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Portfolio optimization with insider's initial information and counterparty risk

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Abstract We study the gain of an insider having private information which concerns the default risk of a counterparty. More precisely, the default time τ is modelled as the first time a stochastic process hits a random threshold L. The insider knows this threshold (as it can be the case for the manager of the counterparty) and this information is modelled by using an initial enlargement of filtration. The standard investors only observe the value of the threshold at the default time and estimate the default event by its conditional density process. The financial market consists of a risk-free asset and a risky asset whose price is exposed to a sudden jump at the default time of the counterparty. All investors aim to maximize the expected utility from terminal wealth given their own information at the initial date. We solve the optimization problem under short-selling and buying constraints and we compare through numerical illustrations the optimal processes for the insider and the standard investors.

Keywords asymmetric information, enlargement of filtrations, counterparty risk, optimal investment, duality, dynamic programming

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1 Introduction

The insider's optimal investment is a classical problem where an investor possessing some extra flow of information aims to maximize the expected utility on the final value of her portfolio. As the insider has more information, she has access to a larger set of available trading strategies, leading to a higher expected utility from terminal wealth. In the literature, the following interesting question has been studied: what is the cost of the extra information? From an indifference point of view, we search for the value at which the investor accepts to buy the information at the initial time, that is, the amount of money she is ready to pay such that this cost is offset by the increase of the maximal expected utility. This is the approach adopted by Amendinger et al. [1], where the authors study the value of an initial information in the setting of a complete default free market. The extra information they consider is a terminal information distorted by an independent noise, for example, a noisy signal of a functional of the final value of the assets. We adopt a more direct manner: we are interested in the gain of the insider from her investment strategy on the portfolio compared to other investors not having access to the extra information. The originality of our paper is to study this problem in the context of credit risks: the insider's information concerns the default risk of a counterparty firm.

During the financial crisis, the counterparty default has become an important source of risk that should be taken into account. Jiao and Pham [15] have considered an optimal investment problem where the risky asset in the portfolio is subjected to the default risk of a counterparty firm and its value may suffer a sudden jump at the counterparty default time τ. This paper is a good benchmark for our study in order to quantify the value of the extra information. In [15], the accessible information for a standard investor is described as in the classical credit risk modelling by Bielecki and Rutkowski [4], using the progressive enlargement of a reference "default-free" filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by the default τ. To analyze the impact of default, the default density framework developed in El Karoui et al. [5] has been adopted.

This current paper concentrates on an insider in comparison with a standard investor. Both agents can invest in the same risk-free asset and risky one and they observe the same market price for each asset. However, the insider possesses more information on the risky asset since it is influenced by the counterparty default on which the insider has additional knowledge. Due to the extra information, the insider may gain larger profit. The insider's information is modelled by using an initial enlargement of filtration as in [1] and in Grorud and Pontier [7]. More precisely, in the credit risk context, we model the default time τ as the first time that a stochastic process hits a random barrier L. The insider knows the barrier from the initial time and the other investors only see its value at the default time. Besides, since the dynamics of the stock prices are modified at the counterparty default time, the extra information of an insider with respect to a standard investor is twofold: the knowledge of L and the knowledge of the modified dynamics of the stock price. In our framework, this total information is called the insider's information, or the full information in Hillairet and Jiao [9], and it is formally described by the initially enlarged filtration.

We shall consider the insider's optimization problem in parallel with the one studied in [15]. The canonical decomposition of processes adapted to the enlarged filtration induces to specify the investment strategies on the two following sets: the before-default one $\{t < \tau\}$ and the after-default one $\{t \geq \tau\}$, which is a similar point to [15]. However, due to the extra knowledge on the default barrier L, the insider's strategy depends on L before the counterparty default, which is not the case for the standard investor. If the default occurs, the insider's strategy will depend on the default time τ. From the methodology point of view, the main difference here is that for the insider, the default time is modelled as in the classical structural approach model since the random barrier L is known, so that τ becomes a predictable stopping time with respect to the insider's filtration. Therefore, the default density hypothesis, which is crucial in [15], fails to hold for the insider and we can no longer adopt the conditional density approach in this situation.

We apply the theory of initial enlargement of filtration, assuming that the conditional law of L given $\mathcal{F}_t$ is equivalent to the law of L. The corresponding Radon–Nikodým derivative process, $(p_t(.), t \geq 0)$ will play a key role in our methodology.

