

Density Approach in Modeling Successive Defaults*

Nicole El Karoui[†], Monique Jeanblanc[‡], and Ying Jiao[§]

Abstract. We apply the density framework developed in [N. El Karoui, M. Jeanblanc, and Y. Jiao, *Stochastic Process. Appl.*, 120 (2010), pp. 1011–1032] to the modeling of successive multiple defaults. Under the hypothesis of existence of the joint density of the ordered default times with respect to a reference filtration, we present general pricing results and establish links with the classical intensity approach; in particular, we emphasize the impact of default events at successive default times. Explicit models, constructed using the methods of change of probability measure or dynamic copula, are proposed.

Key words. default density approach, multiple defaults, contagion risks

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1. Introduction. In credit risk analysis, the dependence of default times is one of the most important issues for both portfolio credit derivatives as well as contagious credit risks management. In the literature, the modeling of multiple credit names is diversified in two directions by using the so-called bottom-up and top-down models. In the former approach, one starts with a model for the marginal distribution of each default time, and then the correlation between them is made precise. In these models, the individual default information is taken into account. Copula models are often used for the correlation structure (see Frey and McNeil [13] for a survey). When it concerns a large number of credit names, computations can become complicated in the bottom-up models. To overcome this difficulty, the top-down models are developed and adapted to the portfolio credit derivatives. This approach consists of describing directly the cumulative loss process and its intensity dynamics by using point processes; see, for example, Arnsdorff and Halperin [1], Cont and Minca [4], Filipović, Overbeck, and Schmidt [11], Giesecke, Goldberg, and Ding [14], Sidenius, Piterbarg, and Andersen [18]. In particular, the Hawkes process has been proposed recently to describe the default contagion such as in Errais, Giesecke, and Goldberg [10] and Dassios and Zhao [6], where the default intensity may depend on the default timings. Although marginal distributions of default times are relatively neglected in the top-down models, this approach provides efficiently tractable and dynamic models for credit portfolio analysis.

In this paper, we consider a family of ordered default times $\sigma_1, \dots, \sigma_n$ as in the top-down

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[†]LPMA, Université Pierre et Marie Curie - Paris 6 and CMAP, Ecole Polytechnique, Paris, France (elkaroui@cmapx.polytechnique.fr).

[‡]Laboratoire de Mathématiques et Modélisation d'Évry (LaMME), Université d'Évry-Val-d'Essonne, UMR CNRS 8071, Evry, France (monique.jeanblanc@univ-evry.fr).

[§]ISFA, Université Claude Bernard - Lyon 1, Lyon, France (ying.jiao@univ-lyon1.fr).

models. We study the credit dependence by using the density process instead of intensity. The motivation is twofold. First, the density approach, which highlights what happens after a default event, adapts naturally to the successive default scenarios $\{t < \sigma_1\}, \{\sigma_1 \leq t < \sigma_2\}, \dots, \{\sigma_n \leq t\}$. The method focuses on analyzing the impact of each default event on the remaining credit names and consists of generalizing in a recursive way the single default case in El Karoui, Jeanblanc, and Jiao [8]. Second, we are interested in the role of information. We distinguish between default-free information and default information, which have quite different natures, and study the role of these two types of information on the credit contagion phenomenon. In the context of multiple default times, the global market information is described by a filtration \mathbb{G} which is the enlargement of the default-free reference filtration \mathbb{F} by adding progressively all the default events. For purposes of pricing and risk management, it is relevant to consider the global market information \mathbb{G} . In the density approach, we suppose that the family of ordered default times $\sigma_1, \dots, \sigma_n$ admits a conditional density with respect to (w.r.t.) the reference filtration \mathbb{F} and deduce general computation results in the enlarged filtration \mathbb{G} . The results that we obtain show the interplay of the \mathbb{F} -density and the past default times, which present the roles of default-free and default information at the default contagion event.

In the literature and in practice, the intensity approach is widely used in credit risk modeling. In the intensity approach, one focuses on the intensity process of the cumulative loss, which is a jump process adapted to the global filtration \mathbb{G} . The contagion phenomenon is described by the size and the frequency of the jumps of the intensity process. In the density approach, the key term is the joint density process. On the one hand, it is adapted to the reference filtration \mathbb{F} , which is often supposed to be a Brownian filtration, so the density can be a continuous process. On the other hand, the joint density describes the dependence structure of the default family and is a stochastic process depending on multiparameters. The impact of each default, such as the contagious jump at default, can be calculated in an intrinsic form using the density process. Compared to the intensity models in the top-down approach, there exist links between density and intensity processes. The intensity process can be deduced completely from the density process. But the inverse is not true in general. Only part of the density can be obtained from the intensity process, which means that the density contains more information. It was shown in [8] that the extra information contained in the density is particularly useful for the after-default analysis. In the multidefault setting, we shall continue to investigate this observation. In particular, we are interested in the immersion property, which implies that any \mathbb{F} -martingale is a \mathbb{G} -martingale. Under this hypothesis, the density can be deduced from the intensity. We note in addition that the intensity process deduced from the density has a general form, which depends on the default timings and does not necessarily satisfy the Markovian property.

The top-down models are often applied to the pricing of credit portfolio derivatives with explicit intensity models proposed in the literature (see [2, 3, 4, 11, 14, 17]). In this paper, the main focus is to present the density framework for ordered multiple defaults and analyze the impact of defaults in a general setting. The implementation of large-sized credit derivative pricing is not directly addressed. However, we present several explicit density models and show through numerical illustrations how the survival probability and the price of credit-sensitive assets can be affected by successive default events.

The paper is organized as follows. We present the density framework for successive defaults

in section 2. Section 3 deals with the relationship of the density model with the cumulative loss process and the classical intensity models in the top-down approach. In section 4, we discuss the change of probability measure for the density. Section 5 is devoted to explicit modeling examples, such as the dynamic copula model. We present our conclusions in section 6.

2. Default density framework. In this section, we present the density framework in credit risk modeling. Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions and representing the “default-free” information. Consider a family of random times $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ taking values in \mathbb{R}_+^n and representing the default times of n firms on the financial market. As in the top-down models, we consider the cumulative loss process and hence are interested in the ordered set of default times. We denote the increasing-ordered permutation of $\boldsymbol{\tau}$ by $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$, such that¹

$$\sigma_1 < \sigma_2 < \dots < \sigma_n.$$

The idea is to work on the n successive sets $\{t < \sigma_1\}, \{\sigma_1 \leq t < \sigma_2\}, \dots, \{\sigma_n \leq t\}$ and to update the conditional laws at the arrival of a default: one starts with the filtration \mathbb{F} , and, at the first default time σ_1 , we enlarge \mathbb{F} with that new knowledge to obtain $\mathbb{G}^{(1)}$. Then we enlarge $\mathbb{G}^{(1)}$ with the knowledge of the second default σ_2 to obtain $\mathbb{G}^{(2)}$, and so on. This progressively arriving information flow will have an impact on the pricing and the risk management problems of the credit portfolio derivatives, which is the main focus of this paper.

2.1. Reminder on single default. In this subsection, we recall some results in the case of a single default (see [8, 9]). Let τ be a finite positive random time on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that there exists a family of nonnegative $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable functions $(\omega, u) \rightarrow \alpha_t(\omega, u)$ such that for any bounded Borel function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$,

$$(1) \quad \mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} f(u)\alpha_t(u)du \quad \forall t \geq 0, \quad \mathbb{P}\text{-a.s.}$$

The \mathbb{F} -conditional survival probability is given by² $\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_{\theta}^{\infty} \alpha_t(u)du$ for all $t, \theta \geq 0$. Roughly speaking, $\alpha_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t)$ and $\alpha(u)$ is the conditional density of τ given the filtration \mathbb{F} . In this approach, the study in the larger filtration can be done in two steps by using the conditional density: before the default, i.e., on the set $\{t < \tau\}$, and after the default, on the set $\{t \geq \tau\}$. The before-default study is classical for the pricing of credit derivatives. The contribution of the density approach is the after-default study, which allows one to analyze the impact of a default event.

Let $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$ be the smallest right-continuous and complete filtration which makes τ a stopping time. It gives the information about default occurrence. The global market information $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ is the smallest right-continuous and complete filtration such that \mathcal{G}_t

¹In the density framework, the probability that two defaults occur at the same time is null. See section 2.2 for more details.

²This equality is valid except on a negligible set which does not depend on t and θ . We often do not mention this important fact.

contains $\mathcal{F}_t \vee \mathcal{D}_t$. In what follows, we shall write, with an abuse of notation, $\mathbb{F} \vee \mathbb{D}$ for this filtration (note that in a general setting $\mathbb{F} \vee \mathbb{D}$ fails to be right-continuous).

