\pi)$ under **(DH)**. Furthermore, by Fubini's theorem and the law of iterated conditional expectations, we then obtain

$$\begin{aligned} J_0(\pi) &= \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T \mathbb{E}\big[U\big(X_T^d(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big]\,d\theta\bigg] \\ &= \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T J_\theta^d\big(X_\theta^d(\theta), \pi^d(\theta)\big)\,d\theta\bigg] \\ &\le \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T V_\theta^d\big(X_\theta^d(\theta)\big)\,d\theta\bigg] \\ &\le \sup_{\pi^{\mathbb{F}}\in\mathcal{A}_{\mathbb{F}}} \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T V_\theta^d\big(X_\theta^{\mathbb{F}}\big(1-\pi_\theta^{\mathbb{F}}\gamma_\theta\big)\big)\,d\theta\bigg] =: \hat{V}_0 \end{aligned} \tag{3.6}$$

by the definitions of J^d, V^d and $X_\theta^d(\theta)$. This proves that $V_0 \le \hat{V}_0$.

To prove the converse inequality, fix an arbitrary $\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}$. By the definition of V^d, for any $\omega \in \Omega$, $\theta \in [0,T]$ and $\varepsilon > 0$, there exists $\pi^{d,\varepsilon,\omega}(\theta) \in \mathcal{A}_d(\theta)$ which is an ε-optimal control for V_θ^d at $(\omega, X_\theta^d(\omega,\theta))$. By a measurable selection result (see, e.g., [16], Sect. 3), one can find $\pi^{d,\varepsilon} \in \mathcal{A}_d$ such that $\pi^{d,\varepsilon}(\omega,\theta) = \pi^{d,\varepsilon,\omega}(\omega,\theta)$ holds $d\mathbb{P} \otimes d\theta$-almost everywhere, and so

$$V_\theta^d\big(X_\theta^d(\theta)\big) - \varepsilon \le J_\theta^d\big(X_\theta^d(\theta), \pi^{d,\varepsilon}(\theta)\big) \quad d\mathbb{P} \otimes d\theta\text{-almost everywhere.}$$

By setting $\pi^\varepsilon = (\pi^{\mathbb{F}}, \pi^{d,\varepsilon}) \in \mathcal{A}$ and using again (3.6), we then get

$$\begin{aligned} V_0 \ge J_0\big(\pi^\varepsilon\big) &= \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T J_\theta^d\big(X_\theta^d(\theta), \pi^{d,\varepsilon}(\theta)\big)\,d\theta\bigg] \\ &\ge \mathbb{E}\bigg[U\big(X_T^{\mathbb{F}}\big)G_T + \int_0^T V_\theta^d\big(X_\theta^d(\theta)\big)\,d\theta\bigg] - \varepsilon. \end{aligned}$$

From the arbitrariness of $\pi^{\mathbb{F}}$ in $\mathcal{A}_{\mathbb{F}}$ and $\varepsilon > 0$, we obtain the required inequality and so the result. □

Remark 3.2 (1) The relation (3.5) can be viewed as a dynamic programming type relation. Indeed, as in the dynamic programming principle (DPP), we look for a relation for the value function by varying the initial states. However, instead of taking two consecutive dates as in the usual DPP, the original feature here is to derive the equation by considering the value function between the initial time and the default time conditionnally on the terminal information, leading to the introduction of an "after-default" and a global before-default optimization problem, the latter involving the former. Each of these optimization problems are performed in market models driven by the Brownian motion and with coefficients adapted to the Brownian reference filtration. The main advantage of this approach is then to reduce the problem to the solution of two optimization problems in complete default-free markets, which are simpler to deal with and give more explicit results than the incomplete market framework studied by the "classical" dynamic programming approach or the convex duality method.

Furthermore, a careful look at the arguments for deriving the relation (3.5) shows that in the decomposition of the optimal trading strategy for the original problem (3.1) which is known to exist a priori under (2.9), we have

$$\hat{\pi}_t = \hat{\pi}_t^{\mathbb{F}} 1_{t \le \tau} + \hat{\pi}_t^d(\tau) 1_{t > \tau}, \quad 0 \le t \le T,$$

$\hat{\pi}^{\mathbb{F}}$ is an optimal control for (3.5), and $\hat{\pi}^d(\theta)$ is an optimal control for $V_\theta^d(\hat{X}_\theta^d(\theta))$ with $\hat{X}_\theta^d(\theta) = \hat{X}_\theta^{\mathbb{F}}(1 - \hat{\pi}_\theta^{\mathbb{F}}\gamma_\theta)$, and $\hat{X}^{\mathbb{F}}$ is the wealth process governed by $\hat{\pi}^{\mathbb{F}}$. In other words, the optimal trading strategy is to follow the trading strategy $\hat{\pi}^{\mathbb{F}}$ before the default time τ, and then to change to the after-default trading strategy $\hat{\pi}^d(\tau)$, which depends on the time where default occurs. In the next section, we focus on the solution of these two optimization problems.

(2) The decomposition result in Theorem 3.1 may be extended to general stochastic optimization problems with state and control processes within a progressively enlarged filtration with several random times satisfying a density hypothesis, and this can be applied to optimal investment problems under multiple counterparty defaults.

4 Solution to the optimal investment problem

In this section, we focus on the solution of the two optimization problems arising from the decomposition of the initial utility maximization problem. We first study the after-default optimal investment problem, and then the global before-default optimization problem.

4.1 The after-default utility maximization problem

Problem (3.4) is an optimal investment problem in a complete market model after default. A specific feature of this model is the dependence of the coefficients (μ^d, σ^d) on the initial time θ when the maximization is performed. This makes the optimization problem time-inconsistent, and the classical dynamic programming method cannot be applied. Another peculiarity in the criterion is the presence of the density term $\alpha_T(\theta)$ weighting the utility function U.

We adapt the convex duality method for solving (3.4). We have to extend this martingale method (in a complete market) to a dynamic framework, since we want to compute the value process at any time $\theta \in [0, T]$. Let us denote by

$$Z_t(\theta) = \exp\left(-\int_\theta^t \frac{\mu_u^d(\theta)}{\sigma_u^d(\theta)}\, dW_u - \frac{1}{2}\int_\theta^t \left|\frac{\mu_u^d(\theta)}{\sigma_u^d(\theta)}\right|^2 du\right), \quad \theta \le t \le T,$$

the (local) martingale density in the market model (2.3) after default. We assume that for all $\theta \in [0, T]$, there exists some $\mathcal{F}_\theta$-measurable strictly positive random variable y_θ such that

$$\mathbb{E}\left[\tilde{U}\left(y_\theta \frac{Z_T(\theta)}{\alpha_T(\theta)}\right)\alpha_T(\theta)\middle|\mathcal{F}_\theta\right] < \infty. \tag{4.1}$$

This assumption is similar to the one imposed in the classical (static) convex duality method for ensuring that the dual problem is well defined and finite.

Theorem 4.1 *Assume that (4.1) and $AE(U) < 1$ hold true. Then the value process to problem (3.4) is finite a.s. and given by*

$$V_\theta^d(x) = \mathbb{E}\left[U\left(I\left(\hat{y}_\theta(x)\frac{Z_T(\theta)}{\alpha_T(\theta)}\right)\right)\alpha_T(\theta)\middle|\mathcal{F}_\theta\right], \quad (\theta, x) \in [0, T] \times (0, \infty),$$

and the corresponding optimal wealth process is equal to

$$\hat{X}_t^{d,x}(\theta) = \mathbb{E}\left[\frac{Z_T(\theta)}{Z_t(\theta)} I\left(\hat{y}_\theta(x)\frac{Z_T(\theta)}{\alpha_T(\theta)}\right)\middle|\mathcal{F}_t\right], \quad \theta \le t \le T, \tag{4.2}$$

where $I = (U')^{-1}$ is the inverse of U' and $\hat{y}_\theta(x)$ is the strictly positive $\mathcal{F}_\theta \otimes \mathcal{B}((0, \infty))$-measurable random variable which solves $\hat{X}_\theta^{d,x}(\theta) = x$.

Proof First observe, similarly as in Theorem 2.2 in [12] that under $AE(U) < 1$, the validity of (4.1) for some or for all $\mathcal{F}_\theta$-measurable strictly positive random variables y_θ is equivalent. By the definition of $Z(\theta)$ and Itô's formula, the process

$\{Z_t(\theta)X_t^{d,x}(\theta), \theta \le t \le T\}$ is a nonnegative $(\mathbb{P}, (\mathcal{F}_t)_{\theta\le t\le T})$-local martingale, hence a supermartingale, for any $\pi^d(\theta) \in \mathcal{A}_d(\theta)$, and so

$$\mathbb{E}\big[X_T^{d,x}(\theta)Z_T(\theta)\big|\mathcal{F}_\theta\big] \le X_\theta^{d,x}(\theta)Z_\theta(\theta) = x.$$

Set $Y_T(\theta) = Z_T(\theta)/\alpha_T(\theta)$. Then, by the definition of $\tilde{U}$, we have for all $\mathcal{F}_\theta$-measurable strictly positive random variables y_θ and $\pi^d(\theta) \in \mathcal{A}_d(\theta)$ that

$$\begin{aligned}\mathbb{E}\big[U\big(X_T^{d,x}(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] &\le \mathbb{E}\big[\tilde{U}\big(y_\theta Y_T(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] \\ &\quad + \mathbb{E}\big[X_T^{d,x}(\theta)y_\theta Y_T(\theta)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] \\ &= \mathbb{E}\big[\tilde{U}\big(y_\theta Y_T(\theta)\big)\alpha_T(\theta)|\mathcal{F}_\theta\big] + y_\theta\mathbb{E}\big[X_T^{d,x}(\theta)Z_T(\theta)\big|\mathcal{F}_\theta\big] \\ &\le \mathbb{E}\big[\tilde{U}\big(y_\theta Y_T(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] + xy_\theta, \end{aligned} \tag{4.3}$$

which proves in particular that $V_\theta^d(x)$ is finite a.s. Now, we recall that under the Inada conditions, the supremum in the definition of $\tilde{U}(y)$ is attained at $I(y)$, i.e., $\tilde{U}(y) = U(I(y)) - yI(y)$. From (4.3), this implies

$$\begin{aligned}\mathbb{E}\big[U\big(X_T^{d,x}(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] &\le \mathbb{E}\big[U\big(I\big(y_\theta Y_T(\theta)\big)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] \\ &\quad - y_\theta\big(\mathbb{E}\big[Z_T(\theta)I\big(y_\theta Y_T(\theta)\big)\big|\mathcal{F}_\theta\big] - x\big).\end{aligned} \tag{4.4}$$

