Conditional Default Probability and Density

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Abstract This paper proposes different methods to construct conditional survival processes, i.e, families of martingales decreasing with respect to a parameter. Conditional survival processes play a pivotal role in the density approach for default risk, introduced by El Karoui et al.[4]. Concrete examples will lead to the construction of dynamic copulae, in particular dynamic Gaussian copulae. It is shown that the change of probability measure methodology is a key tool for that construction. As in Kallianpur and Striebel [10], we apply this methodology in filtering theory to recover in a straightforward way, the classical results when the signal is a random variable.

1 Introduction

The goal of this paper is to give examples of the conditional law of a random variable (or a random vector), given a reference filtration, and methods to construct dynamics of conditional laws, in order to model price processes and/or default risk. This methodology appears in some recent papers (El Karoui et al. [4], Filipovic et

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al. [7]) and it is important to present techniques to build concrete examples. We have chosen to characterize the (conditional) law of a random variable through its (conditional) survival probability or through its (conditional) density, if it exists.

In Section 2, we give the definition of martingale survival processes and density processes. In Section 3, we give standard examples of conditional laws, in particular a Gaussian model, and we give methods to construct other ones. In Section 4, we show that, in the case of random times (i.e., non-negative random variables), the density methodology can be seen as an extension of the Cox model, and we recall a result which allows to construct default times having the same intensity and different conditional laws. We build the change of probability framework in Section 5 and show how it can be applied to filtering theory for computing the conditional law of the random variable which represents the signal.

2 Definitions

Let $(\Omega,\mathscr{A},\mathbf{F},P)$ be a filtered probability space, equipped with a filtration $\mathbf{F}=(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions, where $\mathscr{F}_\infty\subset\mathscr{A}$ and $\mathscr{F}_0$ is the trivial filtration. Let E be equal to one of the following spaces: $\mathbb{R}$, $\mathbb{R}^d$, $\mathbb{R}_+$, or $(\mathbb{R}_+)^d$.

A family of $(P,\mathbf{F})$-martingale survival processes on E is a family of $(P,\mathbf{F})$-martingales $G_\cdot(\theta)$, $\theta\in E$ such that $\theta\to G_t(\theta)$ is decreasing, and for any θ, $G_\cdot(\theta)$ is valued in $[0,1]$. We have used the standard convention for maps from $\mathbb{R}^d$ to $\mathbb{R}$: such a map G is decreasing if $\theta\leq\widetilde{\theta}$ implies $G(\theta)\geq G(\widetilde{\theta})$, where $\theta\leq\widetilde{\theta}$ means that $\forall i=1,\dots,d$, $\theta_i\leq\widetilde{\theta}_i$.

A $(P,\mathbf{F})$ density process on E is a family $g_\cdot(\theta)$, $\theta\in E$ of non-negative, $(P,\mathbf{F})$-martingales such that

$$\int_E g_t(u)du=1,\ \forall t,\ a.s. \tag{1}$$

where du denotes the Lebesgue measure on E. If there is no ambiguity, we shall simply say a martingale survival process and a density process.

If G is a family of martingale survival processes on E, absolutely continuous w.r.t the Lebesgue measure, i.e., $G_t(\theta)=\int_\theta^\infty g_t(u)du$, the family g is a density process (see Jacod [9] for important regularity conditions).

The martingale survival process of an $\mathscr{A}$-measurable random variable X valued in $\mathbb{R}^d$ is the family of càdlàg processes $G_t(\theta)=P(X>\theta|\mathscr{F}_t)$. Obviously, this is a martingale survival process (it is decreasing w.r.t. θ). In particular, assuming regularity conditions, the non-negative function g_0 such that $G_0(\theta)=\int_\theta^\infty g_0(s)ds$ is the probability density of X.

If we are given a family of density processes $g_\cdot(\theta)$, then there exists a random variable X (constructed on an extended probability space) such that

$$P(X>\theta|\mathscr{F}_t)=G_t(\theta)=\int_\theta^\infty g_t(u)du,\quad a.s.$$

where (with an abuse of notation) P is a probability measure on the extended space, which coincides with the given probability measure on $\mathbf{F}$. For the construction, one starts with a random variable X on $\Omega \times I\!R$ independent of $\mathbf{F}$, with probability density g_0 and one checks that $(g_t(X), t \geq 0)$ is an $\mathbf{F} \vee \sigma(X)$-martingale. Then, setting $dQ|_{\mathscr{F}_t \vee \sigma(X)} = \frac{g_t(X)}{g_0(X)} dP|_{\mathscr{F}_t \vee \sigma(X)}$, one obtains, from Bayes' formula that $Q(X > \theta | \mathscr{F}_t) = G_t(\theta)$. This construction was important in Grorud and Pontier [8] and in Amendinger [1] in an initial enlargement of filtration framework for application to insider trading.

In the specific case of random times (non-negative random variables), one has to consider martingale survival processes defined on $I\!R_+$. They can be deduced from martingale survival processes on $I\!R$ by a simple change of variable: if G is the martingale survival process on $I\!R$ of the real valued random variable X and h a strictly increasing function from $I\!R_+$ to $I\!R$, then $G_t^h(u) := G_t(h(u))$ defines a martingale survival process on $I\!R_+$ (corresponding to the change of variable $Y = h^{-1}(X)$). In the case where h is differentiable, the density process is $g^h(u) = g_t(h(u))h'(u)$.

It is important to note that, due to the martingale property, in order to characterize the family $g_t(\theta)$ for any pair $(t,\theta) \in (I\!R_+ \times I\!R)$, it suffices to know this family for any pair (t,θ) such that $\theta \leq t$. Hence, in what follows, we shall concentrate on construction for $\theta \leq t$.

In the paper, the natural filtration of a process Y is denoted by $\mathbf{F}^Y$.

3 Examples of Martingale Survival Processes

We first present two specific examples of conditional law of an $\mathscr{F}^B_\infty$-measurable random variable, when $\mathbf{F}^B$ is the natural filtration of a Brownian motion B. Then we give two large classes of examples, based on Markov processes and diffusion processes.