The pricing problems are related to the computation of \mathbb{G} -conditional expectations. Consider a positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable random variable $Y_T(\cdot)$, T denoting the maturity; then, for any $t \leq T$,

$$(2) \quad \mathbb{E}[Y_T(\tau)|\mathcal{G}_t] = \mathbf{1}_{\{t < \tau\}} \frac{\mathbb{E}\left[\int_t^\infty Y_T(u)\alpha_T(u)du|\mathcal{F}_t\right]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} + \mathbf{1}_{\{\tau \leq t\}} \frac{\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t\right]}{\alpha_t(\theta)} \Bigg|_{\theta=\tau}, \quad \text{a.s.}$$

There are two parts in the above formula: the before-default one on the set $\{t < \tau\}$ and the after-default one on the set $\{t \geq \tau\}$. The default timing τ has an impact on the after-default formula, described by $\mathbb{E}\left[Y_T(\theta)\frac{\alpha_T(\theta)}{\alpha_t(\theta)}|\mathcal{F}_t\right]$ evaluated at $\theta = \tau$. The second term on the right-hand side of (2) can also be written as

$$\mathbf{1}_{\{\tau \leq t\}} \mathbb{E}[Y_T(\tau)|\mathcal{G}_t] = \mathbf{1}_{\{\tau \leq t\}} \mathbb{E}_{\mathbb{P}^\theta}[Y_T(\theta)|\mathcal{F}_t] \Big|_{\theta=\tau},$$

where \mathbb{P}^θ is the probability measure defined on \mathcal{F}_T by³

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = \frac{\alpha_T(\theta)}{\alpha_0(\theta)}.$$

So the impact of a default event after its occurrence time τ can be interpreted as a change of probability measure which depends on the density process.

Let μ_t denote the conditional law of τ given \mathcal{G}_t ; then we have

$$\mu_t(du) = \mathbf{1}_{\{t < \tau\}} \frac{\alpha_t(u)du}{\int_t^\infty \alpha_t(u)du} + \mathbf{1}_{\{t \geq \tau\}} \delta_\tau(du),$$

where δ is the Dirac measure. The conditional expectation (2) can be written as

$$\mathbb{E}[Y_T(\tau)|\mathcal{G}_t] = \int_{\mathbb{R}_+} \mathbb{E}\left[Y_T(u)\frac{\alpha_T(u)}{\alpha_t(u)}|\mathcal{F}_t\right] \mu_t(du) = \int_{\mathbb{R}_+} \mathbb{E}_{\mathbb{P}^u}[Y_T(u)|\mathcal{F}_t] \mu_t(du).$$

The after-default analysis cannot be fully recovered by the classical intensity approach. In particular, the density $\alpha_t(\theta)$ with $t > \theta$, which plays an important role in the after-default formula, cannot be recovered from the intensity in general. This observation shows that the density approach is particularly adapted to the after-default study for the default impact and contagion. More precisely, recall that the \mathbb{G} -intensity of τ (if it exists) is the nonnegative \mathbb{G} -adapted process $\lambda^\mathbb{G}$ such that $(\mathbf{1}_{\{\tau \leq t\}} - \int_0^t \lambda_s^\mathbb{G} ds, t \geq 0)$ is a \mathbb{G} -martingale, and the \mathbb{F} -intensity of τ is the nonnegative \mathbb{F} -adapted process $\lambda^\mathbb{F}$ such that $(\mathbf{1}_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s^\mathbb{F} ds, t \geq 0)$ is a \mathbb{G} -martingale and therefore satisfies $\lambda_t^\mathbb{F} \mathbf{1}_{\{\tau > t\}} = \lambda_t^\mathbb{G}$. The intensity $\lambda^\mathbb{F}$ can be completely deduced from the density by

$$(3) \quad \lambda_t^\mathbb{F} = \frac{\alpha_t(t)}{\mathbb{P}(\tau > t|\mathcal{F}_t)} = \frac{\alpha_t(t)}{\int_t^\infty \alpha_t(u)du}, \quad t \geq 0.$$

³Note that $\alpha(\theta)$ is an \mathbb{F} -martingale.

From the intensity, using the martingale property of $\alpha(\theta)$, we obtain part of the density by

$$(4) \quad \alpha_t(\theta) = \mathbb{E}[\lambda_\theta^\mathbb{G} | \mathcal{F}_t] = \mathbb{E}[\lambda_\theta^\mathbb{F} \mathbf{1}_{\{\theta < \tau\}} | \mathcal{F}_t], \quad t \leq \theta.$$

In classical intensity models such as the Cox model, one often assumes that the immersion property holds between \mathbb{F} and \mathbb{G} , i.e., that \mathbb{F} -martingales remain \mathbb{G} -martingales. This condition, also called the H-hypothesis, is equivalent to $\mathbb{P}(\tau > t | \mathcal{F}_t) = \mathbb{P}(\tau > t | \mathcal{F}_\infty)$ for any $t \geq 0$ and also equivalent to the after-default density satisfying

$$(5) \quad \alpha_t(\theta) = \alpha_\theta(\theta), \quad t > \theta.$$

Hence, under the immersion hypothesis, we obtain the whole density for all positive t and θ from the intensity. Another important consequence of (5) is that the after-default conditional expectation in (2) becomes

$$\mathbb{E}[Y_T(\tau) | \mathcal{G}_t] \mathbf{1}_{\{\tau \leq t\}} = \mathbf{1}_{\{\tau \leq t\}} \mathbb{E}[Y_T(\theta) | \mathcal{F}_t]_{\theta=\tau},$$

which means that under the immersion hypothesis, only the default timing, and not the conditional law of τ , is taken into account in the after-default computation.

2.2. Conditional density for ordered defaults. We consider now the ordered defaults $\sigma_1, \dots, \sigma_n$ and extend the density framework to the multidefault setting. As we have explained previously, the default information is revealed progressively and associated to the successive default times. We first state the density hypothesis for the family of random times $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ w.r.t. the reference filtration \mathbb{F} , and then consider the increasingly enlarged filtrations containing default information.

2.2.1. Density w.r.t. the reference filtration. Recall that $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is a reference filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ representing the default-free information. We present the fundamental hypothesis on the existence of the \mathbb{F} -density of the ordered default times.

Hypothesis 2.1. *The conditional distribution of $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ given \mathbb{F} admits a density with respect to the Lebesgue measure,⁴ i.e., there exists a family of nonnegative $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$ -measurable functions $(\omega, \mathbf{u}) \rightarrow \alpha_t(\mathbf{u}, \omega)$, where $\mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}_+^n$, such that for any positive or bounded Borel function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$,*

$$(6) \quad \mathbb{E}[f(\boldsymbol{\sigma}) | \mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\mathbf{u}) \alpha_t(\mathbf{u}) d\mathbf{u} \quad \forall t \geq 0, \mathbb{P}\text{-a.s.},$$

where $d\mathbf{u}$ denotes $du_1 \cdots du_n$. We call the family $\alpha(\mathbf{u})$ the \mathbb{F} -conditional density (or simply the density when no ambiguity is possible) of $\boldsymbol{\sigma}$.

In this paper, we always assume this hypothesis. The density hypothesis implies that there are no simultaneous defaults; in other words, $\sigma_i \neq \sigma_j$, a.s., if $i \neq j$. By consequence, the sequence $\{\sigma_1, \dots, \sigma_n\}$ is almost surely strictly increasing and the density $\alpha(\mathbf{u})$ is null outside the set $\{u_1 < \dots < u_n\}$. For any fixed $\mathbf{u} \in \mathbb{R}_+^n$, the process $(\alpha_t(\mathbf{u}), t \geq 0)$ is an \mathbb{F} -martingale. The joint conditional survival probability is given, for any $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n$, by

$$\mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta} | \mathcal{F}_t) = \int_{\theta_1}^{\infty} du_1 \cdots \int_{\theta_n}^{\infty} du_n \alpha_t(\mathbf{u}) = \int_{\boldsymbol{\theta}}^{\infty} \alpha_t(\mathbf{u}) d\mathbf{u},$$

⁴This can be generalized to any nonnegative nonatomic measure, which is invariant by permutation.

where the notation $\boldsymbol{\sigma} > \boldsymbol{\theta}$ stands for $\sigma_i > \theta_i$ for all $i \in \{1, \dots, n\}$.

To study successively the default events, we are interested in the subfamily of the first k defaults in the portfolio $\boldsymbol{\sigma}^{(k)} := (\sigma_1, \dots, \sigma_k)$, where $k \leq n$. The marginal density $\alpha_t^{(k)}(\cdot)$ of $\boldsymbol{\sigma}^{(k)}$ w.r.t. \mathcal{F}_t is obtained from the joint density of $\boldsymbol{\sigma}$ as a partial integral by

$$(7) \quad \alpha_t^{(k)}(\mathbf{u}_{(k)}) = \int_{\mathbb{R}_+^{n-k}} \alpha_t(\mathbf{u}) d\mathbf{u}_{(k+1:n)},$$

where for any \mathbf{u} we use the notation $\mathbf{u}_{(k:p)}$ to denote the vector (u_k, \dots, u_p) and $\mathbf{u}_{(p)}$ for $\mathbf{u}_{(1:p)}$. Furthermore, $\mathbf{u}_{(i:i-1)}$ and $\mathbf{u}_{(0)}$ are null vectors.

2.2.2. Density w.r.t. the global filtration. For a single default, the global information is the progressive enlargement of the reference filtration \mathbb{F} by the default time. In the multi-default case, this global information contains the successive enlargements of filtrations by the ordered defaults.

The default information arrives progressively with successive default events. For any $i \in \{1, \dots, n\}$, we denote by $\mathbb{D}^i = (\mathcal{D}_t^i)_{t \geq 0}$ the filtration associated with σ_i , which corresponds to information concerning the i th ordered default. So the increasing filtrations $\mathbb{D}^{(i)} = (\mathcal{D}_t^{(i)})_{t \geq 0} := \mathbb{D}^1 \vee \dots \vee \mathbb{D}^i$ represent the cumulative information flow of the first i defaults, notably, whether the first i defaults have occurred, and the timings of the past default events. In other words, at each default time, we update the information by adding the σ -algebra generated by σ_i . We emphasize that $\mathbb{D}^{(i)}$ coincides with $\mathbb{D}^{(n)}$ stopped at the corresponding default, i.e., $\mathcal{D}_t^{(i)} = \mathcal{D}_{t \wedge \sigma_i}^{(n)}$, $t \geq 0$.