The paper is organized as follows. In section 2, we introduce the model for the counterparty default. We define and compare the informational structure of an insider with respect to a standard investor, who estimates the default event through its conditional density, and a Merton investor, who does not take into account the eventuality of the counterparty default. In Sect. 3, we present the insider's optimal investment problem and we decompose it into an after-default one and a global before-default one, using the Radon Nikodym derivative process. Following the practice of regulation on financial markets exposed to the counterparty risk, we assume that the short-selling is prohibited for all investors and we discuss the possibility and the impact of relaxing this short-selling constraint. In Sect. 4 we solve the two optimization problems: the after-default one through duality methods in a default free complete market, and the global before-default one through dynamic programming approach. To make comparison with the standard investor in [15], we choose to consider CRRA utility function. In Sect. 5, we perform through numerical illustrations the comparison of the optimal value function and wealth process for insider, standard and Merton investors. This gives a numerical study of the gain of the insider, compared to a standard investor.

2 Counterparty default model and information

We consider the default of a counterparty, that will induce a jump in the asset value of a related firm. The model for the counterparty default is a general and standard model in the credit risk analysis. Let us fix a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ which represents the "default-free" information. Without loss of generality, we assume that all the filtrations we deal with in the following satisfy the usual conditions. A finite horizon T is fixed and let τ be a positive random time denoting the default time of the counterparty, which is not necessarily an $\mathbb{F}$-stopping time.

• *The default model*

We consider the default risk of the counterparty in a general barrier model. Let $(\lambda_t, t \geq 0)$ be a positive $\mathbb{F}$-adapted process and $\Lambda_t = \int_0^t \lambda_s ds$. We model the default time as the first passage time of the process Λ to a positive random barrier L, i.e.,

$$\tau = \inf\{t \geq 0 : \Lambda_t \geq L\}.$$

In the widely used Cox process model, Λ is the compensator process of default, L is independent of $\mathcal{F}_\infty$ and follows the uni-exponential law. In the classical structural default models, Λ can represent the cumulated losses of the counterparty and L is constant or deterministic, thus τ is an $\mathbb{F}$-stopping time. In our model, the default threshold L is a positive $\mathfrak{A}$-measurable random variable which can be correlated with the reference filtration $\mathbb{F}$. Let us emphasize that this default model is known to both an insider and a standard investor. In particular, all investors have the full knowledge of the $\mathbb{F}$-conditional distributions of the random variable τ and they observe processes S and Λ. The model with a random threshold L represents the market view about the joint dynamics of the default process and S.

• *Information of the insider*

Besides the information on the "default-free" market, we suppose that the insider has complete information on L: this is the case of the counterparty firm's manager who determines the default threshold according to the financial situation on the market. The choice of the threshold will also depend on the anticipation of the manager on the economic health of her firm. For example, the manager has inside information about the firm's statement of account which will be officially published only in a future date and she may use this information to determine the default threshold L. The manager chooses the level of L at the initial time and keep this benchmark until the date T.[1] For the insider, the information is modeled as the initial enlargement of the filtration $\mathbb{F}$ by L and is denoted by $\mathbb{G}^M = (\mathcal{G}_t^M)_{t\geq 0}$, $\mathcal{G}_t^M = \mathcal{F}_t \vee \sigma(L)$.

• *Information of the standard investor*

A standard investor on the market observes whether the default has occurred or not and if so, the default time τ, together with the information contained in the filtration $\mathbb{F}$. Mathematically, this information is represented by the progressive enlargement of filtration $\mathbb{F}$ by τ, or more precisely, by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$, $\mathcal{D}_t = \sigma(1_{\tau\leq s}, s \leq t)$. This is the standard credit risk modeling for a market investor as in [4]. The standard investor will estimate the default event through the conditional density of τ with respect to the filtration $\mathbb{F}$ and thus establishes a link with the conditional law of L (cf. Remark 4.3).

The investor's information is included in the insider's information flow. We have $\mathcal{G}_t \subseteq \mathcal{G}_t^M$ for any $t \geq 0$. In fact, before the default τ, i.e., on the set $\{t < \tau\}$, the insider has additional information on L, so her information $\mathcal{G}_t^M$ is in general strictly larger than $\mathcal{G}_t$. After the default occurs, both of them observe the default event and subsequently the value of L so that they have equal information flow.