Now, due to the Inada conditions, (4.1) and $AE(U) < 1$, for any $\omega \in \Omega$, $\theta \in [0, T]$, the function $y \in (0,\infty) \mapsto f_\theta(\omega, y) = \mathbb{E}[Z_T(\theta)I(yY_T(\theta))|\mathcal{F}_\theta]$ is a strictly decreasing one-to-one continuous function from $(0,\infty)$ into $(0,\infty)$. Hence, there exists a unique $\hat{y}_\theta(\omega, x) > 0$ such that $f_\theta(\omega, \hat{y}(x)) = x$. Moreover, since $f_\theta(y)$ is $\mathcal{F}_\theta \otimes \mathcal{B}(0,\infty)$-measurable, this value $\hat{y}_\theta(x)$ can be chosen, by a measurable selection argument, as $\mathcal{F}_\theta \otimes \mathcal{B}(0,\infty)$-measurable. With this choice of $y_\theta = \hat{y}_\theta(x)$ and by setting $\hat{X}_T^{d,x}(\theta) = I(\hat{y}_\theta(x)Y_T(\theta))$, the inequality (4.4) yields

$$\mathbb{E}\big[U\big(X_T^{d,x}(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big] \le \mathbb{E}\big[U\big(\hat{X}_T^{d,x}(\theta)\big)\alpha_T(\theta)\big|\mathcal{F}_\theta\big], \quad \forall \pi^d(\theta) \in \mathcal{A}_d(\theta). \tag{4.5}$$

Consider now the process $\hat{X}^{d,x}(\theta)$ defined in (4.2) and leading to $\hat{X}_T^{d,x}(\theta)$ at time T. By definition, the process $\{M_t(\theta) = Z_t(\theta)\hat{X}_t^{d,x}(\theta), \theta \le t \le T\}$ is a strictly positive $(\mathbb{P}, (\mathcal{F}_t)_{\theta\le t\le T})$-martingale. From the martingale representation theorem for the Brownian motion filtration, there exists an $(\mathcal{F}_t)_{\theta\le t\le T}$-adapted process $(\phi_t)_{\theta\le t\le T}$ satisfying $\int_\theta^T |\phi_t|^2\,dt < \infty$ a.s. and such that

$$M_t(\theta) = M_\theta(\theta) + \int_\theta^t \phi_u M_u(\theta)\,dW_u, \quad \theta \le t \le T.$$

Thus, by setting $\hat{\pi}^d(\theta) = (\phi + \frac{\mu^d(\theta)}{\sigma^d(\theta)})/\sigma^d(\theta)$, we see that $\hat{\pi}^d(\theta) \in \mathcal{A}_d(\theta)$, and by Itô's formula, $\hat{X}^{d,x}(\theta) = M(\theta)/Z(\theta)$ satisfies the wealth equation (2.12) controlled by $\hat{\pi}^d(\theta)$. Moreover, by construction of $\hat{y}_\theta(x)$, we have

$$\hat{X}_\theta^{d,x}(\theta) = \mathbb{E}\bigg[Z_T(\theta)I\bigg(\hat{y}_\theta(x)\frac{Z_T(\theta)}{\alpha_T(\theta)}\bigg)\bigg|\mathcal{F}_\theta\bigg] = x.$$

Recalling (4.5), this proves that $\hat{\pi}^d(\theta)$ is a solution to (3.4), with corresponding optimal wealth process $\hat{X}^{d,x}(\theta)$. □

Remark 4.2 Under the **(H)**-hypothesis, $\alpha_T(\theta) = \alpha_\theta(\theta)$ is $\mathcal{F}_\theta$-measurable. In this case, the optimal wealth process for (3.4) is given by

$$\hat{X}_t^{d,x}(\theta) = \mathbb{E}\left[\frac{Z_T(\theta)}{Z_t(\theta)} I\big(\bar{y}_\theta(x) Z_T(\theta)\big)\middle|\mathcal{F}_t\right], \quad \theta \le t \le T,$$

where $\bar{y}_\theta(x)$ is the strictly positive $\mathcal{F}_\theta \otimes \mathcal{B}((0,\infty))$-measurable random variable satisfying $\hat{X}_\theta^{d,x}(\theta) = x$. Hence, the optimal after-default strategy does not depend on the density of the default time.

We illustrate the above results in the case of constant relative risk aversion (CRRA) utility functions.

Example 4.3 (The case of CRRA utility functions) We consider utility functions of the form

$$U(x) = \frac{x^p}{p}, \quad p < 1, p \neq 0, x > 0.$$

In this case, we easily compute the optimal wealth process in (4.2) as

$$\hat{X}_t^{d,x}(\theta) = \frac{x}{\mathbb{E}[\alpha_T(\theta)(\frac{Z_T(\theta)}{\alpha_T(\theta)})^{-q}|\mathcal{F}_\theta]} \frac{\mathbb{E}[\alpha_T(\theta)(\frac{Z_T(\theta)}{\alpha_T(\theta)})^{-q}|\mathcal{F}_t]}{Z_t(\theta)}, \quad \theta \le t \le T,$$

where $q = \frac{p}{1-p}$. The optimal value process is then given for all $x > 0$ by

$$V_\theta^d(x) = \frac{x^p}{p}\left(\mathbb{E}\left[\alpha_T(\theta)\left(\frac{Z_T(\theta)}{\alpha_T(\theta)}\right)^{-q}\middle|\mathcal{F}_\theta\right]\right)^{1-p}, \quad \theta \in [0,T]. \tag{4.6}$$

Notice that the case of the logarithmic utility function $U(x) = \ln x$, $x > 0$, can be either computed directly, or derived as the limiting case of the power utility function case $U(x) = \frac{x^p - 1}{p}$ as p goes to zero. The optimal wealth process is given by

$$\hat{X}_t^{d,x}(\theta) = \frac{x}{\mathbb{E}[\alpha_T(\theta)|\mathcal{F}_\theta]} \frac{\mathbb{E}[\alpha_T(\theta)|\mathcal{F}_t]}{Z_t(\theta)}, \quad \theta \le t \le T,$$

and the optimal value process for all $x > 0$ is equal to

$$V_\theta^d(x) = \mathbb{E}\big[\alpha_T(\theta)\big|\mathcal{F}_\theta\big] \ln\left(\frac{x}{\mathbb{E}[\alpha_T(\theta)|\mathcal{F}_\theta]}\right) + \mathbb{E}\left[\alpha_T(\theta) \ln\left(\frac{\alpha_T(\theta)}{Z_T(\theta)}\right)\middle|\mathcal{F}_\theta\right],$$
$$\theta \in [0,T].$$

4.2 The global before-default optimization problem

In this paragraph, we focus on the solution of the optimization problem (3.5). We already know the existence of an optimal strategy $\hat{\pi}^{\mathbb{F}}$ for this problem (see Remark 3.2) and our main concern is to provide an explicit characterization of the optimal control.

We use a dynamic programming approach. For any $t \in [0, T]$, $\nu \in \mathcal{A}_{\mathbb{F}}$, let us consider the set of controls coinciding with ν until time t, i.e.,

$$\mathcal{A}_{\mathbb{F}}(t, \nu) = \{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}} : \pi^{\mathbb{F}}_{\cdot \wedge t} = \nu_{\cdot \wedge t}\}.$$

Under the standing assumption that $V_0 < \infty$, we then introduce the dynamic version of the optimization problem (3.5) by considering the family of $\mathbb{F}$-adapted processes

$$V_t(\nu) = \operatorname*{ess\,sup}_{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)} \mathbb{E}\Big[U\big(X_T^{\mathbb{F}}\big)G_T + \int_t^T V_\theta^d\big(X_\theta^{\mathbb{F}}\big(1 - \pi_\theta^{\mathbb{F}}\gamma_\theta\big)\big)\,d\theta\Big|\mathcal{F}_t\Big], \quad 0 \le t \le T,$$

so that $V_0 = V_0(\nu)$ for any $\nu \in \mathcal{A}_{\mathbb{F}}$. In the above expression, $X^{\mathbb{F}}$ is the wealth process with dynamics (2.11), controlled by $\pi^{\mathbb{F}} \in \mathcal{A}(t, \nu)$, and starting from X_0. We also denote by $X^{\nu,\mathbb{F}}$ the wealth process with dynamics (2.11), controlled by $\nu \in \mathcal{A}_{\mathbb{F}}$, starting from X_0, so that it coincides with $X^{\mathbb{F}}$ until time t, i.e., $X^{\nu,\mathbb{F}}_{\cdot\wedge t} = X^{\mathbb{F}}_{\cdot\wedge t}$. From the dynamic programming principle (see Appendix A of El Karoui and Quenez [7]), the process $\{V_t(\nu), 0 \le t \le T\}$ can be chosen in its càdlàg version and is such that for any $\nu \in \mathcal{A}_{\mathbb{F}}$,

$$\Big\{V_t(\nu) + \int_0^t V_\theta^d\big(X_\theta^{\nu,\mathbb{F}}(1 - \nu_\theta\gamma_\theta)\big)\,d\theta, 0 \le t \le T\Big\} \quad \text{is a } (\mathbb{P}, \mathbb{F})\text{-supermartingale.} \tag{4.7}$$