The global information $\mathbb{G}^{(n)} = (\mathcal{G}_t^{(n)})_{t \geq 0} := \mathbb{F} \vee \mathbb{D}^{(n)}$ includes both default and default-free information. Any $\mathcal{G}_t^{(n)}$ -measurable random variable X can be written in the form

$$X = \sum_{i=0}^n \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} X_t^i(\boldsymbol{\sigma}^{(i)}),$$

where $X_t^i(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^i)$ -measurable, with convention $\sigma_0 = 0$, $\sigma_{n+1} = \infty$, and $X_t^0(\sigma_{(0)})$ is \mathcal{F}_t -measurable.

If $\mu_t^{(n)}$ is the conditional law of the default times $\boldsymbol{\sigma}$ given the global information $\mathcal{G}_t^{(n)}$, then for any positive or bounded Borel function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$,

$$(8) \quad \mathbb{E}[f(\boldsymbol{\sigma}) | \mathcal{G}_t^{(n)}] = \int_{\mathbb{R}_+^n} f(\mathbf{u}) \mu_t^{(n)}(d\mathbf{u}).$$

By using the \mathbb{F} -density process of $\boldsymbol{\sigma}$, we obtain explicitly

$$(9) \quad \mu_t^{(n)}(d\mathbf{u}) = \sum_{i=0}^n \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \delta_{\boldsymbol{\sigma}^{(i)}}(d\mathbf{u}_{(i)}),$$

with δ denoting the Dirac measure and $\mathbf{t} = (t, \dots, t)$ the vector of $n - i$ copies of t . Observe that on each set $\{\sigma_i \leq t < \sigma_{i+1}\}$, the conditional law $\mu_t^{(n)}$ depends on the first i defaults $\boldsymbol{\sigma}^{(i)}$ which have already occurred.

We now present a general pricing result in the multidefault setting, where $Y_T(\boldsymbol{\sigma})$ is a payoff function of maturity T .

Proposition 2.2. *Let $Y_T(\cdot)$ be a positive $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$ -measurable function on $\Omega \times \mathbb{R}_+^n$; then for any $t \leq T$,*

$$(10) \quad \mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] = \int_{\mathbb{R}_+^n} \mathbb{E}\left[Y_T(\mathbf{u}) \frac{\alpha_T(\mathbf{u})}{\alpha_t(\mathbf{u})} \middle| \mathcal{F}_t\right] \mu_t^{(n)}(d\mathbf{u})$$

or, equivalently,

$$(11) \quad \mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] = \sum_{i=0}^n \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\int_t^\infty \mathbb{E}[Y_T(\mathbf{u}) \alpha_T(\mathbf{u}) | \mathcal{F}_t] d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \Bigg|_{\mathbf{u}_{(i)} = \boldsymbol{\sigma}_{(i)}}, \quad a.s.$$

Proof. We proceed in a recursive way. We introduce the intermediate filtrations $\mathbb{G}^{(i)} = (\mathcal{G}_t^{(i)})_{t \geq 0} := \mathbb{F} \vee \mathbb{D}^{(i)}$, $i \leq n$, and for convenience set $\mathbb{G}^{(0)} = \mathbb{F}$. Indeed, the formula (2) adapts naturally to the successive defaults; in particular, applying the before-default part of formula (2), on the set $\{\sigma_i \leq t < \sigma_{i+1}\}$, to the subfamily of remaining defaults $\boldsymbol{\sigma}_{(i+1:n)} = (\sigma_{i+1}, \dots, \sigma_n)$ and the corresponding filtration $\mathbb{G}^{(i)}$ (as \mathbb{F} in (2)) leads to

$$(12) \quad \begin{aligned} & \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] \\ &= \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\mathbb{E}\left[\int_t^\infty Y_T(\mathbf{u}) \alpha_T^{(i+1:n)|i}(\mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{G}_t^{(i)}\right]}{\mathbb{P}(\sigma_{i+1} > t | \mathcal{G}_t^{(i)})} \Bigg|_{\mathbf{u}_{(i)} = \boldsymbol{\sigma}_{(i)}}, \end{aligned}$$

where $\alpha_t^{(i+1:n)|i}$ denotes the $\mathbb{G}^{(i)}$ -density of $\boldsymbol{\sigma}_{(i+1:n)}$ and is given, on the set $\{\sigma_i \leq t < \sigma_{i+1}\}$, by

$$(13) \quad \alpha_t^{(i+1:n)|i}(\mathbf{u}_{(i+1:n)}) = \frac{\alpha_t(\boldsymbol{\sigma}_{(i)}, \mathbf{u}_{(i+1:n)})}{\alpha_t^{(i)}(\boldsymbol{\sigma}_{(i)})}.$$

We then use the after-default part of (2) to the default subfamily $\boldsymbol{\sigma}_{(i)}$ and the reference filtration \mathbb{F} to obtain, on the set $\{\sigma_i \leq t < \sigma_{i+1}\}$,

$$\begin{aligned} & \mathbb{E}\left[\int_t^\infty Y_T(\mathbf{u}) \alpha_T^{(i+1:n)|i}(\mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{G}_t^{(i)}\right] \Bigg|_{\mathbf{u}_{(i)} = \boldsymbol{\sigma}_{(i)}} \\ &= \frac{\mathbb{E}\left[\alpha_T^{(i)}(\mathbf{u}_{(i)}) \int_t^\infty Y_T(\mathbf{u}) \alpha_T^{(i+1:n)|i}(\mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_t\right]}{\alpha_t^{(i)}(\mathbf{u}_{(i)})} \Bigg|_{\mathbf{u}_{(i)} = \boldsymbol{\sigma}_{(i)}} \\ &= \frac{\mathbb{E}\left[\int_t^\infty Y_T(\mathbf{u}) \alpha_T(\mathbf{u}) d\mathbf{u}_{(i+1:n)} \middle| \mathcal{F}_t\right]}{\alpha_t^{(i)}(\mathbf{u}_{(i)})} \Bigg|_{\mathbf{u}_{(i)} = \boldsymbol{\sigma}_{(i)}}. \end{aligned}$$

Moreover, the denominator in (12) is

$$\mathbb{P}(\sigma_{i+1} > t | \mathcal{G}_t^{(i)}) = \mathbf{1}_{\{\sigma_i > t\}} + \mathbf{1}_{\{\sigma_i \leq t\}} \frac{\int_t^\infty \alpha_t(\boldsymbol{\sigma}_{(i)}, \mathbf{u}_{(i+1:n)}) d\mathbf{u}_{(i+1:n)}}{\alpha_t^{(i)}(\boldsymbol{\sigma}_{(i)})}.$$

This leads to (11). Finally, we obtain the equivalent equality (10) by using the conditional law $\mu_t^{(n)}$ given by (9). ■

Remark 2.3. If we introduce a change of probability measure by $\frac{d\mathbb{P}^{\mathbf{u}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{\alpha_T(\mathbf{u})}{\alpha_0(\mathbf{u})}$, then (10) can be written as $\mathbb{E}_{\mathbb{P}}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] = \int_{\mathbb{R}_+^n} \mathbb{E}_{\mathbb{P}^{\mathbf{u}}}[Y_T(\mathbf{u})|\mathcal{F}_t] \mu_t^{(n)}(d\mathbf{u})$. The Radon–Nikodým derivative of the change of probability plays an important role in the computation.

The default contagion effect is contained in our model through two aspects, as shown in the previous proposition. The first one is that there is a regime switching in the dynamics when each default occurs, which induces a contagious jump at default. The jump size depends on the joint density $\alpha(\mathbf{u})$ and is computed in an intrinsic form. The second one is that the conditional law of defaults given the global information depends explicitly on the past default timings. This last observation is interesting. In the literature, the intensity changes after the occurrence of default, taking into account the number of defaults, but in many models does not keep in memory the timing of each past default (as is the case in Arnsdorff and Halperin [1], Laurent, Cousin, and Fermanian [17], Herbertsson [15], etc.). This allows these models to be Markovian, which appears to be convenient in some cases. Nevertheless, our objective in this paper is to analyze the impact of past defaults so that we take into account the timing of defaults.

In particular, we consider the marginal law of the k th default. The conditional survival probability law of σ_k is given by

$$(14) \quad \begin{aligned} \mathbb{P}(\sigma_k > T | \mathcal{G}_t^{(n)}) &= \int_{\mathbb{R}_+^n} \mathbf{1}_{\{u_k > T\}} \mu_t^{(n)}(d\mathbf{u}) \\ &= \int_t^\infty \mathbf{1}_{\{u_k > T\}} \sum_{i=0}^{k-1} \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \delta_{\sigma^{(i)}}(d\mathbf{u}_{(i)}), \quad T \geq t. \end{aligned}$$

Note that $\mathbb{P}(\sigma_k > T | \mathcal{G}_t^{(n)}) = \mathbb{P}(\sigma_k > T | \mathcal{G}_t^{(k)})$.