The first example, despite its simplicity, will allow us to construct a dynamic copula, in a Gaussian framework; more precisely, we construct, for any t, the (conditional) copula of a family of random times $P(\tau_i > t_i, i = 1, \dots, n | \mathscr{F}_t)$ and we can chose the parameters so that $P(\tau_i > t_i, i = 1, \dots, n)$ equals a given (static) Gaussian copula. To the best of our knowledge, there are very few explicit constructions of such a model.

In Fermanian and Vigneron [5], the authors apply a copula methodology, using a factor Y. However, the processes they use to fit the conditional probabilities $P(\tau_i > t_i, i = 1, \dots, n | \mathscr{F}_t \vee \sigma(Y))$ are not martingales. They show that, using some adequate parametrization, they can produce a model so that $P(\tau_i > t_i, i = 1, \dots, n | \mathscr{F}_t)$ are martingales. Our model will satisfy both martingale conditions.

In [2], Carmona is interested in the dynamics of prices of assets corresponding to a payoff which is a Bernoulli random variable (taking values 0 or 1), in other words, he is looking for examples of dynamics of martingales valued in $[0,1]$, with a given terminal condition. Surprisingly, the example he provides corresponds to the one

we give below in Section , up to a particular choice of the parameters to satisfy the terminal constraint.
In a second example, we construct another dynamic copula, again in an explicit way, with a more complicated dependence.

We then show that a class of examples can be obtained from a Markov model, where the decreasing property is introduced via a change of variable. In the second class of examples, the decreasing property is modeled via the dependence of a diffusion through its initial condition. To close the loop, we show that we can recover the Gaussian model of the first example within this framework.

3.1 A dynamic Gaussian copula model

In this subsection, φ is the standard Gaussian probability density, and Φ the Gaussian cumulative function.

We consider the random variable $X := \int_0^\infty f(s)dB_s$ where f is a deterministic, square-integrable function. For any real number θ and any positive t, one has

$$P(X > \theta|\mathscr{F}_t^B) = P\Big(m_t > \theta - \int_t^\infty f(s)dB_s|\mathscr{F}_t^B\Big)$$

where $m_t = \int_0^t f(s)dB_s$ is $\mathscr{F}_t^B$-measurable. The random variable $\int_t^\infty f(s)dB_s$ follows a centered Gaussian law with variance $\sigma^2(t) = \int_t^\infty f^2(s)ds$ and is independent of $\mathscr{F}_t^B$. Assuming that $\sigma(t)$ does not vanish, one has

$$P(X > \theta|\mathscr{F}_t^B) = \Phi\Big(\frac{m_t - \theta}{\sigma(t)}\Big). \tag{2}$$

In other words, the conditional law of X given $\mathscr{F}_t^B$ is a Gaussian law with mean m_t and variance $\sigma^2(t)$. We summarize the result[1] in the following proposition, and we give the dynamics of the martingale survival process, obtained with a standard use of Itô's rule.

Proposition 1. *Let B be a Brownian motion, f an L^2 deterministic function, $m_t = \int_0^t f(s)dB_s$ and $\sigma^2(t) = \int_t^\infty f^2(s)ds$. The family*

$$G_t(\theta) = \Phi\Big(\frac{m_t - \theta}{\sigma(t)}\Big)$$

is a family of $\mathbf{F}^B$-martingales, valued in $[0,1]$, which is decreasing w.r.t. θ. Moreover

$$dG_t(\theta) = \varphi\Big(\frac{m_t - \theta}{\sigma(t)}\Big)\frac{f(t)}{\sigma(t)}dB_t\,.$$

[1] More results on that model, in an enlargement of filtration setting, can be found in Chaleyat-Maurel and Jeulin in [3] and Yor [17].

The dynamics of the martingale survival process can be written

$$dG_t(\theta) = \varphi\left(\Phi^{-1}(G_t(\theta))\right)\frac{f(t)}{\sigma(t)}\,dB_t\,. \tag{3}$$

We obtain the associated density family by differentiating $G_t(\theta)$ w.r.t. θ,

$$g_t(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma(t)}\exp\Big(-\frac{(m_t-\theta)^2}{2\sigma^2(t)}\Big)$$

and its dynamics

$$dg_t(\theta) = -g_t(\theta)\frac{m_t-\theta}{\sigma^2(t)}f(t)dB_t. \tag{4}$$

Let us emphasize that, starting from (3), it is not obvious to check that the solution is decreasing with respect to the parameter θ, or, as it is done in [5] and [2], to find the solution. In the same way, the solution of (4) with initial condition a probability density g_0, is a density process if and only if $\int_{-\infty}^{\infty} g_t(u)du = 1$, or equivalently, $\int_{-\infty}^{\infty} g_t(\theta)\frac{m_t-\theta}{\sigma^2(t)}f(t)d\theta = 0$. This last equality reduces to

$$\int_{-\infty}^{\infty} g_t(\theta)(m_t-\theta)d\theta = m_t - \int_{-\infty}^{\infty} g_t(\theta)\theta d\theta = 0$$

and we do not see how to check this equality if one does not know the explicit solution.

In order to provide conditional survival probabilities for positive random variables, we consider $\widetilde{X} = \psi(X)$ where ψ is a differentiable, positive and strictly increasing function and let $h = \psi^{-1}$. The conditional law of $\widetilde{X}$ is

$$\tilde{G}_t(\theta) = \Phi\Big(\frac{m_t - h(\theta)}{\sigma(t)}\Big).$$

We obtain

$$\widetilde{g}_t(\theta) = \frac{1}{\sqrt{2\pi}\sigma(t)}h'(\theta)\exp\Big(-\frac{(m_t-h(\theta))^2}{2\sigma^2(t)}\Big)$$

and

$$d\widetilde{G}_t(\theta) = \varphi\Big(\frac{m_t-h(\theta)}{\sigma(t)}\Big)\frac{f(t)}{\sigma(t)}dB_t\,,$$

$$d\widetilde{g}_t(\theta) = -\widetilde{g}_t(\theta)\frac{m_t-h(\theta)}{\sigma(t)}\frac{f(t)}{\sigma(t)}dB_t.$$