Moreover, the optimal strategy $\hat{\pi}^{\mathbb{F}}$ for V_0 is characterized by the martingale property that

$$\Big\{V_t\big(\hat{\pi}^{\mathbb{F}}\big) + \int_0^t V_\theta^d\big(X_\theta^{\hat{\pi}^{\mathbb{F}},\mathbb{F}}\big(1 - \hat{\pi}_\theta^{\mathbb{F}}\gamma_\theta\big)\big)\,d\theta, 0 \le t \le T\Big\} \quad \text{is a } (\mathbb{P}, \mathbb{F})\text{-martingale.}$$

In the sequel, we shall exploit these dynamic programming properties in the particular important case of constant relative risk aversion (CRRA) utility functions. We thus consider utility functions of the form

$$U(x) = \frac{x^p}{p}, \quad p < 1, p \neq 0, x > 0,$$

and we set $q = \frac{p}{1-p}$. Notice that we deal with the economically relevant case when $p < 0$, i.e., the degree of risk aversion $1 - p$ is strictly larger than 1. This will induce some additional technical difficulties that do not exist in the case $p > 0$. For a CRRA utility function, $V^d(x)$ is by (4.6) also of the same power type, i.e.,

$$V_\theta^d(x) = U(x)K_\theta^p \quad \text{with } K_\theta = \bigg(\mathbb{E}\Big[\alpha_T(\theta)\Big(\frac{Z_T(\theta)}{\alpha_T(\theta)}\Big)^{-q}\Big|\mathcal{F}_\theta\Big]\bigg)^{\frac{1}{q}},$$

and we assume that K_θ is finite a.s. for all $\theta \in [0,T]$. The value of the optimization problem (3.5) is written as

$$V_0 = \sup_{\nu \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}\Big[U\big(X_T^{\nu,\mathbb{F}}\big) G_T + \int_0^T U\big(X_\theta^{\nu,\mathbb{F}}\big)(1-\nu_\theta\gamma_\theta)^p K_\theta^p \, d\theta \Big].$$

In the above equality, we may without loss of generality take the supremum over $\mathcal{A}_{\mathbb{F}}(U)$, the set of elements $\nu \in \mathcal{A}_{\mathbb{F}}$ such that

$$\mathbb{E}\Big[U\big(X_T^{\nu,\mathbb{F}}\big) G_T + \int_0^T U\big(X_\theta^{\nu,\mathbb{F}}\big)(1-\nu_\theta\gamma_\theta)^p K_\theta^p \, d\theta \Big] > -\infty, \tag{4.8}$$

and by an abuse of notation, we write $\mathcal{A}_{\mathbb{F}} = \mathcal{A}_{\mathbb{F}}(U)$. For any $\nu \in \mathcal{A}_{\mathbb{F}}$ with corresponding strictly positive wealth process $X^{\nu,\mathbb{F}}$ governed by (2.11) with control ν and starting from X_0, we notice that the càdlàg $\mathbb{F}$-adapted process defined for $0 \le t \le T$ by

$$\begin{aligned} Y_t &:= \frac{V_t(\nu)}{U(X_t^{\nu,\mathbb{F}})} \\ &= p \operatorname*{ess\,sup}_{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)} \mathbb{E}\Big[U\Big(\frac{X_T^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}}\Big) G_T + \int_t^T U\Big(\frac{X_\theta^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}}\Big)\big(1-\pi_\theta^{\mathbb{F}}\gamma_\theta\big)^p K_\theta^p \, d\theta \Big| \mathcal{F}_t \Big] \end{aligned} \tag{4.9}$$

does not depend on $\nu \in \mathcal{A}_{\mathbb{F}}$. It lies in the set $L_+(\mathbb{F})$ of nonnegative càdlàg $\mathbb{F}$-adapted processes. Let us also denote by $L^2_{\text{loc}}(W)$ the set of $\mathbb{F}$-adapted process ϕ such that $\int_0^T |\phi_t|^2 \, dt < \infty$ a.s.

We have the following preliminary properties on this process Y.

Lemma 4.4 *The process Y in (4.9) is strictly positive, i.e., $\mathbb{P}[Y_t > 0, 0 \le t \le T] = 1$. Moreover, for all $\nu \in \mathcal{A}_{\mathbb{F}}$, the process*

$$\xi_t^\nu(Y) := U\big(X_t^{\nu,\mathbb{F}}\big) Y_t + \int_0^t U\big(X_\theta^{\nu,\mathbb{F}}\big)(1-\nu_\theta\gamma_\theta)^p K_\theta^p \, d\theta, \quad 0 \le t \le T,$$

is bounded from below by a martingale.

Proof (1) We first consider the case $p > 0$. Then

$$\begin{aligned} Y_t &= \operatorname*{ess\,sup}_{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)} \mathbb{E}\Big[\Big(\frac{X_T^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}}\Big)^p G_T + \int_t^T \Big(\frac{X_\theta^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}}\Big)^p \big(1-\pi_\theta^{\mathbb{F}}\gamma_\theta\big)^p K_\theta^p \, d\theta \Big| \mathcal{F}_t \Big] \\ &\ge \mathbb{E}\Big[G_T + \int_t^T K_\theta^p \, d\theta \Big| \mathcal{F}_t \Big] > 0, \quad \forall t \in [0,T], \end{aligned} \tag{4.10}$$

by taking in (4.10) the control process $\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)$ defined by $\pi_s^{\mathbb{F}} = \nu_s 1_{s \le t}$. Moreover, since $U(x)$ is nonnegative, the process $\xi^\nu(Y)$ is nonnegative, hence trivially bounded from below by the zero martingale.

(2) We next consider the case $p < 0$. Then

$$\begin{aligned} Y_t &= \operatorname*{ess\,inf}_{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)} \mathbb{E}\bigg[\bigg(\frac{X_T^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}} \bigg)^p G_T + \int_t^T \bigg(\frac{X_\theta^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}} \bigg)^p \big(1 - \pi_\theta^{\mathbb{F}} \gamma_\theta\big)^p K_\theta^p \, d\theta \bigg| \mathcal{F}_t \bigg] \\ &\geq J_t := \operatorname*{ess\,inf}_{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)} \mathbb{E}\bigg[\bigg(\frac{X_T^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}} \bigg)^p G_T \bigg| \mathcal{F}_t \bigg], \quad \forall t \in [0,T]. \end{aligned} \tag{4.11}$$

Notice that the process J can be chosen in its càdlàg modification. By definition, it is obvious that $J \geq 0$. Let us show that for any $t \in [0,T]$, J_t is strictly positive and the infimum for J_t is attained. Fix $t \in [0,T]$, and consider, by a measurable selection argument, a minimizing sequence $(\pi^n)_n$ in $\mathcal{A}_{\mathbb{F}}(t,\nu)$ for J_t, i.e.,

$$\lim_{n \to \infty} \mathbb{E}\bigg[\bigg(\frac{X_T^n}{X_t^{\nu,\mathbb{F}}} \bigg)^p G_T \bigg| \mathcal{F}_t \bigg] = J_t \quad \text{a.s.} \tag{4.12}$$

Here, X^n denotes the wealth process with dynamics (2.11) governed by π^n. Consider the (local) martingale density process

$$Z_s^t = \exp\bigg(-\int_t^s \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}} \, dW_u - \frac{1}{2} \int_t^s \bigg| \frac{\mu_u^{\mathbb{F}}}{\sigma_u^{\mathbb{F}}} \bigg|^2 du \bigg), \quad t \leq s \leq T.$$

By definition of Z^t and Itô's formula, the process $\{Z_s^t X_s^n, t \leq s \leq T\}$ is a nonnegative $(\mathbb{P}, (\mathcal{F}_s)_{t \leq s \leq T})$-local martingale, hence a supermartingale and, therefore, $\mathbb{E}[X_T^n Z_T^t | \mathcal{F}_t] \leq X_t^n Z_t^t = X_t^{\nu,\mathbb{F}}$. By Komlós' lemma applied to the sequence of nonnegative $\mathcal{F}_T$-measurable random variables $(X_T^n)_n$, there exist convex combinations $\tilde{X}_T^n \in \operatorname{conv}(X_T^{k,\mathbb{F}}, k \geq n)$ such that $(\tilde{X}_T^n)_n$ converges a.s. to some nonnegative $\mathcal{F}_T$-measurable random variable $\tilde{X}_T$. By Fatou's lemma, we have $\tilde{X}_t := \mathbb{E}[\tilde{X}_T Z_T^t | \mathcal{F}_t] \leq X_t^{\nu,\mathbb{F}}$. Moreover, by the convexity of $x \mapsto x^p$ and Fatou's lemma, it follows from (4.12) that

$$J_t \geq \mathbb{E}\bigg[\bigg(\frac{\tilde{X}_T}{X_t^{\nu,\mathbb{F}}} \bigg)^p G_T \bigg| \mathcal{F}_t \bigg] \quad \text{a.s.} \tag{4.13}$$

Now, since $p < 0$, $J_t < \infty$ and $G_T > 0$ a.s., we deduce that $\tilde{X}_T > 0$, and so $\tilde{X}_t > 0$ a.s. Consider the process $\bar{X}_s^t = \frac{X_t^{\nu,\mathbb{F}}}{\tilde{X}_t} \mathbb{E}[\frac{Z_T^t}{Z_s^t} \tilde{X}_T | \mathcal{F}_s]$, $t \leq s \leq T$. Then $\{Z_s^t \bar{X}_s^t, t \leq s \leq T\}$ is a strictly positive $(\mathbb{P}, (\mathcal{F}_s)_{t \leq s \leq T})$-martingale, and by the martingale representation theorem for the Brownian filtration, using the same arguments as at the end of proof of Theorem 4.1, we obtain the existence of an $(\mathcal{F}_s)_{t \leq s \leq T}$-adapted process $\bar{\pi}^t = (\bar{\pi}_s^t)_{t \leq s \leq T}$ satisfying $\int_t^T |\bar{\pi}_s^t \sigma_s^{\mathbb{F}}|^2 \, ds < \infty$ such that $\bar{X}^t$ satisfies the wealth process dynamics (2.11) with portfolio $\bar{\pi}^t$ on (t,T), and starting from $\bar{X}_t^t = X_t^{\nu,\mathbb{F}}$. By considering the portfolio process $\bar{\pi} \in \mathcal{A}_{\mathbb{F}}(t,\nu)$ defined by $\bar{\pi}_s = \nu_s 1_{s \leq t} + \bar{\pi}_s^t 1_{s > t}$ for $0 \leq s \leq T$ and denoting by $X^{\bar{\pi},\mathbb{F}}$ the corresponding wealth process, it follows that $X_s^{\bar{\pi},\mathbb{F}} = \bar{X}_s^t$ for $t \leq s \leq T$, and in particular $X_T^{\bar{\pi},\mathbb{F}} = \bar{X}_T^t = \frac{X_t^{\nu,\mathbb{F}}}{\tilde{X}_t} \tilde{X}_T \geq \tilde{X}_T$ a.s.