More generally, we consider the joint law of the first k defaults $\boldsymbol{\sigma}^{(k)}$ in the portfolio. By the fact that $\mathcal{G}_{t \wedge \sigma_k}^{(k)} = \mathcal{G}_{t \wedge \sigma_k}^{(n)}$, the conditional laws of $\boldsymbol{\sigma}^{(k)}$, given the intermediary filtration $\mathcal{G}_t^{(k)}$ and the global filtration $\mathcal{G}_t^{(n)}$, coincide. So it suffices to study for $\boldsymbol{\sigma}^{(k)}$ its $\mathcal{G}_t^{(k)}$ -conditional law, which is denoted by $\mu_t^{(k)}$ and defined as $\mathbb{E}[f(\boldsymbol{\sigma}^{(k)}) | \mathcal{G}_t^{(k)}] = \int_{\mathbb{R}_+^k} f(\mathbf{u}_{(k)}) \mu_t^{(k)}(d\mathbf{u}_{(k)})$. We have, similar to (9), the following formula:

$$(15) \quad \mu_t^{(k)}(d\mathbf{u}_{(k)}) = \sum_{i=0}^{k-1} \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \frac{\alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}} \delta_{\sigma^{(i)}}(d\mathbf{u}_{(i)}) + \mathbf{1}_{\{\sigma_k \leq t\}} \delta_{\boldsymbol{\sigma}^{(k)}}(d\mathbf{u}_{(k)}).$$

3. Successive defaults and top-down models. In this section, we investigate the links between successive defaults and the loss-counting process in top-down models and, in particular, the links between the density of $\boldsymbol{\sigma}$ and the loss intensity in the multidefault setting.

3.1. Default information—ordered defaults and loss-counting process. Recall that τ is a family of random times and $\boldsymbol{\sigma}$ is the associated ordered sequence. The knowledge of the

default counting process is given by

$$N_t := \sum_{i=1}^n \mathbf{1}_{\{\tau_i \leq t\}},$$

which can also be given by the ordered default times and equals $N_t = \sum_{i=1}^n \mathbf{1}_{\{\sigma_i \leq t\}}$. At a given time $t \geq 0$, the information generated by the random variable N_t , i.e., $\sigma(N_t)$, gives the number of defaults up to time t ; and the filtration $\mathbb{D}^N = (\mathcal{D}_t^N)_{t \geq 0}$ generated by the counting process N , i.e., $\mathcal{D}_t^N = \sigma(N_s, s \leq t)$, gives the number of past defaults together with their timings. In addition, \mathbb{D}^N coincides with the information flow generated by the ordered default times, i.e., $\mathcal{D}_t^{(n)} = \mathcal{D}_t^N$, and satisfies $\mathcal{D}_{t \wedge \sigma_i}^N = \mathcal{D}_t^{(i)}$. Clearly, the global information $\mathbb{G}^N = (\mathcal{G}_t^N)_{t \geq 0}$ given as $\mathbb{F} \vee \mathbb{D}^N$ coincides with $\mathbb{G}^{(n)}$.

A useful observation gives the link between the ordered defaults and the counting process N : for any $k = 0, 1, \dots, n-1$ and any $t \geq 0$,

$$\{N_t \leq k\} = \{\sigma_{k+1} > t\}.$$

Using the marginal law of ordered defaults and the fact that \mathbb{G}^N and $\mathbb{G}^{(n)}$ coincide, we have the conditional loss distribution for $t \leq T$,

$$(16) \quad P_k(t, T) := \mathbb{P}(N_T \leq k | \mathcal{G}_t^N) = \mathbb{P}(\sigma_{k+1} > T | \mathcal{G}_t^N) = \int_{\mathbb{R}_+^{k+1}} \mathbf{1}_{\{u_{k+1} > T\}} \mu_t^{(k+1)}(d\mathbf{u}_{(k+1)}),$$

where $\mu_t^{(k+1)}$ is the $\mathcal{G}_t^{(k+1)}$ -conditional law of $\sigma_{(k+1)}$ given by (15). Obviously $\mathbb{P}(N_T = k | \mathcal{G}_t^N) = P_k(t, T) - P_{k-1}(t, T)$. In the literature, the counting process N is often supposed to be Markovian. This is not the case here. Note that the loss distribution (16) depends on the number of defaults and the timing of each default.

For a collateralized debt obligation (CDO) tranche pricing, a standard hypothesis on the market is the constant recovery rate R (equal to 40% in practice) for all defaults. So $L_T = (1 - R)N_T$. Then using the equality $(N_T - K)^+ = \int_K^\infty \mathbf{1}_{\{N_T > u\}} du$ together with (16), we can compute $\mathbb{E}[(N_T - K)^+ | \mathcal{G}_t^N]$, which is the key term for the CDO pricing.

3.2. Links with intensity approach. Most top-down models are based on the intensity approach. The loss intensity is the \mathbb{G}^N -adapted process λ^N such that $(N_t - \int_0^t \lambda_s^N ds, t \geq 0)$ is a \mathbb{G}^N -martingale. This quantity is often used in the top-down models to characterize the loss distribution. In the following, we show that the loss intensity, which is equal to the sum of all the ordered default intensities, can be computed from the density.

Recall that the $\mathbb{G}^{(i)}$ -intensity of σ_i is the positive $\mathbb{G}^{(i)}$ -adapted process λ^i such that $(M_t^i := \mathbf{1}_{\{\sigma_i \leq t\}} - \int_0^t \lambda_s^i ds, t \geq 0)$ is a $\mathbb{G}^{(i)}$ -martingale.⁵ It is well known that M^i is stopped at σ_i and that the $\mathbb{G}^{(i)}$ -intensity satisfies $\lambda_t^i = 0$ on $\{t \geq \sigma_i\}$. The following lemma shows that the $\mathbb{G}^{(i)}$ -intensity of σ_i coincides with its \mathbb{G}^N -intensity.

Lemma 3.1. *For $i = 1, \dots, n$, any $\mathbb{G}^{(i)}$ -martingale stopped at σ_i is a \mathbb{G}^N -martingale.*

⁵Note that it is different from the \mathbb{G}^i -intensity of σ_i .

Proof. We prove that any $\mathbb{G}^{(1)}$ -martingale stopped at σ_1 is a $\mathbb{G}^{(2)}$ -martingale. The result will follow. Let M be a $\mathbb{G}^{(1)}$ -martingale stopped at σ_1 , i.e., $M_t = M_{t \wedge \sigma_1}$ for any $t \geq 0$. For $s < t$,

$$\mathbb{E}[M_{t \wedge \sigma_1} | \mathcal{G}_s^{(2)}] = 1_{\{\sigma_2 \leq s\}} M_{\sigma_1} + 1_{\{\sigma_2 > s\}} \frac{\mathbb{E}[M_{t \wedge \sigma_1} 1_{\{\sigma_2 > s\}} | \mathcal{G}_s^{(1)}]}{\mathbb{P}(\sigma_2 > s | \mathcal{G}_s^{(1)})}.$$

Obviously, since $\sigma_1 < \sigma_2$, one has

$$\mathbb{E}[M_{t \wedge \sigma_1} 1_{\{\sigma_2 > s\}} | \mathcal{G}_s^{(1)}] = 1_{\{\sigma_1 > s\}} \mathbb{E}[M_{t \wedge \sigma_1} | \mathcal{G}_s^{(1)}] + 1_{\{\sigma_1 \leq s\}} \mathbb{E}[M_{\sigma_1} 1_{\{\sigma_2 > s\}} | \mathcal{G}_s^{(1)}].$$

The $\mathbb{G}^{(1)}$ -martingale property of M yields

$$1_{\{\sigma_1 > s\}} \mathbb{E}[M_{t \wedge \sigma_1} | \mathcal{G}_s^{(1)}] = 1_{\{\sigma_1 > s\}} M_{s \wedge \sigma_1}.$$

Moreover,

$$1_{\{\sigma_1 \leq s\}} \mathbb{E}[M_{\sigma_1} 1_{\{\sigma_2 > s\}} | \mathcal{G}_s^{(1)}] = 1_{\{\sigma_1 \leq s\}} M_{\sigma_1} \mathbb{P}(\sigma_2 > s | \mathcal{G}_s^{(1)}).$$

Hence $\mathbb{E}[M_{t \wedge \sigma_1} | \mathcal{G}_s^{(2)}] = M_{s \wedge \sigma_1}$. The result follows. \blacksquare

Proposition 3.2. *The loss intensity is equal to the sum of $\mathbb{G}^{(i)}$ -intensities of all ordered defaults σ_i , i.e., for any $t \geq 0$,*

$$(17) \quad \lambda_t^N = \sum_{i=1}^n \lambda_t^i, \quad a.s.,$$

where

$$(18) \quad \lambda_t^i = 1_{\{\sigma_{i-1} \leq t < \sigma_i\}} \frac{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i:n)}} \Big|_{\substack{\mathbf{u}_{(i-1)} = \boldsymbol{\sigma}_{(i-1)} \\ u_i = t}}, \quad a.s.$$

Remark 3.3. The intensity of σ_i is null outside the set $\{\sigma_{i-1} \leq t < \sigma_i\}$ and depends on the density $\alpha_t(\mathbf{u})$ with $t \leq u_i$ and on the past defaults $\boldsymbol{\sigma}_{(i-1)}$.