Introducing an n-dimensional standard Brownian motion $B = (B^i, i = 1,\dots,n)$ and a factor Y, independent of $\mathbf{F}^B$, gives a dynamic copula approach, as we present now. For h_i an increasing function, mapping $\mathbb{R}^+$ into $\mathbb{R}$, and setting $\tau_i =$

$(h_i)^{-1}(\sqrt{1-\rho_i^2}\int_0^\infty f_i(s)dB^i_s+\rho_i Y)$, for $\rho_i\in(-1,1)$, an immediate extension of the Gaussian model leads to

$$P(\tau_i>t_i,\forall i=1,\dots,n|\mathscr{F}^B_t\vee\sigma(Y))=\prod_{i=1}^n\Phi\left(\frac{1}{\sigma_i(t)}\left(m^i_t-\frac{h_i(t_i)-\rho_i Y}{\sqrt{1-\rho_i^2}}\right)\right)$$

where $m^i_t=\int_0^t f_i(s)dB^i_s$ and $\sigma_i^2(t)=\int_t^\infty f_i^2(s)ds$. It follows that

$$P(\tau_i>t_i,\forall i=1,\dots,n|\mathscr{F}^B_t)=\int_{-\infty}^{\infty}\prod_{i=1}^n\Phi\left(\frac{1}{\sigma_i(t)}\left(m^i_t-\frac{h_i(t_i)-\rho_i y}{\sqrt{1-\rho_i^2}}\right)\right)f_Y(y)dy\,.$$

Note that, in that setting, the random times $(\tau_i, i=1,\dots,n)$ are conditionally independent given $\mathbf{F}^B\vee\sigma(Y)$, a useful property which is not satisfied in Fermanian and Vigneron model. For $t=0$, choosing f_i so that $\sigma_i(0)=1$, and Y with a standard Gaussian law, we obtain

$$P(\tau_i>t_i,\forall i=1,\dots,n)=\int_{-\infty}^{\infty}\prod_{i=1}^n\Phi\left(-\frac{h_i(t_i)-\rho_i y}{\sqrt{1-\rho_i^2}}\right)\varphi(y)dy$$

which corresponds, by construction, to the standard Gaussian copula ($h_i(\tau_i)=\sqrt{1-\rho_i^2}X_i+\rho_i Y$, where X_i,Y are independent standard Gaussian variables).

Relaxing the independence condition on the components of the process B leads to more sophisticated examples.

3.2 A Gamma model

Here, we present another model, where the processes involved are no more Gaussian ones. Consider $A^{(\mu)}_t:=\int_0^t e^{2B^{(\mu)}_s}ds$ where $B^{(\mu)}_t=B_t+\mu t$, μ being a positive constant. Matsumoto and Yor [15] have established that $A^{(-\mu)}_\infty=A^{(-\mu)}_t+e^{2B^{(-\mu)}_t}\widetilde{A}^{(-\mu)}_\infty$ where $\widetilde{A}^{(-\mu)}_\infty$ is independent of $\mathscr{F}^B_t$, with the same law as $A^{(-\mu)}_\infty$. The law of $A^{(-\mu)}_\infty$ is proved to be the law of $1/(2\gamma_\mu)$, γ_μ being a Gamma random variable with parameter μ. The survival probability of $A^{(-\mu)}_\infty$ is $\Upsilon(x)=\frac{1}{\Gamma(\mu)}\int_0^{1/(2x)}y^{\mu-1}e^{-y}dy$, where Γ is the Gamma function. Then, one obtains

$$G_t(\theta)=P(A^{(-\mu)}_\infty>\theta|\mathscr{F}^B_t)=\Upsilon\Big(\frac{\theta-A^{(-\mu)}_t}{e^{2B^{(-\mu)}_t}}\Big)1\!\!1_{\theta>A^{(-\mu)}_t}+1\!\!1_{\theta\le A^{(-\mu)}_t}.$$

This gives a family of martingale survival processes G, similar to (5), with gamma structure. It follows that, on $\{\theta>A^{(-\mu)}_t\}$

$$dG_t(\theta) = \frac{1}{2^{\mu-1}\Gamma(\mu)} e^{-\frac{1}{2}Z_t(\theta)} (Z_t(\theta))^{\mu} dB_t$$

where $Z_t(\theta) = \frac{e^{2B_t^{(-\mu)}}}{\theta - A_t^{(-\mu)}}$ (to have light notation, we do not specify that this process Z depends on μ). One can check that $G_t(\cdot)$ is differentiable w.r.t. θ, so that $G_t(\theta) = \int_\theta^\infty g_t(u)du$, where

$$g_t(\theta) = \mathbb{1}_{\theta > A_t^{(-\mu)}} \frac{1}{2^{\mu}\Gamma(\mu)} (Z_t(\theta))^{\mu+1} e^{-\frac{1}{2}Z_t(\theta) - 2B_t^{(-\mu)}}.$$

Again, introducing an n-dimensional Brownian motion, a factor Y and the r.vs $\alpha_i A_\infty^{(-\mu,i)} + \rho_i Y$, where α_i and ρ_i are constants, will give an example of a dynamic copula.

3.3 Markov processes

Let X be a real-valued Markov process with transition probability $p_T(t,x,y)dy = P(X_T \in dy | X_t = x)$, and Ψ a family of functions $\mathbb{R} \times \mathbb{R} \to [0,1]$, decreasing w.r.t. the second variable, such that

$$\Psi(x, -\infty) = 1, \Psi(x, \infty) = 0.$$

Then, for any T,

$$G_t(\theta) := E(\Psi(X_T, \theta) | \mathscr{F}_t^X) = \int_{-\infty}^{\infty} p_T(t, X_t, y) \Psi(y, \theta) dy$$

is a family of martingale survival processes on $\mathbb{R}$. While modeling $(T;x)$-bond prices, Filipovic et al. [6] have used this approach in an affine process framework. See also Keller-Ressel et al. [13].