From (4.13), the nonincreasing property of $x \mapsto x^p$ and the definition of J_t, we deduce that

$$J_t = \tilde{J}_t := \mathbb{E}\bigg[\bigg(\frac{X_T^{\bar{\pi},\mathbb{F}}}{X_t^{\nu,\mathbb{F}}}\bigg)^p G_T \bigg| \mathcal{F}_t\bigg] \quad \text{a.s.} \tag{4.14}$$

and as a byproduct that $X_T^{\bar{\pi},\mathbb{F}} = \tilde{X}_T$. The equality (4.14) means that the process $J = (J_t)_{t\in[0,T]}$ is a modification of the process $\tilde{J} = (\tilde{J}_t)_{t\in[0,T]}$. Since J and $\tilde{J}$ are càdlàg, they are then indistinguishable, i.e., $\mathbb{P}[J_t = \tilde{J}_t,\ 0 \le t \le T] = 1$. We deduce that the process J, and consequently Y, inherit the strict positivity of the process $\tilde{J}$.

From (4.11), we have for all $\nu \in \mathcal{A}_\mathbb{F}$ and all $t \in [0,T]$ that

$$\begin{aligned}\xi_t^\nu(Y) &= \operatorname*{ess\,sup}_{\pi^\mathbb{F}\in\mathcal{A}_\mathbb{F}(t,\nu)} \mathbb{E}\bigg[U\big(X_T^\mathbb{F}\big)G_T + \int_0^T U\big(X_\theta^\mathbb{F}\big)\big(1-\pi_\theta^\mathbb{F}\gamma_\theta\big)^p K_\theta^p\, d\theta \bigg| \mathcal{F}_t\bigg] \\ &\ge M_t^\nu := \mathbb{E}\bigg[U\big(X_T^{\nu,\mathbb{F}}\big)G_T + \int_0^T U\big(X_\theta^{\nu,\mathbb{F}}\big)\big(1-\nu_\theta\gamma_\theta\big)^p K_\theta^p\, d\theta \bigg| \mathcal{F}_t\bigg],\end{aligned} \tag{4.15}$$

by taking in (4.15) the control process $\pi^\mathbb{F} = \nu \in \mathcal{A}_\mathbb{F}(t,\nu)$. The negative process $(M_t^\nu)_{t\in[0,T]}$ is an integrable (recall (4.8)) martingale, and the assertions of the lemma are proved. □

In the sequel, we denote by $L_+^b(\mathbb{F})$ the set of processes $\tilde{Y}$ in $L_+(\mathbb{F})$ such that for all $\nu \in \mathcal{A}_\mathbb{F}$, the process $\xi^\nu(\tilde{Y})$ is bounded from below by a martingale. The next result gives a characterization of the process Y in terms of a backward stochastic differential equation (BSDE) and of the optimal strategy for problem (3.5) which is already known to exist (see Remark 3.2).

Theorem 4.5 *When $p > 0$ (resp. $p < 0$), the process Y in (4.9) is the smallest (resp. largest) solution in $L_+^b(\mathbb{F})$ to the BSDE*

$$Y_t = G_T + \int_t^T f(\theta, Y_\theta, \phi_\theta)\, d\theta - \int_t^T \phi_\theta\, dW_\theta, \quad 0 \le t \le T, \tag{4.16}$$

for some $\phi \in L^2_{\mathrm{loc}}(W)$, and where

$$f(t, Y_t, \phi_t) = p \operatorname*{ess\,sup}_{\nu\in\mathcal{A}_\mathbb{F}} \bigg[\big(\mu_t^\mathbb{F} Y_t + \sigma_t^\mathbb{F}\phi_t\big)\nu_t - \frac{1-p}{2} Y_t \big|\nu_t\sigma_t^\mathbb{F}\big|^2 + K_t^p \frac{(1-\nu_t\gamma_t)^p}{p}\bigg]. \tag{4.17}$$

The optimal strategy $(\hat{\pi}_t^\mathbb{F})_{t\in[0,T]}$ to problem (3.5) exists and attains the supremum in (4.17). Moreover, under the integrability condition $\int_0^T |\frac{K_t}{\sigma_t^\mathbb{F}}|^{\frac{2p}{2-p}}\, dt < \infty$ a.s., the supremum in (4.17) can be taken pointwise, i.e.,

$$f(t, Y_t, \phi_t) = p \operatorname*{ess\,sup}_{\pi\in\mathbb{R}:\pi\gamma_t<1} \bigg[\big(\mu_t^\mathbb{F} Y_t + \sigma_t^\mathbb{F}\phi_t\big)\pi - \frac{1-p}{2} Y_t \big|\pi\sigma_t^\mathbb{F}\big|^2 + K_t^p \frac{(1-\pi\gamma_t)^p}{p}\bigg],$$

while the optimal strategy exists and is given by

$$\hat{\pi}_t^{\mathbb{F}} = \underset{\pi \in \mathbb{R}: \pi \gamma_t < 1}{\arg\max} \left[\left(\mu_t^{\mathbb{F}} Y_t + \sigma_t^{\mathbb{F}} \phi_t\right)\pi - \frac{1-p}{2} Y_t \left|\pi \sigma_t^{\mathbb{F}}\right|^2 + K_t^p \frac{(1-\pi\gamma_t)^p}{p} \right]$$

for $0 \le t \le T$. *It satisfies for* $0 \le t \le T$ *the estimates*

$$\begin{aligned} &\left[\min\left(\pi_t^M, \frac{1}{\gamma_t}\right) - \rho_t\right] 1_{\gamma_t \ge 0} + \max\left(\pi_t^M, \frac{1}{\gamma_t}\right) 1_{\gamma_t < 0} \\ &\quad \le \hat{\pi}_t^{\mathbb{F}} \le \min\left(\pi_t^M, \frac{1}{\gamma_t}\right) 1_{\gamma_t \ge 0} + \left[\max\left(\pi_t^M, \frac{1}{\gamma_t}\right) + \rho_t\right] 1_{\gamma_t < 0}, \end{aligned} \tag{4.18}$$

where

$$\pi_t^M = \frac{\mu_t^{\mathbb{F}}}{(1-p)|\sigma_t^{\mathbb{F}}|^2} + \frac{\phi_t}{(1-p)Y_t\sigma_t^{\mathbb{F}}} \quad \textit{and} \quad \rho_t = \left(\frac{|\gamma_t|^p K_t^p}{(1-p)Y_t|\sigma_t^{\mathbb{F}}|^2}\right)^{\frac{1}{2-p}}. \tag{4.19}$$

Proof By Lemma 4.4, the process Y lies in $L_+^b(\mathbb{F})$. From (4.7), we know that for any $\nu \in \mathcal{A}_{\mathbb{F}}$, the process $\xi^\nu(Y)$ is a $(\mathbb{P}, \mathbb{F})$-supermartingale. In particular, by taking $\nu = 0$, we see that the process $\{Y_t + \int_0^t K_\theta^p \, d\theta, 0 \le t \le T\}$ is a $(\mathbb{P}, \mathbb{F})$-supermartingale. From the Doob–Meyer decomposition and the martingale representation theorem for the Brownian motion filtration, we get the existence of $\phi \in L^2_{\text{loc}}(W)$ and an increasing $\mathbb{F}$-adapted process A such that

$$dY_t = \phi_t \, dW_t - dA_t, \quad 0 \le t \le T. \tag{4.20}$$

From (2.11) and Itô's formula, we deduce that the finite variation process in the decomposition of the $(\mathbb{P}, \mathbb{F})$-supermartingale $\xi^\nu(Y)$, $\nu \in \mathcal{A}_{\mathbb{F}}$, is given by $-A^\nu$ with

$$\begin{aligned} dA_t^\nu = \left(X_t^{\nu,\mathbb{F}}\right)^p \bigg\{ \frac{1}{p} dA_t - \bigg[\left(\mu_t^{\mathbb{F}} Y_t + \sigma_t^{\mathbb{F}} \phi_t\right)\nu_t - \frac{1-p}{2} Y_t \left|\nu_t \sigma_t^{\mathbb{F}}\right|^2 \\ + K_t^p \frac{(1-\nu_t\gamma_t)^p}{p} \bigg] dt \bigg\}. \end{aligned}$$

Now the supermartingale property of $\xi^\nu(Y)$, $\nu \in \mathcal{A}_{\mathbb{F}}$, means that A^ν is nondecreasing. Combining this with the martingale property of $\xi^{\hat{\pi}^{\mathbb{F}}}(Y)$, i.e., $A^{\hat{\pi}^{\mathbb{F}}} = 0$, we obtain

$$\begin{aligned} dA_t &= p \left[\left(\mu_t^{\mathbb{F}} Y_t + \sigma_t^{\mathbb{F}} \phi_t\right)\hat{\pi}_t^{\mathbb{F}} - \frac{1-p}{2} Y_t \left|\hat{\pi}_t^{\mathbb{F}} \sigma_t^{\mathbb{F}}\right|^2 + K_t^p \frac{(1-\hat{\pi}_t^{\mathbb{F}}\gamma_t)^p}{p} \right] dt \\ &= p \operatorname*{ess\,sup}_{\nu \in \mathcal{A}_{\mathbb{F}}} \left[\left(\mu_t^{\mathbb{F}} Y_t + \sigma_t^{\mathbb{F}} \phi_t\right)\nu_t - \frac{1-p}{2} Y_t \left|\nu_t \sigma_t^{\mathbb{F}}\right|^2 + K_t^p \frac{(1-\nu_t\gamma_t)^p}{p} \right] dt. \end{aligned}$$

Observing from (4.9) that $Y_T = G_T$, this proves together with (4.20) that (Y, ϕ) solves the BSDE (4.16). In particular, the process Y is continuous.