Proof. Since $(1_{\{\sigma_i \leq t\}} - \int_0^t \lambda_s^i ds, t \geq 0)$ is a $\mathbb{G}^{(i)}$ -martingale stopped at σ_i , it is, from Lemma 3.1, a \mathbb{G}^N -martingale; hence the sum $(N_t - \int_0^t \sum_{i=1}^n \lambda_s^i ds, t \geq 0)$ is a \mathbb{G}^N -martingale. It follows that $\lambda^N = \sum_{i=1}^n \lambda^i$. In order to prove (18), we use a recursive method with $\mathbb{G}^{(i)} = \mathbb{D}^i \vee \mathbb{G}^{(i-1)}$. As a consequence of (3),

$$\lambda_t^i = 1_{\{t < \sigma_i\}} \frac{\alpha_t^{i|i-1}(t)}{\mathbb{P}(\sigma_i > t | \mathcal{G}_t^{(i-1)})},$$

where

$$(19) \quad \mathbb{P}(\sigma_i > t | \mathcal{G}_t^{(i-1)}) = \int_t^\infty \alpha_t^{i|i-1}(u) du.$$

We have that the $\mathbb{G}^{(i-1)}$ -density of σ_i satisfies $1_{\{t < \sigma_{i-1}\}} \alpha_t^{i|i-1}(t) = 0$, and on the set $\{\sigma_{i-1} \leq t\}$,

$$\alpha_t^{i|i-1}(t) = \frac{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\alpha_t^{(i-1)}(\mathbf{u}_{(i-1)})} \Big|_{\substack{\mathbf{u}_{(i-1)} = \boldsymbol{\sigma}_{(i-1)} \\ u_i = t}}.$$

In addition

$$\mathbb{P}(\sigma_i > t | \mathcal{G}_t^{(i-1)}) = \mathbf{1}_{\{t < \sigma_{i-1}\}} + \mathbf{1}_{\{\sigma_{i-1} \leq t\}} \frac{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i:n)}}{\alpha_t^{(i-1)}(\mathbf{u}_{(i-1)})} \Big|_{\mathbf{u}_{(i-1)} = \sigma_{(i-1)}},$$

so the result follows. \blacksquare

By the above proposition, we note that there exists a family of processes $\lambda^{i, \mathbb{F}}$ such that

$$\lambda_t^{i, \mathbb{F}}(\mathbf{u}_{(i-1)}) = \frac{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i+1:n)}}{\int_t^\infty \alpha_t(\mathbf{u}) d\mathbf{u}_{(i:n)}} \Big|_{u_i=t}$$

is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^{i-1})$ -measurable and that the $\mathbb{G}^{(i)}$ -intensity of σ_i is given in the form

$$(20) \quad \lambda_t^i = \mathbf{1}_{\{\sigma_{i-1} \leq t < \sigma_i\}} \lambda_t^{i, \mathbb{F}}(\sigma_{(i-1)}).$$

Without loss of generality, we can suppose that $\lambda_t^{i, \mathbb{F}}(\mathbf{u}_{(i-1)}) = 0$ if $\mathbf{u}_{(i-1)}$ is outside the set $\{u_1 < \dots < u_{i-1}\}$ or if $t < u_{i-1}$ or $t > u_i$.

Explicit models of loss intensity have been proposed in the literature where λ^N is supposed to be a function of the loss N ; see, for example, Brigo, Pallavicini, and Torresetti [3], Filipović, Overbeck, and Schmidt [11], and Sidenius, Piterbarg, and Andersen [18]. In Frey and Backhaus [12], λ^N depends on an auxiliary Markov chain. In Arnsdorff and Halperin [1], it depends on some contagion factors (but not on the default timings in these models). In Errais, Giesecke, and Goldberg [10] and Dassios and Zhao [6], the loss intensity depends on the timing of defaults using Hawkes processes. In the density approach, the key term is the joint density process $\alpha(\mathbf{u})$; then the loss intensity process λ^N can be computed, and it will depend on the joint law of defaults and the default timings by Proposition 3.2.

3.3. Ordered defaults and immersion hypothesis. In this subsection, we are interested in the immersion property in the density framework for the ordered default times. For a family of ordered defaults, the immersion hypothesis holds between all the successive filtrations, that is, between $\mathbb{G}^{(i)}$ and $\mathbb{G}^{(i+1)}$ for all $i = 0, \dots, n-1$ if and only if \mathbb{F} is immersed in \mathbb{G}^N (see Ehlers and Schönbucher [7]). We now characterize this property by using the density.

Proposition 3.4. *The immersion property between \mathbb{F} and \mathbb{G}^N is equivalent to the following conditions: for any $i = 1, \dots, n$ and any $\mathbf{u} \in \mathbb{R}_+^n$, $u_1 < \dots < u_n$, the marginal density of $\sigma_{(i)}$ satisfies*

$$(21) \quad \alpha_t^{(i)}(\mathbf{u}_{(i)}) = \alpha_{u_i}^{(i)}(\mathbf{u}_{(i)}) \quad \forall t > u_i,$$

where $\alpha^{(i)}$ is given by (7).

Proof. The result holds for $n = 1$. We shall prove it for $n = 2$, and the general result will follow by recurrence. In fact, it's easy to verify that under the conditions (21), we have $\alpha_t^{1|0}(u) = \alpha_u^{1|0}(u)$ and $\alpha_t^{2|1}(u) = \alpha_u^{2|1}(u)$ for $t > u$ (the quantity $\alpha^{i+1|i}$ is defined in (19)), and these two equalities are equivalent to the immersion property between \mathbb{F} and $\mathbb{G}^{(1)}$ and between $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$, respectively, and hence to the immersion property between \mathbb{F} and $\mathbb{G}^{(2)}$. \blacksquare

The immersion property is often assumed in the intensity models in the literature. A standard method to construct a family of ordered random times satisfying the immersion

property is to generalize the Cox model in the top-down model setting. More precisely, let λ^i be a family of positive processes of the form (20) such that $\int_0^\infty \lambda_s^i ds = +\infty$; then using a recurrence procedure establishes that \mathbb{F} is immersed in \mathbb{G}^N if

$$(22) \quad \sigma_i = \inf \left\{ t \geq 0 : \int_{\sigma_{i-1}}^t \lambda_s^i ds \geq \eta_i \right\}, \quad i = 1, \dots, n,$$

where η_i is a positive random variable independent of $\mathbb{G}^{(i-1)}$, and hence of $\eta_1, \dots, \eta_{i-1}$. The case where (η_1, \dots, η_n) is a family of mutually independent unieponential random variables corresponds to the model described in [7].

The loss distribution $P_k(t, T)$, whose general form is given in (16), has a more familiar expression under immersion. The following result generalizes a well-known result in the single default case. Note that under immersion, the probability of having less than k defaults in the portfolio depends only on the intensity of σ_{k+1} or, equivalently, on the loss intensity restricted to the set $\{\sigma_k \leq t < \sigma_{k+1}\}$.

Proposition 3.5. *Under the immersion hypothesis between \mathbb{F} and \mathbb{G}^N , the conditional loss distribution is given by*

$$(23) \quad P_k(t, T) = \mathbf{1}_{\{t < \sigma_{k+1}\}} \mathbb{E} \left[\exp \left\{ - \int_t^T \lambda_s^{k+1, \mathbb{F}}(\boldsymbol{\sigma}^{(k)}) ds \right\} \middle| \mathcal{G}_t^{(k)} \right].$$

Proof. By the immersion hypothesis and a recursive argument,

$$P_k(t, T) = \mathbb{E}[\mathbf{1}_{\{\sigma_{k+1} > T\}} | \mathcal{G}_t^N] = \mathbb{E}[\mathbf{1}_{\{\sigma_{k+1} > T\}} | \mathcal{G}_t^{(k+1)}] = \mathbf{1}_{\{\sigma_{k+1} > t\}} \frac{\mathbb{E}[S_T^{k+1|k} | \mathcal{G}_t^{(k)}]}{S_t^{k+1|k}},$$

where $S_t^{k+1|k} = \mathbb{P}(\sigma_{k+1} > t | \mathcal{G}_t^{(k)})$. Since the immersion property holds between $\mathbb{G}^{(k)}$ and $\mathbb{G}^{(k+1)}$ by [7], applying a classical result in the single default case to σ_{k+1} , we have $S_t^{k+1|k} = \exp(-\int_0^t \lambda_s^{k+1}(\boldsymbol{\sigma}^{(k)}) ds)$, and the result follows. ■

The following proposition gives the joint density of $\boldsymbol{\sigma}$ in terms of their marginal intensities. In the density framework, the immersion property does not necessarily hold. The knowledge of intensities cannot imply the whole family of the density unless the immersion property holds. We note that in the following proposition formula (24) is true in general. However, the equality $\alpha_t(\mathbf{u}) = \alpha_{u_n}(\mathbf{u})$ for all $t > u_n$ is valid only under the immersion property.