• *Information of a Merton type investor*

In a classical optimization problem, the default event is not taken into account. This corresponds to the Merton's problem, solved in [16] for CRRA and CARA utility functions and extended for more general utility functions in [17]. In this case, the filtration $\mathbb{F}$ represents the available information for the Merton type investor.

[1] We let for a future work the case where the threshold can be adjusted dynamically.

3 Insider's optimization problem

3.1 Portfolio investment strategy and wealth process

A finite horizon T being fixed, all investment strategies take place from time 0 to time T. The insider has access to the same financial market as the standard investor, more precisely, she can invest in two types of financial assets. The first one is a risk-free bond with strictly positive values. We choose it as the numéraire and assume, without loss of generality that the value of this bond is equal to 1. The other asset is a risky one which is affected by the default risk of the counterparty firm on which the insider has extra information.

The price of this risky asset is observable by all investors on market at any time $t \in [0, T]$. Since it is subject to the counterparty default risk, the price process may have a jump at the default time τ. Thus it is modelled by a $\mathbb{G}$-adapted process S, which admits the decomposition form (cf. the result of Jeulin [14] Lemma 4.4 that we recall in Annex) :

$$S_t = S_t^0 1_{t<\tau} + S_t^1(\tau) 1_{t\geq\tau}, \quad 0 \leq t \leq T$$

where S^0 is $\mathbb{F}$-adapted and $S^1(\cdot)$ is $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted, $\mathcal{B}(\mathbb{R}_+)$ being the Borel σ-algebra. We suppose that the asset admits a contagious jump at the default time of the counterparty, that is,

$$S_\theta^1(\theta) = S_{\theta-}^0(1 - \gamma_\theta).$$

The process γ is $\mathbb{F}$-adapted and represents the proportional jump at default. We suppose $\gamma < 1$, so that the risky asset price remains strictly positive after the counterparty default. The case of a positive jump (gain) of S corresponds to a negative γ (for example the case of a duopoly competition) and conversely, a positive γ induces a contagious loss of S (for example if the asset is positively correlated with the counterparty). We suppose that the sign of the jump γ remains unchanged and is known by all investors.

We consider the trading strategy of the insider, who adjusts her portfolio of assets according to her information accessibility. Therefore, her investment strategy process is characterized by a $\mathbb{G}^M$-predictable process π which represents the proportion of wealth invested in the risky asset and is of the form (cf. [14] Lemma 3.13 and 4.4)

$$\pi_t = 1_{t\leq\tau}\pi_t^0(L) + 1_{t>\tau}\pi_t^1(\tau),$$

where $\pi^0(\cdot)$ and $\pi^1(\cdot)$ are $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable processes, $\mathcal{P}(\mathbb{F})$ being the predictable σ-algebra associated to the filtration $\mathbb{F}$. Starting from an initial wealth $X_0 \in \mathbb{R}_+$, the total wealth of the insider's portfolio is then a $\mathbb{G}^M$-adapted process given by

$$X_t = 1_{t<\tau} X_t^0(L) + 1_{t\geq\tau} X_t^1(\tau)$$

where the before-default wealth process satisfies the self-financing equation

$$dX_t^0(L) = X_t^0(L)\pi_t^0(L)\frac{dS_t^0}{S_t^0}, \quad 0 \leq t \leq T$$

and after the default τ, the wealth process has a change of regime in its dynamics and satisfies

$$dX_t^1(\tau) = X_t^1(\tau)\pi_t^1(\tau)\frac{dS_t^1(\tau)}{S_t^1(\tau)}, \quad t \in [\![\tau, T]\!].$$

At the default time, the wealth jumps. Therefore, at time τ, the initial value of the after-default wealth process is

$$X_\tau^1(\tau) = X_{\tau-}^0(L)\big(1 - \pi_\tau^0(L)\gamma_\tau\big). \tag{3.1}$$

We suppose that $\pi_\tau^0(L)\gamma_\tau < 1$, so that the wealth remains strictly positive after the jump due to the counterparty default.