Example 1. Let X be a Brownian motion, and $\Psi(x,\theta) = e^{-\theta x^2} \mathbf{1}_{\theta \geq 0} + \mathbf{1}_{\theta \leq 0}$. We obtain a martingale survival process on $\mathbb{R}_+$, defined for $\theta \geq 0$ and $t < T$ as,

$$G_t(\theta) = E\big[\exp(-\theta X_T^2) | \mathscr{F}_t^X\big] = \frac{1}{\sqrt{1+2(T-t)\theta}} \exp\big(-\frac{\theta X_t^2}{1+2(T-t)\theta}\big)$$

The construction given above provides a martingale survival process $G(\theta)$ on the time interval $[0,T]$. Using a (deterministic) change of time, one can easily deduce a martingale survival process on the whole interval $[0,\infty[$: setting

$$\hat{G}_t(\theta) = G_{h(t)}(\theta)$$

for a differentiable increasing function h from $[0,\infty]$ to $[0,T]$, and assuming that $dG_t(\theta) = G_t(\theta) K_t(\theta) dB_t, t < T$, one obtains

$$d\hat{G}_t(\theta) = \hat{G}_t(\theta) K_{h(t)}(\theta) \sqrt{h'(t)} dW_t$$

where W is a Brownian motion.
One can also randomize the terminal date and consider T as an exponential random variable independent of $\mathbf{F}$. Noting that the previous $G_t(\theta)$'s depend on T, one can write them as $G_t(\theta, T)$ and consider

$$\widetilde{G}_t(\theta) = \int_0^\infty G_t(\theta, z) e^{-z} dz$$

which is a martingale survival process. The same construction can be done with a random time T with any given density, independent of $\mathbf{F}$.

3.4 Diffusion-based model with initial value

Proposition 2. *Let Ψ be a cumulative distribution function of class C^2, and Y the solution of*

$$dY_t = a(t, Y_t)dt + \nu(t, Y_t)dB_t,\ Y_0 = y_0$$

where a and ν are deterministic functions smooth enough to ensure that the solution of the above SDE is unique. Then, the process $(\Psi(Y_t), t \geq 0)$ is a martingale, valued in $[0,1]$, if and only if

$$a(t,y)\Psi'(y) + \frac{1}{2}\nu^2(t,y)\Psi''(y) = 0\,. \tag{5}$$

Proof. The result follows by applying Itô's formula and noting that $\Psi(Y_t)$ being a (bounded) local martingale is a martingale.

We denote by $Y_t(y)$ the solution of the above SDE with initial condition $Y_0 = y$. Note that, from the uniqueness of the solution, $y \to Y_t(y)$ is increasing (i.e., $y_1 > y_2$ implies $Y_t(y_1) \geq Y_t(y_2)$). It follows that

$$G_t(\theta) := 1 - \Psi(Y_t(\theta))$$

is a family of martingale survival processes.

Example 2. Let us reduce our attention to the case where Ψ is the cumulative distribution function of a standard Gaussian variable. Using the fact that $\Phi''(y) = -y\Phi'(y)$, Equation (5) reduces to

$$a(t,y) - \frac{1}{2} y \nu^2(t,y) = 0$$

In the particular the case where $\nu(t,y) = \nu(t)$, straightforward computation leads to

$$Y_t(y) = e^{\frac{1}{2}\int_0^t \nu^2(s)ds}(y + \int_0^t e^{-\frac{1}{2}\int_0^s \nu^2(u)du}\nu(s)dB_s).$$

Setting $f(s) = -\nu(s)\exp(-\frac{1}{2}\int_0^s \nu^2(u)du)$, one deduces that $Y_t(y) = \frac{y-m_t}{\sigma(t)}$, where $\sigma^2(t) = \int_t^\infty f^2(s)ds$ and $m_t =: \int_0^t f(s)dB_s$, and we recover the Gaussian example of Subsection 3.1.

4 Density Models

In this section, we are interested in densities on $\mathbb{R}_+$ in order to give models for the conditional law of a random time τ. We recall the classical constructions of default times as first hitting time of a barrier, independent of the reference filtration, and we extend these constructions to the case where the barrier is no more independent of the reference filtration. It is then natural to characterize the dependence of this barrier and the filtration by means of its conditional law.

In the literature on credit risk modeling, the attention is mostly focused on the intensity process, i.e., to the process Λ such that $1\!\!1_{\tau\le t} - \Lambda_{t\wedge\tau}$ is a $\mathbf{G} = \mathbf{F}\vee\mathbf{H}$-martingale, where $\mathscr{H}_t = \sigma(t\wedge\tau)$. We recall that the intensity process Λ is the only increasing predictable process such that the survival process $G_t := P(\tau > t|\mathscr{F}_t)$ admits the decomposition $G_t = N_t e^{-\Lambda_t}$ where N is a local martingale. We recall that the intensity process can be recovered form the density process as $d\Lambda_s = \frac{g_s(s)}{G_s(s)}ds$ (see [4]). We end the section giving an explicit example of two different martingale survival processes having the same survival processes (hence the intensities are equal).

4.1 Structural and reduced-form models

In the literature, models for default times are often based on a threshold: the default occurs when some driving process X reaches a given barrier. Based on this observation, we consider the random time on $\mathbb{R}_+$ in a general threshold model. Let X be a stochastic process and Θ be a barrier which we shall precise later. Define the random time as the first passage time

$$\tau := \inf\{t : X_t \ge \Theta\}.$$

In classical structural models, the process X is an $\mathbf{F}$-adapted process associated with the value of a firm and the barrier Θ is a constant. So, τ is an $\mathbf{F}$-stopping time. In this case, the conditional distribution of τ does not have a density process, since $P(\tau > \theta|\mathscr{F}_t) = 1\!\!1_{\theta<\tau}$ for $\theta \le t$.

To obtain a density process, the model has to be changed, for example one can stipulate that the driving process X is not observable and that the observation is a filtration $\mathbf{F}$, smaller than the filtration $\mathbf{F}^X$, or a filtration including some noise. The goal is again to compute the conditional law of the default $P(\tau > \theta|\mathscr{F}_t)$, using for example filtering theory.