Consider now another solution $(\tilde{Y}, \tilde{\phi}) \in L_+^b(\mathbb{F}) \times L^2_{\text{loc}}(W)$ to the BSDE (4.16), and define the family of nonnegative $\mathbb{F}$-adapted processes $\tilde{\xi}^\nu(\tilde{Y})$, $\nu \in \mathcal{A}_\mathbb{F}$, by

$$\xi_t^\nu(\tilde{Y}) = U\big(X_t^{\nu,\mathbb{F}}\big)\tilde{Y}_t + \int_0^t U\big(X_\theta^{\nu,\mathbb{F}}\big)(1-\nu_\theta\gamma_\theta)^p K_\theta^p \, d\theta, \quad 0 \le t \le T. \tag{4.21}$$

By Itô's formula, the same calculations as above yield $d\xi_t^\nu(\tilde{Y}) = d\tilde{M}_t^\nu - d\tilde{A}_t^\nu$, where $\tilde{A}^\nu$ is a nondecreasing $\mathbb{F}$-adapted process and $\tilde{M}^\nu$ is a local $(\mathbb{P},\mathbb{F})$-martingale as a stochastic integral with respect to the Brownian motion W. By Fatou's lemma, using that $\tilde{Y} \in L_+^b(\mathbb{F})$, this implies that the process $\xi^\nu(\tilde{Y})$ is a $(\mathbb{P},\mathbb{F})$-supermartingale for any $\nu \in \mathcal{A}_\mathbb{F}$. Recalling that $\tilde{Y}_T = G_T$, we deduce that for all $\nu \in \mathcal{A}_\mathbb{F}$,

$$\mathbb{E}\bigg[U\big(X_T^{\nu,\mathbb{F}}\big)G_T + \int_t^T U\big(X_\theta^{\nu,\mathbb{F}}\big)(1-\nu_\theta\gamma_\theta)^p K_\theta^p \, d\theta \bigg| \mathcal{F}_t\bigg] \le U\big(X_t^{\nu,\mathbb{F}}\big)\tilde{Y}_t \tag{4.22}$$

for all $t \in [0,T]$. If $p > 0$ (resp. $p < 0$), then by dividing the above inequalities by $U(X_t^{\nu,\mathbb{F}})$ which is positive (resp. negative), we deduce by the definition of Y (see (4.10) and (4.11)) and the arbitrariness of $\nu \in \mathcal{A}_\mathbb{F}$ that $Y_t \le$ (resp. $\ge$) $\tilde{Y}_t$, $0 \le t \le T$. This shows that Y is the smallest (resp. largest) solution to the BSDE (4.16).

Next, we impose the additional integrability condition that

$$\int_0^T \left|\frac{K_t}{\sigma_t^\mathbb{F}}\right|^{\frac{2p}{2-p}} dt < \infty \quad \text{a.s.} \tag{4.23}$$

Consider the function F defined on $\{(\omega,t,\pi) \in \Omega \times [0,T] \times \mathbb{R} : \pi\gamma_t(\omega) < 1\}$ by

$$F(t,\pi) = \big(\mu_t^\mathbb{F} Y_t + \sigma_t^\mathbb{F}\phi_t\big)\pi - \frac{1-p}{2}Y_t\big|\pi\sigma_t^\mathbb{F}\big|^2 + K_t^p\frac{(1-\pi\gamma_t)^p}{p}$$

(as usual, we omit the dependence of F on ω), and define for any $(\omega,t) \in \Omega \times [0,T]$ the set $\Gamma_t = \{\pi \in \mathbb{R} : \pi\gamma_t < 1\}$, which is equal to $(-\infty, 1/\gamma_t)$ if $\gamma_t \ge 0$ and $(1/\gamma_t, \infty)$ otherwise. By definition, we clearly have almost surely

$$\frac{1}{p}f(t,Y_t,\phi_t) \le \operatorname*{ess\,sup}_{\pi\in\Gamma_t} F(t,\pi), \quad 0 \le t \le T. \tag{4.24}$$

Let us prove the converse inequality. Observe that almost surely, for all $t \in [0,T]$, the function $\pi \mapsto F(t,\pi)$ is strictly concave (recall that the process Y is strictly positive) and C^2 on Γ_t with

$$\frac{\partial F}{\partial \pi}(t,\pi) = \big(\mu_t^\mathbb{F} Y_t + \sigma_t^\mathbb{F}\phi_t\big) - (1-p)Y_t\big|\sigma_t^\mathbb{F}\big|^2\pi - \gamma_t K_t^p(1-\pi\gamma_t)^{p-1},$$

and satisfies in the case where $\gamma_t \ge 0$, i.e., $\Gamma_t = (-\infty, 1/\gamma_t)$, that

$$\lim_{\pi\to-\infty} F(t,\pi) = -\infty, \qquad \lim_{\pi\to-\infty}\frac{\partial F}{\partial\pi}(t,\pi) = \infty, \qquad \lim_{\pi\nearrow 1/\gamma_t}\frac{\partial F}{\partial\pi}(t,\pi) = -\infty,$$

and in the other case where $\gamma_t < 0$, i.e., $\Gamma_t = (1/\gamma_t, \infty)$, that

$$\lim_{\pi \searrow 1/\gamma_t} \frac{\partial F}{\partial \pi}(t,\pi) = \infty, \qquad \lim_{\pi \to \infty} F(t,\pi) = -\infty, \qquad \lim_{\pi \to \infty} \frac{\partial F}{\partial \pi}(t,\pi) = -\infty.$$

We deduce that almost surely, for all $t \in [0,T]$, the function $\pi \mapsto F(t,\pi)$ attains its maximum at some point $\hat{\pi}_t^{\mathbb{F}}$, which satisfies

$$\frac{\partial F}{\partial \pi}\big(t, \hat{\pi}_t^{\mathbb{F}}\big) = 0.$$

By a measurable selection argument, this defines an $\mathbb{F}$-adapted process $\hat{\pi}^{\mathbb{F}} = (\hat{\pi}_t^{\mathbb{F}})_{t\in[0,T]}$. In order to prove that we have equality in (4.24), it suffices to show that such a $\hat{\pi}^{\mathbb{F}}$ lies in $\mathcal{A}_{\mathbb{F}}$, and this will be checked under the condition (4.23). For this, consider the $\mathbb{F}$-adapted processes π^M in (4.19) and $\tilde{\pi}^{M,\gamma}$ defined by

$$\tilde{\pi}_t^{M,\gamma} = \min\left(\pi_t^M, \frac{1}{\gamma_t}\right) 1_{\gamma_t \geq 0} + \max\left(\pi_t^M, \frac{1}{\gamma_t}\right) 1_{\gamma_t < 0} \in \Gamma_t, \quad 0 \leq t \leq T.$$

Notice that by (2.5), continuity of the paths of Y and since $\phi \in L^2_{\text{loc}}(W)$, we have $\int_0^T |\pi_t^M \sigma_t^{\mathbb{F}}|^2\, dt < \infty$ a.s. Moreover, we have $|\tilde{\pi}^{M,\gamma}| \leq |\pi^M|$, and thus $\tilde{\pi}^{M,\gamma}$ lies in $\mathcal{A}_{\mathbb{F}}$. Fix $\omega \in \Omega$, and for $t \in [0,T]$, let us distinguish the following two cases:

– (i) $\gamma_t \geq 0$: if $\pi_t^M < 1/\gamma_t$, and so $\tilde{\pi}^{M,\gamma} = \pi^M$, we have

$$\frac{\partial F}{\partial \pi}\big(t, \tilde{\pi}_t^{M,\gamma}\big) = -\gamma_t K_t^p \big(1 - \pi_t^M \gamma_t\big)^{p-1} \leq 0,$$

and thus by strict concavity of $F(t,\pi)$ in π that $\hat{\pi}_t^{\mathbb{F}} \leq \tilde{\pi}_t^{M,\gamma}$. If $\pi_t^M \geq 1/\gamma_t$, and since $\hat{\pi}_t^{\mathbb{F}} < 1/\gamma_t$, we also get $\hat{\pi}_t^{\mathbb{F}} \leq \tilde{\pi}_t^{M,\gamma}$ $(= 1/\gamma_t)$.

– (ii) $\gamma_t < 0$: if $\pi_t^M > 1/\gamma_t$, and so $\tilde{\pi}^{M,\gamma} = \pi^M$, we have

$$\frac{\partial F}{\partial \pi}\big(t, \tilde{\pi}_t^{M,\gamma}\big) = -\gamma_t K_t^p \big(1 - \pi_t^M \gamma_t\big)^{p-1} \geq 0,$$

and thus by strict concavity of $F(t,\pi)$ in π that $\hat{\pi}_t^{\mathbb{F}} \geq \tilde{\pi}_t^{M,\gamma}$. If $\pi_t^M \leq 1/\gamma_t$, and since $\hat{\pi}_t^{\mathbb{F}} > 1/\gamma_t$, we also get $\hat{\pi}_t^{\mathbb{F}} \geq \tilde{\pi}_t^{M,\gamma}$ $(= 1/\gamma_t)$.