Proposition 3.6. *Under the immersion hypothesis between \mathbb{F} and \mathbb{G}^N , the density is given by*

$$\alpha_t(\mathbf{u}) = \begin{cases} \mathbb{E}[\alpha_{u_n}(\mathbf{u}) | \mathcal{F}_t], & 0 \leq t \leq u_n, \\ \alpha_{u_n}(\mathbf{u}), & t > u_n, \end{cases}$$

where

$$(24) \quad \alpha_{u_n}(\mathbf{u}) = \prod_{i=1}^n \lambda_{u_i}^{i, \mathbb{F}}(\mathbf{u}_{(i-1)}) \exp \left\{ - \int_{u_{i-1}}^{u_i} \lambda_s^{i, \mathbb{F}}(\mathbf{u}_{(i-1)}) ds \right\}.$$

Proof. The case where $n = 1$ holds, as recalled in (4) and (5). We shall prove the proposition for $n = 2$, and the general result will follow. Considering σ_2 , we apply (4) to σ_2 and $\mathbb{G}^{(1)}$ and obtain

$$\alpha_t^{2|1}(u_2) = \mathbb{E} \left[\lambda_{u_2}^{2,\mathbb{F}}(\sigma_1) \exp \left(- \int_0^{u_2} \lambda_s^{2,\mathbb{F}}(\sigma_1) ds \right) \middle| \mathcal{G}_t^{(1)} \right].$$

Identifying both sides of the equality on the set $\{t < \sigma_1\}$ by (2) implies

$$\begin{aligned} \int_t^\infty du_1 \alpha_t(u_1, u_2) &= \mathbb{E} \left[\mathbf{1}_{\{\sigma_1 > t\}} \lambda_{u_2}^{2,\mathbb{F}}(\sigma_1) \exp \left(- \int_0^{u_2} \lambda_s^{2,\mathbb{F}}(\sigma_1) ds \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\int_t^\infty du_1 \alpha_{u_2}^{1|0}(u_1) \lambda_{u_2}^{2,\mathbb{F}}(u_1) \exp \left(- \int_0^{u_2} \lambda_s^{2,\mathbb{F}}(u_1) ds \right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Under immersion between \mathbb{F} and $\mathbb{G}^{(1)}$, $\alpha_{u_2}^{1|0}(u_1) = \alpha_{u_1}^{1|0}(u_1) = \lambda_{u_1}^{1,\mathbb{F}} \exp(-\int_0^{u_1} \lambda_s^{1,\mathbb{F}} ds)$, which allows us to conclude the proof. ■

4. Change of probability measure. In this section, we study the change of probability for the density family. On the one hand, the density is affected by a change of probability; on the other hand, we can construct a density process by using the methodology of a change of probability measure. Very often, hypotheses concerning default times are made under the historical probability, and, for pricing purpose, one needs to work under some equivalent martingale measure. Hence, it is important to understand the role of a change of probability.

4.1. Density under a change of probability. We begin with a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where the immersion property holds between \mathbb{F} and $\mathbb{G}^{(n)}$ and σ has the \mathbb{F} -density $\alpha(\mathbf{u})$. Let W be a (\mathbb{P}, \mathbb{F}) -Brownian motion and hence a $(\mathbb{P}, \mathbb{G}^{(n)})$ -Brownian motion. We describe the Radon–Nikodým derivative by using a general martingale process. Let Q be a positive $(\mathbb{P}, \mathbb{G}^{(n)})$ -martingale with expectation 1, solution of the SDE

$$dQ_t = Q_t \left(\Psi_t dW_t + \sum_{i=1}^n \Phi_t^i dM_t^i \right), \quad Q_0 = 1,$$

where $(M_t^i = \mathbf{1}_{\{\sigma_i \leq t\}} - \int_0^{t \wedge \sigma_i} \lambda_s^{i,\mathbb{F}}(\sigma_{(i-1)}) ds, t \geq 0)$, $i = 1, \dots, n$, are $(\mathbb{P}, \mathbb{G}^{(n)})$ -martingales of pure jump, Ψ and Φ^i are square-integrable $\mathbb{G}^{(n)}$ -predictable processes which can be written in the form $\Psi_t = \sum_{i=0}^n \mathbf{1}_{\{\sigma_i < t \leq \sigma_{i+1}\}} \psi_t^i(\sigma_{(i)})$ and $\Phi_t^i = \sum_{k=0}^{i-1} \mathbf{1}_{\{\sigma_k < t \leq \sigma_{k+1}\}} \phi_t^{i,k}(\sigma_{(k)})$ with $\phi^{i,k} > -1$. Then the process Q can be written in the form

$$Q_t = \sum_{i=0}^n \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} q_t^i(\sigma_{(i)}),$$

where for all $i = 0, 1, \dots, n-1$

$$q_t^i(\mathbf{u}_{(i)}) = q_{u_i}^i(\mathbf{u}_{(i)}) \exp \left(\int_{u_i}^t \psi_s^i(\mathbf{u}_{(i)}) dW_s - \frac{1}{2} \int_{u_i}^t (\psi_s^i(\mathbf{u}_{(i)}))^2 ds - \int_{u_i}^t \phi_s^{i+1,i}(\mathbf{u}_{(i)}) \lambda_s^{i+1,\mathbb{F}}(\mathbf{u}_{(i)}) ds \right),$$

$t > u_i,$

and

$$q_t^n(\mathbf{u}) = q_{u_n}^n(\mathbf{u}) \exp\left(\int_{u_n}^t \psi_s^n(\mathbf{u}) dW_s - \frac{1}{2} \int_{u_n}^t (\psi_s^n(\mathbf{u}))^2 ds\right), \quad t > u_n,$$

with initial values

$$q_0^0 = 1 \quad \text{and} \quad q_{u_i}^i(\mathbf{u}_{(i)}) = q_{u_i}^{i-1}(\mathbf{u}_{(i-1)}) (1 + \phi_{u_i}^{i,i-1}(\mathbf{u}_{(i-1)})), \quad i = 1, \dots, n.$$

Since the immersion property holds under the probability \mathbb{P} , the density $\alpha(\mathbf{u})$ of σ satisfies (21) and, in particular, $\alpha_t(\mathbf{u}) = \alpha_{u_n}(\mathbf{u})$ if $t > u_n$. We now consider a new probability measure \mathbb{Q} defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Q_t \quad \text{on} \quad \mathcal{G}_t^{(n)}.$$

Then the density of σ under \mathbb{Q} is given by

$$(25) \quad \alpha_t^{\mathbb{Q}}(\mathbf{u}) = \begin{cases} \frac{1}{Q_t^{\mathbb{F}}} \mathbb{E}\left[q_{u_n}^n(\mathbf{u}) \alpha_{u_n}(\mathbf{u}) | \mathcal{F}_t\right], & t \leq u_n, \\ \frac{1}{Q_t^{\mathbb{F}}} q_t^n(\mathbf{u}) \alpha_{u_n}(\mathbf{u}), & t > u_n, \end{cases}$$

where $Q^{\mathbb{F}}$ is the restriction of Q on \mathbb{F} , i.e., $Q_t^{\mathbb{F}} := \mathbb{E}[Q_t | \mathcal{F}_t]$. We observe that after the change of probability, the immersion property no longer holds.

An example concerns the successive Cox model where σ is constructed as in (22). In this case, the immersion property holds between \mathbb{F} and $\mathbb{G}^{(n)}$ and the density $\alpha_t(\mathbf{u})$ of σ under \mathbb{P} is given by Proposition 3.6. Then choosing carefully the term $q_{u_n}^n(\mathbf{u})$ and computing (25) allows us to obtain the density $\alpha^{\mathbb{Q}}(\mathbf{u})$ of σ under a new probability measure \mathbb{Q} .

4.2. Dynamic copula. The default dependence structure can be described by a change of probability. We can construct the conditional density process in a dynamic way by starting from the unconditional law of σ . This gives a dynamic copula viewpoint: the idea is to diffuse the initial dependence structure of defaults and appeal to the change of probability measure.

We begin with a probability measure \mathbb{P} where σ is independent of the filtration \mathbb{F} and admits a probability density function $\alpha_0 : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$. Then the conditional survival probability satisfies $\mathbb{P}(\sigma > \theta | \mathcal{F}_t) = \mathbb{P}(\sigma > \theta) = \int_{\theta}^{\infty} \alpha_0(\mathbf{u}) d\mathbf{u}$. The density function $\alpha_0(\mathbf{u})$ can be modeled by using a copula model and represents the initial dependence between the default times σ . To describe the dynamic correlation structure, we introduce a new probability measure. Let $(\beta_t(\mathbf{u}), t \geq 0)$ be a family of nonnegative (\mathbb{P}, \mathbb{F}) -martingales with $\beta_0(\mathbf{u}) = 1$ for all $\mathbf{u} \in \mathbb{R}_+^n$. For example, let $\beta_t(\mathbf{u}) = \mathcal{E}(\int Z(\mathbf{u}) dW)_t$, where \mathcal{E} denotes the Doléans–Dade exponential and $Z(\mathbf{u})$ is an \mathbb{F} -adapted process such that the Novikov condition $\mathbb{E}_{\mathbb{P}}[\exp(\int_0^t \frac{1}{2} |Z_s(\mathbf{u})|^2 ds)] < \infty$ is satisfied. Due to the assumed independence, $(\beta_t(\sigma), t \geq 0)$ is a \mathbb{P} -martingale w.r.t. the filtration $\mathbb{G}^{\sigma} = \mathbb{F} \vee \sigma(\sigma)$ and defines a probability measure \mathbb{Q} by

$$(26) \quad \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_t^{\sigma}} = \beta_t(\sigma).$$

From the Bayes formula, the density of σ under \mathbb{Q} is given by

$$(27) \quad \alpha_t^{\mathbb{Q}}(\mathbf{u}) = \frac{1}{m_t^{\beta}} \beta_t(\mathbf{u}) \alpha_0(\mathbf{u}),$$

where $m_t^\beta := \mathbb{E}_{\mathbb{P}}[\beta_t(\boldsymbol{\sigma})|\mathcal{F}_t] = \int_0^\infty \beta_t(\mathbf{u})\alpha_0(\mathbf{u})d\mathbf{u}$. Since $\beta_0(\mathbf{u}) = 1$, $\mathbb{Q}(\boldsymbol{\sigma} > \boldsymbol{\theta}) = \mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta})$, so the unconditional law of $\boldsymbol{\sigma}$ is the same under the two probabilities.