Another method is to consider a right-continuous $\mathbf{F}$-adapted increasing process Γ and to randomize the barrier. The easiest way is to take the barrier Θ as an $\mathscr{A}$-measurable random variable independent of $\mathbf{F}$, and to consider

$$\tau := \inf\{t : \Gamma_t \geq \Theta\}. \tag{6}$$

If Γ is continuous, τ is the inverse of Γ taken at Θ, and $\Gamma_\tau = \Theta$. The $\mathbf{F}$-conditional law of τ is

$$P(\tau > \theta|\mathscr{F}_t) = G^\Theta(\Gamma_\theta),\ \theta \leq t$$

where G^Θ is the survival probability of Θ given by $G^\Theta(t) = P(\Theta > t)$. We note that in this particular case, $P(\tau > \theta|\mathscr{F}_t) = P(\tau > \theta|\mathscr{F}_\infty)$ for any $\theta \leq t$, which means that the H-hypothesis is satisfied[2] and that the martingale survival processes remain constant after θ (i.e., $G_t(\theta) = G_\theta(\theta)$ for $t \geq \theta$). This result is stable by increasing transformation of the barrier, so that we can assume without loss of generality that the barrier is the standard exponential random variable $-\log(G^\Theta(\Theta))$.

If the increasing process Γ is assumed to be absolutely continuous w.r.t. the Lebesgue measure with Radon-Nikodým density γ and if G^Θ is differentiable, then the random time τ admits a density process given by

$$\begin{aligned} g_t(\theta) &= -(G^\Theta)'(\Gamma_\theta)\gamma_\theta = g_\theta(\theta),\ \theta \leq t \\ &= E(g_\theta(\theta)|\mathscr{F}_t),\ \theta > t. \end{aligned} \tag{7}$$

Example (Cox process model) In the widely used Cox process model, the independent barrier Θ follows the exponential law and $\Gamma_t = \int_0^t \gamma_s ds$ represents the default compensator process. As a direct consequence of (7),

$$g_t(\theta) = \gamma_\theta e^{-\Gamma_\theta},\ \theta \leq t.$$

4.2 Generalized threshold models

In this subsection, we relax the assumption that the threshold Θ is independent of $\mathscr{F}_\infty$. We assume that the barrier Θ is a strictly positive random variable whose conditional distribution w.r.t. $\mathbf{F}$ admits a density process, i.e., there exists a family of $\mathscr{F}_t \otimes \mathscr{B}(\mathbb{R}_+)$-measurable functions $p_t(u)$ such that

$$G_t^\Theta(\theta) := P(\Theta > \theta|\mathscr{F}_t) = \int_\theta^\infty p_t(u)du. \tag{8}$$

[2] We recall that H-hypothesis stands for any $\mathbf{F}$-martingale is a $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ martingale.

We assume in addition that the process Γ is absolutely continuous w.r.t. the Lebesgue measure, i.e., $\Gamma_t = \int_0^t \gamma_s ds$. We still consider τ defined as in (6) by $\tau = \Gamma^{-1}(\Theta)$ and we say that a random time constructed in such a setting is given by *a generalized threshold*.

Proposition 3. *Let τ be given by a generalized threshold. Then τ admits the density process $g(\theta)$ where*

$$g_t(\theta) = \gamma_\theta p_t(\Gamma_\theta),\ \theta \le t. \tag{9}$$

Proof. By definition and by the fact that Γ is strictly increasing and absolutely continuous, we have for $t \ge \theta$,

$$G_t(\theta) := P(\tau > \theta|\mathscr{F}_t) = P(\Theta > \Gamma_\theta|\mathscr{F}_t) = G_t^\Theta(\Gamma_\theta) = \int_{\Gamma_\theta}^\infty p_t(u)du = \int_\theta^\infty p_t(\Gamma_u)\gamma_u du,$$

which implies $g_t(\theta) = \gamma_\theta p_t(\Gamma_\theta)$ for $t \ge \theta$.

Obviously, in the particular case where the threshold Θ is independent of $\mathscr{F}_\infty$, we recover the classical results (7) recalled above.

Conversely, if we are given a density process g, then it is possible to construct a random time τ by a generalized threshold, that is, to find Θ such that the associated τ has g as density, as we show now. It suffices to define $\tau = \inf\{t\ :\ t \ge \Theta\}$ where Θ is a random variable with conditional density $p_t = g_t$. Of course, for any increasing process Γ, $\tau = \inf\{t : \Gamma_t \ge \Delta\}$ where $\Delta := \Gamma_\Theta$ is a different way to obtain a solution!

4.3 An example with same survival processes

We recall that, starting with a survival martingale process $\widetilde{G}_t(\theta)$, one can construct other survival martingale processes $G_t(\theta)$ admitting the same survival process (i.e., $\widetilde{G}_t(t) = G_t(t)$), in particular the same intensity. The construction is based on the general result obtained in Jeanblanc and Song [11]: for any supermartingale Z valued in $[0,1[$, with multiplicative decomposition $Ne^{-\Lambda}$, where Λ is continuous, the family

$$G_t(\theta) = 1 - (1 - Z_t)\exp\left(-\int_\theta^t \frac{Z_s}{1 - Z_s} d\Lambda_s\right)\ \ 0 < \theta \le t \le \infty,$$

is a martingale survival process (called the basic martingale survival process) which satisfies $G_t(t) = Z_t$ and, if N is continuous, $dG_t(\theta) = \frac{1-G_t(\theta)}{1-Z_t} e^{-\Lambda_t} dN_t$. In particular, the associated intensity process is Λ (we emphasize that the intensity process does not contain enough information about the conditional law).