To sum up the cases (i) and (ii), we have almost surely, for all $t \in [0,T]$, that

$$\hat{\pi}_t^{\mathbb{F}} \leq \tilde{\pi}_t^{M,\gamma} \quad \text{if } \gamma_t \geq 0, \quad \text{and} \quad \hat{\pi}_t^{\mathbb{F}} \geq \tilde{\pi}_t^{M,\gamma} \quad \text{if } \gamma_t < 0. \tag{4.25}$$

Next, consider the $\mathbb{F}$-adapted process $\bar{\pi}$ defined by

$$\bar{\pi}_t = \big(\tilde{\pi}_t^{M,\gamma} - \rho_t\big) 1_{\gamma_t \geq 0} + \big(\tilde{\pi}_t^{M,\gamma} + \rho_t\big) 1_{\gamma_t < 0}, \quad 0 \leq t \leq T \tag{4.26}$$

for some $\mathbb{F}$-adapted nonnegative process $\rho = (\rho_t)_{t\in[0,T]}$ to be determined. Fix $\omega \in \Omega$, and for $t \in [0,T]$, let us again distinguish the following two cases:

– (i′) $\gamma_t \geq 0$: if $\pi_t^M < 1/\gamma_t$, and so $\bar{\pi}_t = \pi_t^M - \rho_t$, we have

$$\frac{\partial F}{\partial \pi}(t,\bar{\pi}_t) = (1-p)Y_t\big|\sigma_t^{\mathbb{F}}\big|^2\rho_t - \gamma_t K_t^p\big(1-\pi_t^M\gamma_t+\rho_t\gamma_t\big)^{p-1}$$
$$\geq (1-p)Y_t\big|\sigma_t^{\mathbb{F}}\big|^2\rho_t - \gamma_t K_t^p(\rho_t\gamma_t)^{p-1}. \tag{4.27}$$

If $\pi_t^M \geq 1/\gamma_t$, and so $\bar{\pi}_t = \frac{1}{\gamma_t} - \rho_t$, the inequality (4.27) also holds true. Hence, by choosing ρ_t such that the right-hand side of (4.27) vanishes, i.e.,

$$\rho_t = \left(\frac{|\gamma_t|^p K_t^p}{(1-p)Y_t|\sigma_t^{\mathbb{F}}|^2}\right)^{\frac{1}{2-p}}, \tag{4.28}$$

we obtain $\frac{\partial F}{\partial \pi}(t,\bar{\pi}_t) \geq 0$, and so by strict concavity of F in π that $\bar{\pi}_t \leq \hat{\pi}_t^{\mathbb{F}}$.

– (ii′) $\gamma_t < 0$: if $\pi_t^M > 1/\gamma_t$, and so $\bar{\pi}_t = \pi_t^M + \rho_t$, we have

$$\frac{\partial F}{\partial \pi}(t,\bar{\pi}_t) = -(1-p)Y_t\big|\sigma_t^{\mathbb{F}}\big|^2\rho_t - \gamma_t K_t^p\big(1-\pi_t^M\gamma_t-\rho_t\gamma_t\big)^{p-1}$$
$$\leq (1-p)Y_t\big|\sigma_t^{\mathbb{F}}\big|^2\rho_t - \gamma_t K_t^p\big(\rho_t|\gamma_t|\big)^{p-1}. \tag{4.29}$$

If $\pi_t^M \leq 1/\gamma_t$, and so $\bar{\pi}_t = \frac{1}{\gamma_t} + \rho_t$, the inequality (4.29) also holds true. Hence, by choosing ρ_t as in (4.28), we see that the right-hand side of (4.29) vanishes, and so $\frac{\partial F}{\partial \pi}(t,\bar{\pi}_t) \leq 0$. By strict concavity of F in π, we deduce that $\bar{\pi}_t \geq \hat{\pi}_t^{\mathbb{F}}$.

Let us then consider the process $\rho = (\rho_t)_{t\in[0,T]}$ defined by (4.28) for all $0 \leq t \leq T$. Under (4.23), and recalling that Y is continuous and $\gamma < 1$, we easily see that ρ satisfies the integrability condition $\int_0^T |\rho_t\sigma_t^{\mathbb{F}}|^2\,dt < \infty$ a.s., and so $\bar{\pi}$ in (4.26) lies in $\mathcal{A}_{\mathbb{F}}$. Moreover, from the analysis in the cases (i′) and (ii′), we have almost surely, for all $t \in [0,T]$,

$$\hat{\pi}_t^{\mathbb{F}} \geq \bar{\pi}_t \quad \text{if } \gamma_t \geq 0, \quad \text{and} \quad \hat{\pi}_t^{\mathbb{F}} \leq \bar{\pi}_t \quad \text{if } \gamma_t < 0.$$

Together with (4.25), this proves that $\hat{\pi}^{\mathbb{F}}$ lies in $\mathcal{A}_{\mathbb{F}}$ and satisfies the estimates (4.18). □

Remark 4.6 The driver $f(t,Y_t,\phi_t)$ of the BSDE (4.16) is in general not Lipschitz in the arguments in (Y_t,ϕ_t), and we are not able to prove by standard arguments that there exists a unique solution to this BSDE. Actually, when $\gamma = 0$, the driver is equal to $f(t,Y_t,\phi_t) = \frac{1}{2}\frac{(\mu_t^{\mathbb{F}}Y_t+\sigma_t^{\mathbb{F}}\phi_t)^2}{(1-p)\sigma_t^{\mathbb{F}}Y_t}$, hence not even quadratic in (Y_t,ϕ_t), and to the best of our knowledge, there do not exist uniqueness results for such BSDEs. However, in the case $p < 0$ and under (4.23), one can show the uniqueness[1] of a solution in $L_+^b(\mathbb{F})$ to the BSDE (4.16). Indeed, let us consider another solution $(\tilde{Y},\tilde{\phi}) \in L_+^b(\mathbb{F}) \times L_{\mathrm{loc}}^2(W)$ to (4.16), and take a pointwise maximizer of $f(t,\tilde{Y}_t,\tilde{\phi}_t)$. Under (4.23) and by the same arguments as at the end of the proof of Theorem 4.5

[1] We thank Marcel Nutz for pointing out this argument.

with Y replaced by $\tilde{Y}$, this defines a control $\nu^* \in \mathcal{A}_{\mathbb{F}}$. Then, by construction of the BSDE, the process $\xi^{\nu^*}(\tilde{Y})$ in (4.21) is a local martingale. Since $p<0$, we see that $\xi^{\nu^*}(\tilde{Y})$ is nonpositive, and hence a submartingale. This implies that the relation (4.22) holds now with the opposite inequality for $\nu=\nu^*$. By using the definition (4.11) of Y, we deduce that $\tilde{Y} \geq Y$, i.e., Y is the smallest solution. Since we already know that it is the largest solution, this shows that Y is unique in $L_+^b(\mathbb{F})$.

Remark 4.7 We provide some comments and interpretation on the form of the optimal before-default strategy. Let us consider a default-free stock market model with drift and volatility coefficients $\mu^{\mathbb{F}}$ and $\sigma^{\mathbb{F}}$ and an investor with CRRA utility function $U(x)=x^p/p$, looking at the optimal investment problem

$$V_0^M = \sup_{\pi \in \mathcal{A}_{\mathbb{F}}} \mathbb{E}\big[U\big(X_T^{\mathbb{F}}\big)\big],$$

where $X^{\mathbb{F}}$ is the wealth process in (2.11) and $\mathcal{A}_{\mathbb{F}}$ is the set of $\mathbb{F}$-adapted processes $\pi^{\mathbb{F}}$ satisfying $\int_0^T |\pi_t^{\mathbb{F}}\sigma_t^{\mathbb{F}}|^2\,dt<\infty$ and $\pi_t\gamma_t<1$, $0\leq t\leq T$, a.s. In other words, V_0^M is the Merton optimal investment problem for strategies constrained to be upper-bounded (resp. lower-bounded) in proportion by $1/\gamma_t$ when $\gamma_t \geq 0$ (resp. <0). By considering similarly as in (4.9), the process

$$Y_t^M = p \operatorname*{ess\,sup}_{\pi^{\mathbb{F}} \in \mathcal{A}_{\mathbb{F}}(t,\nu)} \mathbb{E}\left[U\left(\frac{X_T^{\mathbb{F}}}{X_t^{\nu,\mathbb{F}}}\right)\middle|\mathcal{F}_t\right], \quad 0 \leq t \leq T$$

and arguing similarly as in Theorem 4.5, one can prove that when $p>0$ (resp. $p<0$), Y^M is the smallest (resp. largest) solution to the BSDE

$$Y_t^M = 1 + \int_t^T f^M\big(\theta, Y_\theta^M, \phi_\theta^M\big)\,d\theta - \int_t^T \phi_\theta^M\,dW_\theta, \quad 0 \leq t \leq T,$$

for some $\phi^M \in L^2_{\mathrm{loc}}(W)$, where

$$f^M\big(t, Y_t^M, \phi_t^M\big) = p \operatorname*{ess\,sup}_{\pi \in \mathbb{R}: \pi\gamma_t<1} \left[\big(\mu_t^{\mathbb{F}} Y_t^M + \sigma_t^{\mathbb{F}}\phi_t^M\big)\pi - \frac{1-p}{2} Y_t^M \big|\pi\sigma_t^{\mathbb{F}}\big|^2\right],$$

while the optimal strategy for V_0^M is given by

$$\begin{aligned}\hat{\pi}_t^{M,\gamma} &= \min\left(\frac{\mu_t^{\mathbb{F}}}{(1-p)|\sigma_t^{\mathbb{F}}|^2} + \frac{\phi_t^M}{(1-p)Y_t^M\sigma_t^{\mathbb{F}}}, \frac{1}{\gamma_t}\right)1_{\gamma_t \geq 0} \\ &\quad + \max\left(\frac{\mu_t^{\mathbb{F}}}{(1-p)|\sigma_t^{\mathbb{F}}|^2} + \frac{\phi_t^M}{(1-p)Y_t^M\sigma_t^{\mathbb{F}}}, \frac{1}{\gamma_t}\right)1_{\gamma_t<0}.\end{aligned}$$