The dynamic dependence under the new probability \mathbb{Q} is introduced through the change of probability measure using the normalized (\mathbb{Q}, \mathbb{F}) -martingale $\frac{\beta_t(\mathbf{u})}{m_t^\beta}$. The normalization factor m_t^β introduces a nonlinear dependence of $\alpha_t^\mathbb{Q}(\mathbf{u})$ with respect to the initial density.

We can construct a sequence of successive defaults $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ with a given density $\alpha_t(\mathbf{u})$ by using the method of change of probability. We start from a family of random times $\boldsymbol{\sigma}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ which is independent of \mathbb{F} , so the (\mathbb{P}, \mathbb{F}) -density $\alpha(\mathbf{u})$ coincides with its initial value $\alpha_0(\mathbf{u})$. Define the probability measure \mathbb{Q} on $(\Omega, \mathbb{G}^\boldsymbol{\sigma})$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_t^\boldsymbol{\sigma}} = \frac{\alpha_t(\boldsymbol{\sigma})}{\alpha_0(\boldsymbol{\sigma})}.$$

Then $\boldsymbol{\sigma}$ admits $\alpha_t(\mathbf{u})$ as the conditional density under the new probability measure \mathbb{Q} by (27).

5. Modeling of density process. The joint density process plays an essential role in the density approach. In this section, we show several explicit examples of the density process for ordered default times. We also present a link between the ordered and nonordered default times. The individual default information can be used to construct a joint density process for nonordered defaults. Then using order statistics allows us to obtain the density for ordered defaults.

5.1. Nonordered defaults and order statistics. Let us consider the family of nonordered default times $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$. We assume that the \mathbb{F} -density of $\boldsymbol{\tau}$ exists and is denoted by $\gamma(\cdot)$ such that for any bounded Borel function $f : \mathbb{R}_+^n \rightarrow \mathbb{R}$,

$$\mathbb{E}[f(\boldsymbol{\tau})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{\theta})\gamma_t(\boldsymbol{\theta})d\boldsymbol{\theta}, \quad t \geq 0.$$

The ordered defaults $\boldsymbol{\sigma}$ is the increasing permutation of $\boldsymbol{\tau}$. There exists an explicit relationship between $\gamma(\cdot)$ and the density $\alpha(\cdot)$ of $\boldsymbol{\sigma}$ by using the order statistics. For any $\mathbf{u} \in \mathbb{R}_+^n$ such that $u_1 < \dots < u_n$ and any $t \geq 0$,

$$(28) \quad \alpha_t(u_1, \dots, u_n) = \mathbf{1}_{\{u_1 < \dots < u_n\}} \sum_{\Pi} \gamma_t(u_{\Pi(1)}, \dots, u_{\Pi(n)}),$$

where $(\Pi(1), \dots, \Pi(n))$ is a permutation of $(1, \dots, n)$. The individual default information can be incorporated into the density of ordered defaults by using equality (28). In particular, if $\boldsymbol{\tau}$ is exchangeable (see, e.g., Frey and McNeil [13]), that is, if $(\tau_1, \dots, \tau_n) \stackrel{d}{=} (\tau_{\Pi(1)}, \dots, \tau_{\Pi(n)})$ for any permutation where $\stackrel{d}{=}$ signifies the equality in distribution, then

$$\alpha_t(u_1, \dots, u_n) = \mathbf{1}_{\{u_1 < \dots < u_n\}} n! \gamma_t(u_1, \dots, u_n).$$

This implies that all subfamilies of $\boldsymbol{\tau}$ with the same cardinal have the same distribution. In other words, the portfolio is homogeneous.

5.2. Several explicit examples. In this subsection, we give several explicit examples for the \mathbb{F} -conditional survival probability $G_t(\boldsymbol{\theta}) := \mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_t)$ of nonordered defaults, from which we can deduce the density $\gamma(\boldsymbol{\theta})$.

The following example is a backward one based on the Cox process model. The correlation structure is fixed for the final time by a copula function, and $G_t(\boldsymbol{\theta})$ is obtained by taking conditional expectation. By choosing different copula functions, we obtain a large family of joint densities.

Example 5.1. Let $\tau_i, i \in \{1, \dots, n\}$, be defined as in the Cox process model in Lando [16]. That is, $\tau_i = \inf\{t \geq 0 : \Phi_t^i \geq \xi_i\}$, where Φ^i is an \mathbb{F} -adapted increasing process satisfying $\Phi_0^i = 0$ and $\lim_{t \rightarrow \infty} \Phi_t^i = +\infty$, and ξ_i is an \mathcal{A} -measurable random variable of exponential law with parameter 1 and is independent of \mathcal{F}_∞ . In this model, the marginal survival process is given by $G_t^i = \mathbb{P}(\tau_i > t | \mathcal{F}_t) = e^{-\Phi_t^i} = \mathbb{P}(\tau_i > t | \mathcal{F}_\infty)$. Let the correlation of defaults be represented by a copula function $\mathcal{C} : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ such that $\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_\infty) = \mathcal{C}(G_{\theta_1}^1, \dots, G_{\theta_n}^n)$. Then

$$G_t(\boldsymbol{\theta}) = \mathbb{E}[\mathcal{C}(G_{\theta_1}^1, \dots, G_{\theta_n}^n) | \mathcal{F}_t].$$

The Gaussian copula model is the standard market model for the CDO pricing, where the correlation between defaults is described by a standard Gaussian random variable representing the common market factor. We now generalize the Gaussian copula model in a dynamic way. In the following, the first example is a backward one where the factor is a Cox–Ingersoll–Ross (CIR) process and the second one is a forward constructive model (see also Crépey, Jeanblanc, and Wu [5]).

Example 5.2. Consider a family of processes $X = (X^1, \dots, X^n)$, where $X^i = Y^0 + Y^i$ ($i = 1, \dots, n$) is a fundamental process of the i th firm depending on two factors: the process Y^0 can be interpreted as a common factor of the market, and Y^i is the individual factor of each firm where $Y^i, i = 0, 1, \dots, n$, are independent CIR processes

$$dY_t^i = \kappa_i(\mu_i - Y_t^i)dt + \sigma_i \sqrt{Y_t^i} dB_t^i, \quad Y_0^i > 0, \quad i = 0, 1, \dots, n.$$

We assume moreover that $2\kappa_i\mu_i > \sigma_i^2$ so that Y^i does not vanish. Let the filtration \mathbb{F} be generated by the multidimensional Brownian motion $B = (B^0, B^1, \dots, B^n)$. Let $T \geq 0$ be a terminal time. We define the conditional survival probability for $0 \leq t < T$ and $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n$ by

$$G_t(\boldsymbol{\theta}) = \mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_t) = \mathbb{E}[e^{-\boldsymbol{\theta} \cdot X_T} | \mathcal{F}_t] = \mathbb{E}[e^{-\sum_{i=0}^n \theta_i Y_T^i} | \mathcal{F}_t],$$

where $\theta_0 = \theta_1 + \dots + \theta_n$. Then classical results on affine processes yield

$$G_t(\boldsymbol{\theta}) = \exp\left(-\sum_{i=0}^n \varphi_i(T-t, \theta_i) Y_t^i - \sum_{i=0}^n \psi_i(T-t, \theta_i)\right),$$

with φ_i and ψ_i being functions $\mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ given explicitly by

$$\varphi_i(s, v) = \frac{2\kappa_i v}{(2\kappa_i + \sigma_i^2 v)e^{\kappa_i s} - \sigma_i^2 v}, \quad \psi_i(s, v) = \frac{2\kappa_i \mu_i}{\sigma_i^2} \left(\ln \frac{(2\kappa_i + \sigma_i^2 v)e^{\kappa_i s} - \sigma_i^2 v}{2\kappa_i} - \kappa_i s \right).$$

The correlation between default times is characterized by the process Y^0 . The case $Y^0 = 0$ provides an example where the default times are independent. Moreover, given the process Y^0 , the default times satisfy the standard conditional independence condition.