We illustrate this construction in the Gaussian example presented in Section 3.1 where we set $Y_t = \frac{m_t - h(t)}{\sigma(t)}$. The multiplicative decomposition of the supermartingale $\widetilde{G}_t = P(\tau > t|\mathscr{F}_t^B)$ is $\widetilde{G}_t = N_t \exp\left(-\int_0^t \lambda_s ds\right)$ where

$$dN_t = N_t \frac{\varphi(Y_t)}{\sigma(t)\Phi(Y_t)} dm_t, \quad \lambda_t = \frac{h'(t)\,\varphi(Y_t)}{\sigma(t)\,\Phi(Y_t)}.$$

Using the fact that $\widetilde{G}_t(t) = \Phi(Y_t)$, one checks that the basic martingale survival process satisfies

$$dG_t(\theta) = (1 - G_t(\theta)) \frac{f(t)\varphi(Y_t)}{\sigma(t)\Phi(-Y_t)} dB_t, t \geq \theta, \quad G_\theta(\theta) = \Phi(Y_\theta)$$

which provides a new example of martingale survival processes, with density process

$$g_t(\theta) = (1 - G_t) e^{-\int_\theta^t \frac{G_s}{1-G_s}\lambda_s ds} \frac{G_\theta \lambda_\theta}{1 - G_\theta}, \theta \leq t.$$

Other constructions of martingale survival processes having a given survival process can be found in [12], as well as constructions of local-martingales N such that $Ne^{-\Lambda}$ is valued in $[0,1]$ for a given increasing continuous process Λ.

5 Change of Probability Measure and Filtering

In this section, our goal is to show how, using a change of probability measure, one can construct density processes. The main idea is that, starting from the (unconditional) law of τ, we construct a conditional density in a dynamic way using a change of probability. This methodology is a very particular case of the general change of measure approach developed in [4]. Then, we apply the idea of change of probability framework to a filtering problem (due to Kallianpur and Striebel [10]), to obtain the Kallianpur-Striebel formula for the conditional density (see also Meyer [16]). Our results are established in a very simple way, in a general filtering model, when the signal is a random variable, and contain, in the simple case, the results of Filipovic et al. [7]. We end the section with the example of the traditional Gaussian filtering problem.

5.1 Change of measure

One starts with the elementary model where, on the filtered probability space $(\Omega, \mathscr{A}, \mathbf{F}, P)$, an $\mathscr{A}$-measurable random variable X is independent from the reference filtration $\mathbf{F} = (\mathscr{F}_t)_{t \geq 0}$ and its law admits a density probability g_0, so that

$$P(X > \theta | \mathscr{F}_t) = P(X > \theta) = \int_\theta^\infty g_0(u) du.$$

We denote by $\mathbf{G}^X = \mathbf{F} \vee \sigma(X)$ the filtration generated by $\mathbf{F}$ and X.
Let $(\beta_t(u), t \in \mathbb{R}_+)$ be a family of positive $(P, \mathbf{F})$-martingales such that $\beta_0(u) = 1$ for all $u \in \mathbb{R}$. Note that, due to the assumed independence of X and $\mathbf{F}$, the process $(\beta_t(X), t \geq 0)$ is a $\mathbf{G}^X$-martingale and one can define a probability measure Q on $(\Omega, \mathscr{G}_t^X)$, by $dQ = \beta_t(X)dP$. Since $\mathbf{F}$ is a subfiltration of $\mathbf{G}^X$, the positive $\mathbf{F}$-martingale

$$m_t^\beta := E(\beta_t(X)|\mathscr{F}_t) = \int_0^\infty \beta_t(u) g_0(u) du$$

is the Radon-Nikodým density of the measure Q, restricted to $\mathscr{F}_t$ with respect to P (note that $m_0^\beta = 1$). Moreover, the Q-conditional density of X with respect to $\mathscr{F}_t$ can be computed, from the Bayes' formula

$$Q(X \in B|\mathscr{F}_t) = \frac{1}{E(\beta_t(X)|\mathscr{F}_t)} E(\mathbf{1}_B(X)\beta_t(X)|\mathscr{F}_t) = \frac{1}{m_t^\beta} \int_B \beta_t(u) g_0(u) du$$

where we have used, in the last equality the independence between X and $\mathbf{F}$, under P. Let us summarize this simple but important result:

Proposition 4. *If X is a r.v. with probability density g_0, independent from $\boldsymbol{F}$ under P, and if Q is a probability measure, equivalent to P on $\boldsymbol{F} \vee \sigma(X)$ with Radon-Nikodým density $\beta_t(X), t \geq 0$, then the $(Q, \boldsymbol{F})$ density process of X is*

$$g_t^Q(u)du := Q(X \in du|\mathscr{F}_t) = \frac{1}{m_t^\beta} \beta_t(u) g_0(u) du \tag{10}$$

where m^β is the normalizing factor $m_t^\beta = \int_{-\infty}^\infty \beta_t(u) g_0(u) du$. In particular

$$Q(\tau \in du) = P(\tau \in du) = g_0(u) du\,.$$

The right-hand side of (10) can be understood as the ratio of $\beta_t(u)g_0(u)$ (the change of probability times the P probability density) and a normalizing coefficient m_t^β. One can say that $(\beta_t(u)g_0(u), t \geq 0)$ is the un-normalized density, obtained by a linear transformation from the initial density. The normalization factor $m_t^\beta = \int \beta_t(u) g_0(u) du$ introduces a nonlinear dependance of $g_t^Q(u)$ with respect to the initial density. The example of the filtering theory provides an explicit form to this dependence when the martingales $\beta_t(u)$ are stochastic integrals with respect to a Brownian motion.

Remark 1. We present here some important remarks.
(1) If, for any t, $m_t^\beta = 1$, then the probability measures P and Q coincide on $\mathbf{F}$. In that case, the process $(\beta_t(u)g_0(u), t \geq 0)$ is a density process.
(2) Let $\mathbf{G} = (\mathscr{G}_t)_{t \geq 0}$ be the usual right-continuous and complete filtration in the default framework (i.e. when $X = \tau$ is a non negative r.v.) generated by $\mathscr{F}_t \vee \sigma(\tau \wedge t)$. Similar calculation may be made with respect to $\mathscr{G}_t$. The only difference is that the conditional distribution of τ is a Dirac mass on the set $\{t \geq \tau\}$. On the set $\{\tau > t\}$, and under Q, the distribution of τ admits a density given by:

$$Q(\tau \in du|\mathscr{G}_t) = \beta_t(u)g_0(u)\frac{1}{\int_t^\infty \beta_t(\theta)g_0(\theta)d\theta}du\,.$$

(3) This methodology can be easily extended to a multivariate setting: one starts with an elementary model, where the $\tau_i, i = 1,\dots,d$ are independent from $\mathbf{F}$, with joint density $g(u_1,\dots,u_d)$. With a family of non-negative martingales $\beta(\theta_1,\dots,\theta_d)$, the associated change of probability provides a multidimensional density process.