Notice that when the coefficients $\mu^{\mathbb{F}}$, $\sigma^{\mathbb{F}}$ and γ are deterministic, then Y^M is also deterministic, i.e., $\phi^M=0$, and is the positive solution to the ordinary differential equation

$$Y_t^M = 1 + \int_t^T f^M\big(\theta, Y_\theta^M\big)\,d\theta, \quad 0 \leq t \leq T,$$

with $f^M(t,y) = py \sup_{\pi:\pi\gamma_t<1}(\mu_t^{\mathbb{F}}\pi - \frac{1-p}{2}|\pi\sigma_t^{\mathbb{F}}|^2) =: pyc(t)$, so that we then have $Y_t^M = \exp(p\int_t^T c(\theta)\,d\theta)$. In particular, when there is no constraint on trading strategies, i.e., $\gamma = 0$, we also recover the usual expression of the optimal Merton trading strategy as $\hat{\pi}_t^{M,0} = \frac{\mu_t^{\mathbb{F}}}{(1-p)|\sigma_t^{\mathbb{F}}|^2}$.

In our with-default stock market model, the optimal before-default strategy $\hat{\pi}^{\mathbb{F}}$ satisfies the estimates (4.18), which have the following interpretation. The process $\hat{\pi}^{M,\gamma}$ has a similar form as the optimal Merton trading strategy described above, but includes further, through the process Y and K, the eventuality of a default of the stock price, inducing a jump of size γ, and then a switch of the coefficients of the stock price from $(\mu^{\mathbb{F}}, \sigma^{\mathbb{F}})$ to (μ^d, σ^d). In the case of a loss at default, i.e., positive jump γ, the optimal trading strategy $\hat{\pi}^{\mathbb{F}}$ is upper-bounded by $\hat{\pi}^{M,\gamma}$, with a spread measured by the term ρ varying increasingly with the loss size γ, and converging to zero when the loss goes to zero, as expected since in this case the model behaves as a no-default market. Symmetrically, in the case of gain at default, i.e., negative jump γ, the optimal trading strategy $\hat{\pi}^{\mathbb{F}}$ is lower-bounded by $\hat{\pi}^{M,\gamma}$, with a spread also measured by the term ρ.

4.3 Example and numerical illustrations

In this paragraph, we present a simple illustrative example to show quantitatively the impact of the counterparty default probability as a function of the default intensity and the loss/gain given default on optimal investment. We consider the special case where $\mu^{\mathbb{F}}, \sigma^{\mathbb{F}}, \gamma$ are constants, $\mu^d(\theta)$ and $\sigma^d(\theta)$ are only deterministic functions of θ, and the default time τ is independent of $\mathbb{F}$, so that $\alpha_t(\theta)$ is simply a known deterministic function $\alpha(\theta)$ of $\theta \in \mathbb{R}_+$ and the survival probability

$$G(t) = \mathbb{P}[\tau > t|\mathcal{F}_t] = \mathbb{P}[\tau > t] = \int_t^\infty \alpha(\theta)\,d\theta$$

is a deterministic function. We also choose a CRRA utility function $U(x) = \frac{x^p}{p}$, $p < 1$, $p \neq 0$, $x > 0$. Notice that $V_\theta^d(x) = v^d(\theta, x) = U(x)k(\theta)^p$ with

$$k(\theta) = \left(\mathbb{E}\left[\alpha_T(\theta)\left(\frac{Z_T(\theta)}{\alpha_T(\theta)}\right)^{-q}\right]\right)^{\frac{1}{q}} = \alpha(\theta)^{\frac{1}{p}} \exp\left(\frac{1}{2}\left|\frac{\mu^d(\theta)}{\sigma^d(\theta)}\right|^2 \frac{1}{1-p}(T-\theta)\right).$$

Moreover, the optimal wealth process after default does not depend on the default time density, and the optimal strategy after default is given, similarly as in the (unconstrained) Merton case, by

$$\hat{\pi}^d(\theta) = \frac{\mu^d(\theta)}{(1-p)|\sigma^d(\theta)|^2}.$$

On the other hand, from the above results and discussion, we know that in this Markovian case, the value function of the global before-default optimization problem is of the form $V_0 = v(0, X_0)$ with

$$v(t,x) = U(x)Y(t),$$

where Y is a deterministic function of time, solving the first-order ordinary differential equation (ODE)

$$Y(t) = G(T) + \int_t^T f\big(\theta, Y(\theta)\big)\, d\theta, \quad t \in [0, T] \tag{4.30}$$

with

$$f(t, y) = p \sup_{\pi\gamma<1} \left[\left(\mu^{\mathbb{F}}\pi - \frac{1-p}{2}\left|\pi\sigma^{\mathbb{F}}\right|^2 \right) y + k(t)^p \frac{(1-\pi\gamma)^p}{p} \right]. \tag{4.31}$$

There is no explicit solution to this ODE, and so we give some numerical illustrations.

The following numerical results are based on the model parameters described below. We suppose that the survival probability follows an exponential distribution with constant default intensity, i.e., $G(t) = e^{-\lambda t}$ where $\lambda > 0$, and thus the density function is $\alpha(\theta) = \lambda e^{-\lambda\theta}$. In the case where $\gamma > 0$ (loss at default), we consider functions $\mu^d(\theta)$ and $\sigma^d(\theta)$ of the form

$$\mu^d(\theta) = \mu^{\mathbb{F}} \frac{\theta}{T}, \quad \sigma^d(\theta) = \sigma^{\mathbb{F}} \left(2 - \frac{\theta}{T} \right), \quad \theta \in [0, T],$$

which have the following economic interpretation. The ratio between the after- and before-default rate of return is smaller than one, meaning that the asset is less competitive after the loss at default. Moreover, this ratio increases linearly with the default time; so the after-default rate of return drops to zero when the default time occurs near the initial date, and converges to the before-default rate of return when the default time occurs near the finite investment horizon. We have a similar interpretation for the volatility, but with a symmetric relation: The ratio between the after- and before-default volatility is larger than one, meaning that the market is more volatile after default. Moreover, this ratio decreases linearly with the default time, converging to the double (resp. initial) value of the before-default volatility when the default time goes to the initial (resp. terminal horizon) time. When $\gamma < 0$, we choose the reciprocal model for μ^d, that is, $\mu^d(\theta) = \mu^{\mathbb{F}}(2 - \frac{\theta}{T})$, $\theta \in [0, T]$, which means that the asset is more competitive in this case, and we suppose that σ^d is still defined as above.

Notice that in this example, $k(t)/\sigma^{\mathbb{F}}$ is a deterministic continuous function on $[0, T]$ so that the integrability condition (4.23) is obviously satisfied. To solve numerically the ODE (4.30), we apply the Howard algorithm, which consists in iterating in (4.31) the control value π at each step of the ODE resolution. We initialize the algorithm by choosing the constrained Merton strategy

$$\hat{\pi}^{M,\gamma} = \min\left(\frac{\mu^{\mathbb{F}}}{(1-p)|\sigma^{\mathbb{F}}|^2}, \frac{1}{\gamma} \right) 1_{\gamma \ge 0} + \max\left(\frac{\mu^{\mathbb{F}}}{(1-p)|\sigma^{\mathbb{F}}|^2}, \frac{1}{\gamma} \right) 1_{\gamma < 0}.$$

We perform numerical results with $\mu^{\mathbb{F}} = 0.03$, $\sigma^{\mathbb{F}} = 0.2$, $T = 1$, for various degrees of risk aversion $1 - p$ (smaller, close to and larger than one) and by varying both the intensity of default λ and the jump size γ. The numerical tests show that except in some extreme cases where both the default probability and the loss or gain given default are large, the optimal strategy is quite invariant with respect to time in most

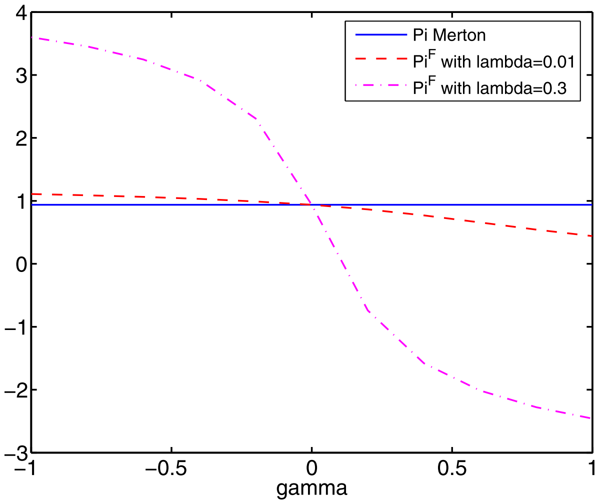


Fig. 1 Optimal strategy $\hat{\pi}^{\mathbb{F}}$ vs. Merton $\hat{\pi}^{M,\gamma}$: $p = 0.2$, $\lambda = 0.01$ and 0.3, respectively

considered cases. So we give below instead of the optimal strategy its average value over time.