In particular, for a homogeneous portfolio, $\kappa_i = \kappa$, $\mu_i = \mu$, and $\sigma_i = \sigma$ for $i = 0, \dots, n$. So the functions satisfy $\varphi_i = \varphi$ and $\psi_i = \psi$ and

$$G_t(\boldsymbol{\theta}) = \exp \left(- \sum_{i=0}^n \varphi(T-t, \theta_i) Y_t^i - \sum_{i=0}^n \psi(T-t, \theta_i) \right).$$

Example 5.3. Let h_i be a family of increasing functions mapping \mathbb{R}_+ into \mathbb{R} , let $B = (B^i, i = 1, \dots, n)$ be an n -dimensional standard Brownian motion, and let Y be a random variable independent of \mathbb{F} . We set

$$\tau_i = (h_i)^{-1} \left(\sqrt{1 - \rho_i^2} \int_0^\infty f_i(s) dB_s^i + \rho_i Y \right)$$

for $\rho_i \in (-1, 1)$ and f_i a family of deterministic square-integrable functions. An immediate extension of the Gaussian model leads to

$$\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_t \vee \sigma(Y)) = \prod_{i=1}^n \Phi \left(\frac{1}{\sigma_i(t)} \left(m_t^i - \frac{h_i(\theta_i) - \rho_i Y}{\sqrt{1 - \rho_i^2}} \right) \right),$$

where $m_t^i = \int_0^t f_i(s) dB_s^i$ and $\sigma_i^2(t) = \int_t^\infty f_i^2(s) ds$. It follows that

$$\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_t) = \int_{-\infty}^\infty \prod_{i=1}^n \Phi \left(\frac{1}{\sigma_i(t)} \left(m_t^i - \frac{h_i(\theta_i) - \rho_i y}{\sqrt{1 - \rho_i^2}} \right) \right) f_Y(y) dy,$$

where f_Y denotes the probability density function of Y . Note that, in this setting, the random times $\boldsymbol{\tau}$ are conditionally independent given the factor Y , similar to the standard Gaussian copula model. In the particular case $t = 0$, choosing f_i so that $\sigma_i(0) = 1$, and Y with a standard Gaussian law, we obtain

$$P(\boldsymbol{\tau}_i > \boldsymbol{\theta}) = \int_{-\infty}^\infty \prod_{i=1}^n \Phi \left(- \frac{h_i(\theta_i) - \rho_i y}{\sqrt{1 - \rho_i^2}} \right) \varphi(y) dy,$$

which corresponds, by construction, to the standard Gaussian copula ($h_i(\tau_i) = \sqrt{1 - \rho_i^2} X_i + \rho_i Y$, where X_i, Y are independent standard Gaussian variables).

Relaxing the independence condition on the components of the process B leads to more sophisticated examples.

5.3. Numerical illustrations. In this subsection, we present a simple numerical example to illustrate quantitatively the impact of successive defaults on the survival probability of remaining credit names and the role of a change of probability measure. We consider a family of nonordered random times $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ and its ordered permutation $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$.

We suppose that τ_1, \dots, τ_n are correlated by a Gumbel copula, which is suitable to characterize heavy tail dependence and is often used for insurance portfolios. More precisely, the joint survival probability is given by $\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta}) = \exp\left(-\left(\sum_{i=1}^n (\lambda_i \theta_i)^\xi\right)^{1/\xi}\right)$ with $\lambda_i > 0$ and $\xi \geq 1$. In this model, each default time τ_i follows the exponential law with intensity λ_i , and the default correlation is characterized by the constant parameter ξ . The case $\xi = 1$ corresponds to the independent case (which, however, does not imply the independence between $\sigma_1, \dots, \sigma_n$), and a larger value of ξ implies a stronger correlation between the credit names.

We suppose that under the initial probability \mathbb{P} , the default times are independent of the reference filtration \mathbb{F} . So the \mathbb{F} -density of $\boldsymbol{\sigma}$ is a deterministic function and the immersion property is satisfied under \mathbb{P} . In the following numerical tests, we let $n = 3$, $T = 1$ and let the intensity parameters be $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$. Figure 1 plots a simulation of the conditional survival probability of the third default $\mathbb{P}(\sigma_3 > T | \mathcal{G}_t^{(3)})$ given in (14) as a function of time t for different values of the correlation parameter ξ . In this scenario, the first two defaults σ_1 and σ_2 occur before T . We observe that the survival probability of σ_3 is decreasing w.r.t. ξ . Before the first default, all three curves are increasing with time. At each default time, there is a negative jump in the survival probability of the third one σ_3 , and the jump size is increasing w.r.t. the correlation.

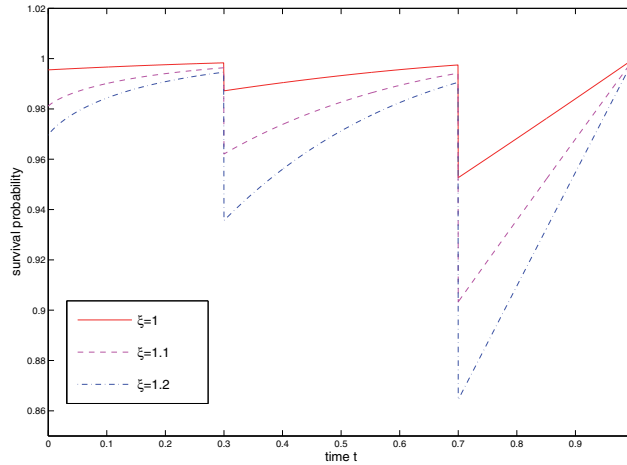


Figure 1. Conditional survival probability $\mathbb{P}(\sigma_3 > T | \mathcal{G}_t^{(3)})$.

Consider a change of probability as in subsection 4.2 and let the Radon–Nikodým density in (26) be given as $\beta_t(\mathbf{u}) = \exp\left(-\left(\sum_{i=1}^n u_i\right)M_t - \frac{1}{2}\left(\sum_{i=1}^n u_i\right)^2 \langle M \rangle_t\right)$, where $(M_t := \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), t \geq 0)$ is a positive (\mathbb{P}, \mathbb{F}) -martingale, W being a (\mathbb{P}, \mathbb{F}) -Brownian motion. We will compare the survival probability under the two probability measures. In Figure 2, we plot the survival probability $\mathbb{Q}(\sigma_3 > T | \mathcal{G}_t^{(3)})$ under the new probability measure \mathbb{Q} . The parameters are the same as in the previous test, and we let the volatility parameter be equal to 0.3. We observe that with the same scenario of default times σ_1 and σ_2 , the survival conditional probability of σ_3 has similar behaviors under the two probability measures. However, the value of the survival probability is perturbed by the change of probability.

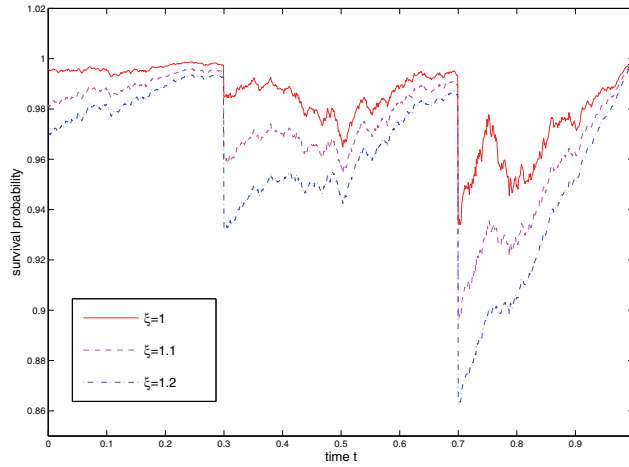


Figure 2. Conditional survival probability $\mathbb{Q}(\sigma_3 > T | \mathcal{G}_t^{(3)})$.

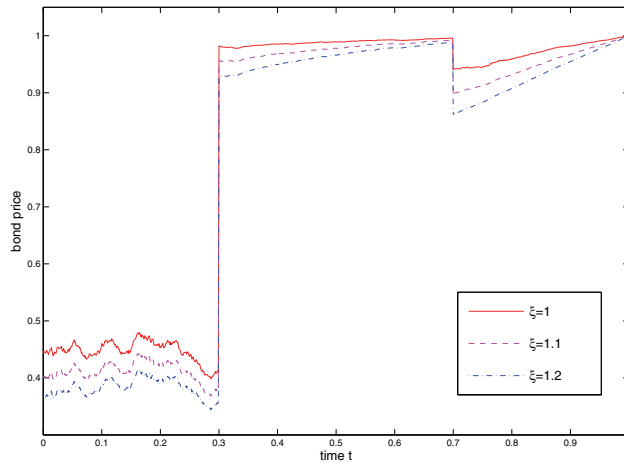


Figure 3. Value of the defaultable bond.

In the third test, we consider a defaultable zero-coupon bond which is written on the last default σ_3 and will be activated only if the first default σ_1 occurs before the maturity T . We suppose that \mathbb{Q} is a risk-neutral pricing probability measure (with the same parameters as in Figure 2) and that the interest rate is zero. Figure 3 plots the value of this bond at time t in a scenario where both σ_1 and σ_2 occur before the maturity T and σ_3 is larger than T . Because of the constraint on σ_1 , the bond price is low before the first default. When the first default occurs, the bond becomes a standard defaultable bond on σ_3 and the price jumps to a high value. At the second default time σ_2 , we observe a contagious phenomenon and the bond price has a negative jump.

6. Conclusion and perspective. In this paper, we present the density framework to study ordered default times. We assume that the family of ordered defaults admits a joint density w.r.t. the reference filtration and apply the before-default and after-default analysis to successive default scenarios. The main contribution of this approach is to study in detail the impact of each default event on the remaining credit names such as the regime change and the contagious jump at each default. In addition, we distinguish the roles of reference information and default information, and the default timings are explicitly included in the results obtained. The density approach provides flexible choices on default correlation structure. In particular, the immersion property hypothesis, which is often assumed in intensity models, is not necessary in the density approach.

Besides the theoretical results, the perspective of the density approach consists of developing explicit and tractable density models such that the pricing formulas can be implemented numerically for credit portfolio derivatives, and in particular for the CDOs.

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