5.2 Filtering theory

The change of probability approach presented in the previous subsection 5.1 is based on the idea that one can restrict our attention to the simple case where the random variable is independent from the filtration and use a change of probability. The same idea is the building block of filtering theory as we present now.

Let W be a Brownian motion on the probability space $(\Omega,\mathscr{A},P)$, and X be a random variable independent of W, with probability density g_0. We denote by

$$dY_t = a(t,Y_t,X)dt + b(t,Y_t)dW_t \tag{11}$$

the observation process, where a and b are smooth enough to have a solution and where b does not vanish. The goal is to compute the conditional density of X with respect to the filtration $\mathbf{F}^Y$. The way we shall solve the problem is to construct a probability Q, equivalent to P, such that, under Q, the signal X and the observation $\mathbf{F}^Y$ are independent, and to compute the density of X under P by means of the change of probability approach of the previous section. It is known in nonlinear filtering theory as the Kallianpur-Striebel methodology [10], a way to linearize the problem.
Note that, from the independence assumption between X and W, we see that W is a $\mathbf{G}^X = \mathbf{F}^W \vee \sigma(X)$-martingale under P.

5.2.1 Simple case

We start with the simple case where the dynamics of the observation is

$$dY_t = a(t,X)dt + dW_t\,.$$

We assume that a is smooth enough so that the solution of

$$d\beta_t(X) = -\beta_t(X)a(t,X)dW_t,\, \beta_0(X) = 1$$

is a $(P,\mathbf{G}^X)$-martingale, and we define a probability measure Q on $\mathscr{G}_t^X$ by $dQ = \beta_t(X)dP$. Then, by Girsanov's theorem, the process Y is a $(Q,\mathbf{G}^X)$-Brownian motion, hence is independent from $\mathscr{G}_0^X = \sigma(X)$, under Q. Then, we apply our change of probability methodology, writing

$$dP = \frac{1}{\beta_t(X)} dQ =: \ell_t(X) dQ$$

with

$$d\ell_t(X) = \ell_t(X) a(t,X) dY_t,\ \ell_0(X) = 1$$

(in other words, $\ell_t(u) = \frac{1}{\beta_t(u)} = \exp\left(\int_0^t a(s,u)dY_s - \frac{1}{2}\int_0^t a^2(s,u)ds\right)$) and we get from Proposition 4 that the density of X under P, with respect to $\mathbf{F}^Y$, is $g_t(u)$, given by

$$P(X \in du | \mathscr{F}_t^Y) = g_t(u)du = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)du$$

where $m_t^\ell = E_Q(\ell_t(X)|\mathscr{F}_t^Y) = \int_{-\infty}^{\infty} \ell_t(u) g_0(u) du$. Since

$$dm_t^\ell = \left(\int_{-\infty}^{\infty} \ell_t(u) a(t,u) g_0(u) du\right) dY_t = m_t^\ell \left(\int_{-\infty}^{\infty} g_t(u) a(t,u) du\right) dY_t$$

and setting

$$\widehat{a}_t := E(a(t,X)|\mathscr{F}_t^Y) = \int_{-\infty}^{\infty} g_t(u) a(t,u) du\,,$$

Girsanov's theorem implies that the process B given by

$$dB_t = dY_t - \widehat{a}_t dt = dW_t + (a(t,X) - \widehat{a}_t)\, dt$$

is a $(P, \mathbf{F}^Y)$ Brownian motion (it is the innovation process). From Itô's calculus, it is easy to show that the density process satisfies the nonlinear filtering equation

$$\begin{aligned} dg_t(u) &= g_t(u) \left(a(t,u) - \frac{1}{m_t^\ell} \int_{-\infty}^{\infty} dy\, g_0(y) a(t,y) \ell_t(y)\right) dB_t \\ &= g_t(u)\, (a(t,u) - \widehat{a}_t)\, dB_t\,. \end{aligned} \tag{12}$$

Remark 2. Observe that conversely, given a solution $g_t(u)$ of (12), and the process μ solution of $d\mu_t = \mu_t \widehat{a}_t dY_t$, then $h_t(u) = \mu_t g_t(u)$ is solution of the linear equation $dh_t(u) = h_t(u) a(t,u) dY_t$.

5.2.2 General case

Using the same ideas, we now solve the filtering problem in the case where the observation follows (11). Let $\beta(X)$ be the $\mathbf{G}^X$ local martingale, solution of

$$d\beta_t(X) = \beta_t(X)\sigma_t(X) dW_t,\, \beta_0(X) = 1$$

with $\sigma_t(X) = -\frac{a(t,Y_t,X)}{b(t,Y_t)}$. We assume that a and b are smooth enough so that β is a martingale. Let Q be defined on $\mathscr{G}_t^X$ by $dQ = \beta_t(X) dP$.

From Girsanov's theorem, the process $\widehat{W}$ defined as

$$d\widehat{W}_t = dW_t - \sigma_t(X)dt = \frac{1}{b(t,Y_t)}dY_t$$

is a $(Q,\mathbf{G}^X)$-Brownian motion, hence $\widehat{W}$ is independent from $\mathscr{G}_0^X = \sigma(X)$. Being $\mathbf{F}^Y$-adapted, the process $\widehat{W}$ is a $(Q,\mathbf{F}^Y)$-Brownian motion, X is independent from $\mathbf{F}^Y$ under Q, and, as mentioned in Proposition 4, admits, under Q, the probability density g_0.
We now assume that the natural filtrations of Y and $\widehat{W}$ are the same. To do so, note that it is obvious that $\mathbf{F}^{\widehat{W}} \subseteq \mathbf{F}^Y$. If the SDE $dY_t = b(t,Y_t)d\widehat{W}_t$ has a strong solution (e.g., if b is Lipschitz, with linear growth) then $\mathbf{F}^Y \subseteq \mathbf{F}^{\widehat{W}}$ and the equality between the two filtrations holds.
Then, we apply our change of probability methodology, with $\mathbf{F}^Y$ as the reference filtration, writing $dP = \ell_t(X)dQ$ with $d\ell_t(X) = -\ell_t(X)\sigma_t(X)d\widehat{W}_t$ (which follows from $\ell_t(X) = \frac{1}{\beta_t(X)}$) and we get that the density of X under P, with respect to $\mathbf{F}^Y$ is $g_t(u)$ given by