Figure 1 plots the graph of the optimal proportion $\hat{\pi}^{\mathbb{F}}$ (which takes thus into account the counterparty risk) invested in stock before default as a function of the jump size γ. When γ equals zero, it is clear that the optimal strategy coincides with the Merton one. When there is a loss at default, i.e., $\gamma > 0$, the optimal strategy is always smaller than the Merton one, and the situation is reversed when there is a gain at default, i.e., $\gamma < 0$. Moreover, the strategy is decreasing with respect to γ, which means that one should reduce the stock investment when the loss given default is increasing, while one should increase investment when the gain at default increases. This behavior of the optimal trading strategy is consistent with the estimates in (4.18). These observations are more manifest when λ (and hence the default probability) is large. Moreover, we see that when λ is small, $\hat{\pi}^{\mathbb{F}}$ approaches the Merton strategy.

Table 1 shows the impact of the default intensity λ, or equivalently of the default probability of the counterparty up to T, i.e., of PD $= \mathbb{P}(\tau \leq T) = 1 - e^{-\lambda T}$, on the optimal strategy $\hat{\pi}^{\mathbb{F}}$ compared to the Merton strategy $\hat{\pi}^{M,\gamma}$. We also compute numerical results by varying the degree of risk aversion $1 - p$ and for different values of γ. First observe, as expected, that when the agent is more risk-averse, i.e., p is decreasing, then the proportion invested in stock is also decreasing. Secondly, under loss at default, i.e., $\gamma > 0$, we see that the optimal strategy $\hat{\pi}^{\mathbb{F}}$ decreases when the probability of default increases, and this monotonicity is reversed under gain at default, i.e., $\gamma < 0$. Notice also, as already mentioned for Fig. 1, that the optimal strategy is increasing with the size $|\gamma|$ of the gain at default (when $\gamma < 0$), and decreasing with the size of the loss at default (when $\gamma > 0$). In this last case, we even observe that under a big loss at default, it is optimal to be short in the stock. For example, we see that for a proportional loss $\gamma = 0.5$, an intensity of default $\lambda = 0.3$ and with $p = 0.2$, the optimal strategy is $\hat{\pi}^{\mathbb{F}} = -1.83$. The economic interpretation is the following: The investor knows that there is a large probability of default, which will induce a

Table 1 Optimal strategy $\hat{\pi}^{\mathbb{F}}$ with various λ and γ

$\hat{\pi}^{M,\gamma}$	$p=0.2$		$p\to 0$		$p=-0.2$	
	$\gamma=0.1$	$\gamma=0.5$	$\gamma=0.1$	$\gamma=0.5$	$\gamma=0.1$	$\gamma=0.5$
	0.94	0.94	0.75	0.75	0.63	0.63
$\lambda=0.01$ PD $=0.01$	0.90	0.72	0.72	0.57	0.60	0.48
$\lambda=0.05$ PD $=0.05$	0.77	0.12	0.62	0.09	0.51	0.08
$\lambda=0.1$ PD $=0.10$	0.61	−0.41	0.49	−0.33	0.41	−0.27
$\lambda=0.3$ PD $=0.26$	0.00	−1.83	0.00	−1.43	0.00	−1.18
$\hat{\pi}^{M,\gamma}$	$\gamma=-0.1$	$\gamma=-0.5$	$\gamma=-0.1$	$\gamma=-0.5$	$\gamma=-0.1$	$\gamma=-0.5$
	0.94	0.94	0.75	0.75	0.63	0.63
$\lambda=0.01$ PD $=0.01$	0.97	1.05	0.77	0.84	0.64	0.70
$\lambda=0.05$ PD $=0.05$	1.08	1.44	0.87	1.15	0.72	0.95
$\lambda=0.1$ PD $=0.10$	1.22	1.86	0.98	1.47	0.81	1.22
$\lambda=0.3$ PD $=0.26$	1.76	3.10	1.41	2.44	1.17	2.00

big loss on its asset. Then it is intuitively clear that she should sell her stock before the default.

Finally, we compare the value function, i.e., the performance of the optimal investment strategy, in our counterparty risk model to that in the classical Merton model. This is equivalent here to compare the solution $Y(t)$ of the ODE (4.30) with the function $Y^M(t)$ deduced with $k(t)=0$ and $G(T)=1$. Figure 2 represents the curves of Y for different values of loss at default $\gamma>0$ and for a given small intensity of default $\lambda=0.01$. It appears that the value function $Y(t)$ obtained with counterparty risk is always below the Merton one Y^M, and Y is decreasing with respect to the proportional loss γ, which is a priori in accordance with economic intuition. We also observe that Y is decreasing in time (as the Merton value Y^M) and converges at $T=1$ to $G(T)=e^{-\lambda T}\approx 0.99$, the survival probability. Figure 3 provides similar results, but for different values of gain at default $\gamma<0$ and with a given small intensity of default. In contrast with the situation in Fig. 2, we observe here that the value function is larger than the Merton one in the beginning and becomes smaller when one approaches the final horizon T since it converges to $G(T)<1$. This confirms the intuition that the investor improves her optimal performance in the beginning by making

Fig. 2 Value function Y for loss at default vs. Merton Y^M: $p=0.2$, $\lambda=0.01$ and γ positive

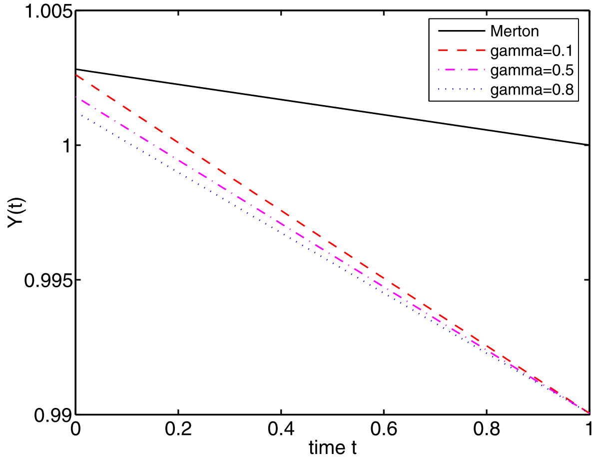
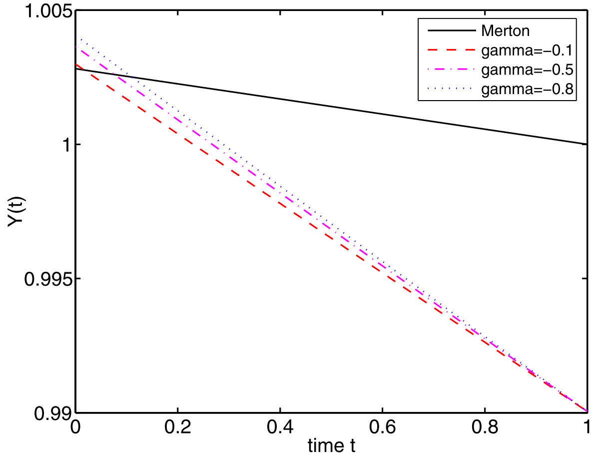


Fig. 3 Value function Y for gain at default vs. Merton Y^M: $p=0.2$, $\lambda=0.01$ and γ negative



profit from the rise of the asset value after default. Actually, as shown by Fig. 4, one can also outperform the Merton strategy in the case of loss at default under extreme situations when the intensity of default is large, e.g., $\lambda = 0.5$ (corresponding approximately to a default probability of PD $= 40\%$ per year) by taking short positions in the asset, and this benefit is increasing with the size of the loss γ. For example, with a proportional loss of 80%, we find a relative ratio of outperformance in the beginning equal to $(Y - Y^M)(0)/Y^M(0) = 6.9\%$. This may be interpreted as follows: The investor knows that there is a high probability of default, and she takes advantage of this information to sell short in the beginning her positions on the asset, and then to buy off the asset after default at a low price, improving consequently her optimal performance, at least far from the final horizon.

The comparison of Figs. 2 and 4 reveals an interesting feature in the case of loss at default, i.e., $\gamma > 0$. By doing more numerical tests, we observed that there is a critical level of default intensity λ (around 0.1, corresponding approximately to a default probability of 10%) from which the optimal performance Y exceeds the Merton one

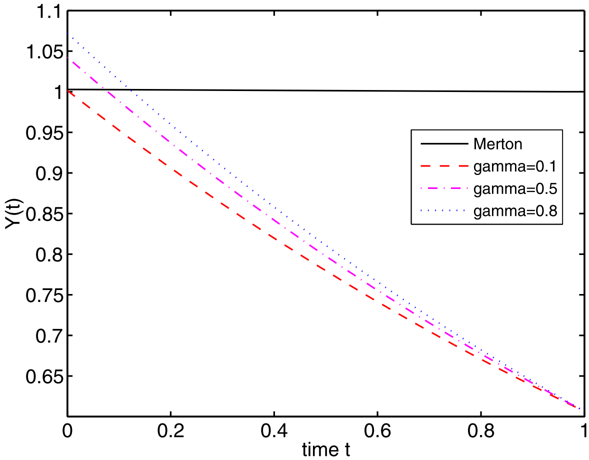


Fig. 4 Value function Y for loss at default vs. Merton Y^M: $p = 0.2$, $\lambda = 0.5$ and γ positive

in the beginning. Furthermore, the monotonicity of Y with respect to γ switches from a decreasing to an increasing property.

5 Conclusion

This paper studies an optimal investment problem under the presence of counterparty risk for the traded stock. By adopting a conditional density approach for the default time, we derive a suitable decomposition in the reference filtration of the initial utility maximization problem into an after-default and a global before-default one, the solution to the latter depending on the former. This makes the resolution of the optimization problem more explicit and provides a detailed description of the optimal trading strategy, emphasizing the impact of default time and loss or gain given default. The density approach can be used for studying other optimal portfolio problems, like mean-variance hedging or pricing by utility indifference, with counterparty risk. A further important topic is the optimal investment problem with two assets (names) exposed both to bilateral counterparty risk, and the conditional density approach should be relevant for such a study planned for future research.

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