$$g_t(u) = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)$$

with dynamics

$$\begin{aligned} dg_t(u) &= -g_t(u)\left(\sigma_t(u) - \frac{1}{m_t^\ell}\int_{-\infty}^{\infty} dy\, g_0(y)\sigma_t(y)\ell_t(y)\right)dB_t \\ &= g_t(u)\left(\frac{a(t,Y_t,u)}{b(t,Y_t)} - \frac{1}{b(t,Y_t)}\int_{-\infty}^{\infty} dy\, g_t(y)a(t,Y_t,y)\right)dB_t \\ &= g_t(u)\left(\frac{a(t,Y_t,u)}{b(t,Y_t)} - \frac{\widehat{a}_t}{b(t,Y_t)}\right)dB_t\,. \end{aligned} \tag{13}$$

Here B is a $(P,\mathbf{F}^Y)$ Brownian motion (the innovation process) given by

$$dB_t = dW_t + \left(\frac{a(t,Y_t,X)}{b(t,Y_t)} - \frac{\widehat{a}_t}{b(t,Y_t)}\right)dt\,,$$

where $\widehat{a}_t = E(a(t,Y_t,X)|\mathscr{F}_t^Y)$.

Proposition 5. *If the signal X has probability density $g_0(u)$ and is independent from the Brownian motion W, and if the observation process Y follows*

$$dY_t = a(t,Y_t,X)dt + b(t,Y_t)dW_t\,,$$

then, the conditional density of X given $\mathscr{F}_t^Y$ is

$$P(X \in du|\mathscr{F}_t^Y) = g_t(u)du = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)du \tag{14}$$

where $\ell_t(u) = \exp\left(\int_0^t \frac{a(s,Y_s,u)}{b^2(s,Y_s)} dY_s - \frac{1}{2}\int_0^t \frac{a^2(s,Y_s,u)}{b^2(s,Y_s)} ds\right)$, $m_t^\ell = \int_{-\infty}^{\infty} \ell_t(u) g_0(u) du$, *and its dynamics is given in (13).*

5.3 Gaussian filter

We apply our results to the well known case of Gaussian filter. Let W be a Brownian motion, X a random variable (the signal) with density probability g_0 a Gaussian law with mean m_0 and variance γ_0, independent of the Brownian motion W and let Y (the observation) be the solution of

$$dY_t = (a_0(t,Y_t) + a_1(t,Y_t)X)dt + b(t,Y_t)dW_t,$$

Then, from the previous results, the density process $g_t(u)$ is of the form

$$\frac{1}{m_t^\ell} \exp\left(\int_0^t \frac{a_0(s,Y_s) + a_1(s,Y_s)u}{b^2(s,Y_s)}\, dY_t - \frac{1}{2}\int_0^t \left(\frac{a_0(s,Y_s) + a_1(s,Y_s)u}{b(s,Y_s)}\right)^2 ds\right) g_0(u)$$

The logarithm of $g_t(u)$ is a quadratic form in u with stochastic coefficient, so that $g_t(u)$ is a Gaussian density, with mean m_t and variance γ_t (as proved already by Liptser and Shiryaev [14]). A tedious computation, purely algebraic, shows that

$$\gamma_t = \frac{\gamma_0}{1 + \gamma_0 \int_0^t \frac{a_1^2(s,Y_s)}{b^2(s,Y_s)} ds}, \quad m_t = m_0 + \int_0^t \gamma_s \frac{a_1(s,Y_s)}{b(s,Y_s)}\, dB_s$$

with $dB_t = dW_t + \frac{a_1(t,Y_t)}{b(t,Y_t)}(X - E(X|\mathscr{F}_t^Y))dt$.

Back to the Gaussian example Section 3.1: In the case where the coefficients of the process Y are deterministic functions of time, i.e.,

$$dY_t = (a_0(t) + a_1(t)X)dt + b(t)dW_t$$

the variance $\gamma(t)$ is deterministic and the mean is an $\mathbf{F}^Y$-Gaussian martingale

$$\gamma(t) = \frac{\gamma_0}{1 + \gamma_0 \int_0^t \alpha^2(s)ds}, \quad m_t = m_0 + \int_0^t \gamma(s)\alpha(s)dB_s$$

where $\alpha = a_1/b$. Furthermore, $\mathbf{F}^Y = \mathbf{F}^B$.
Choosing $f(s) = \frac{\gamma(s)a_1(s)}{b(s)}$ in the example of Section 3.1 leads to the same conditional law (with $m_0 = 0$); indeed, it is not difficult to check that this choice of parameter leads to $\int_t^\infty f^2(s)ds = \sigma^2(t) = \gamma(t)$ so that the two variances are equal.
The similarity between filtering and the example of Section 3.1 can be also explained as follows. Let us start from the setting of Section 3.1 where $X = \int_0^\infty f(s)dB_s$ and introduce $\mathbf{G}^X = \mathbf{F}^B \vee \sigma(X)$, where B is the given Brownian motion. Standard

results of enlargement of filtration (see Jacod [9]) show that

$$W_t := B_t + \int_0^t \frac{m_s - X}{\sigma^2(s)} f(s) ds$$

is a $\mathbf{G}^X$-BM, hence is a $\mathbf{G}^W$-BM independent of X. So, the example presented in Section 3.1 is equivalent to the following filtering problem: the signal is X a Gaussian variable, centered, with variance $\gamma(0) = \int_0^\infty f^2(s) ds$ and the observation

$$dY_t = f(t) X dt + \left(\int_t^\infty f^2(s) ds \right) dW_t = f(t) X dt + \sigma^2(t) dW_t \, .$$

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