We consider the following dynamics for the asset price S on the before-default set $\{t < \tau\}$ for S^0 and on the after-default set $\{t \geq \tau\}$ for S^1:

$$\begin{aligned} dS_t^0 &= S_t^0(\mu_t^0 dt + \sigma_t^0 dW_t), \quad 0 \leq t \leq T \\ dS_t^1(\theta) &= S_t^1(\theta)(\mu_t^1(\theta)dt + \sigma_t^1(\theta)dW_t), \quad \theta \leq t \leq T \end{aligned}$$

where the coefficients μ^0 and σ^0 are $\mathbb{F}$-adapted processes, $\mu^1(\theta)$ and $\sigma^1(\theta)$ are $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted processes, and W is an $\mathbb{F}$-Brownian motion. In addition, we suppose the integrability condition

$$\int_0^T \Big|\frac{\mu_t^0}{\sigma_t^0}\Big|^2 dt + \int_\theta^T \Big|\frac{\mu_t^1(\theta)}{\sigma_t^1(\theta)}\Big|^2 dt + \int_0^T |\sigma_t^0|^2 dt + \int_\theta^T |\sigma_t^1(\theta)|^2 dt < \infty.$$

So the values of the before-default and after-default wealth satisfy the dynamics

$$dX_t^0(L) = X_t^0(L)\pi_t^0(L)(\mu_t^0 dt + \sigma_t^0 dW_t), \quad 0 \leq t \leq T \tag{3.2}$$

$$dX_t^1(\tau) = X_t^1(\tau)\pi_t^1(\tau)(\mu_t^1(\tau)dt + \sigma_t^1(\tau)dW_t), \quad t \in [\![\tau, T]\!] \tag{3.3}$$

and the jump at default of the wealth process is given by the equality (3.1).

We discuss firstly the constraints on the investment in the risky assets. On the one hand, in practice, there are buying constraints: for diversification and regulation reasons, investors are limited to invest in a given asset below a certain level. Otherwise, they must make reporting to the authorities. On the other hand, after the financial crisis in 2008, regulators prohibited short-selling on several equity markets. Nowadays, these restrictions have been relaxed for liquidity reason. Based on these observations, in our framework, the regulators are concerned about the default risk of the counterparty and its impact on the risky asset. Thus they impose a buying constraint δ_b and they forbid short-selling as long as the default has not occurred yet. After the counterparty default, those restrictions will be relaxed.

This motivates us to propose the following admissible trading strategy family $\mathcal{A}_L$ as the set of pairs $(\pi^0(\cdot), \pi^1(\cdot))$, where $\pi^0(\cdot)$ and $\pi^1(\cdot)$ are $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable processes such that

$$\forall\, l > 0, \left(\int_0^{\tau_l \wedge T} |\pi_t^0(l)\sigma_t^0|^2 dt + \int_{\tau_l \wedge T}^T |\pi_t^1(\tau_l)\sigma_t^1(\tau_l)|^2 dt\right) < \infty, \quad a.s.$$

$$0 \leq \pi^0 \leq \delta_b \text{ and } \pi_{\tau_l}^0(l)\gamma_{\tau_l} < 1,$$

where τ_l is the $\mathbb{F}$-stopping time defined by $\tau_l := \inf\{t \,:\, \Lambda_t \geq l\}$.

Remark 3.1 Let $\mathcal{A}$ denote the set of all $\mathbb{G}^M$-predictable processes π such that $\int_0^T |\pi_t\sigma_t|^2 dt < \infty$, $0 \le \pi 1_{\{t\le\tau\}} \le \delta_b$ and $\pi_\tau\gamma_\tau < 1$ (where $\sigma_t = \sigma_t^0 1_{\{\tau>t\}} + \sigma_t^1(\tau)1_{\{\tau\le t\}}$). If $(\pi^0(\cdot), \pi^1(\cdot))$ is an element in $\mathcal{A}_L$, then $(\pi_t = \pi_t^0(L)1_{\tau\ge t} + \pi_t^1(\tau_L)1_{\tau<t}, t \ge 0)$ is a processus in the set $\mathcal{A}$. Conversely, given a process $\pi \in \mathcal{A}$, there exists a pair $(\pi^0(\cdot), \pi^1(\cdot)) \in \mathcal{A}_L$ such that $\pi_t = \pi_t^0(L)1_{\tau\ge t} + \pi_t^1(\tau_L)1_{\tau<t}$ for any $t \ge 0$, thanks to Lemma 6.1 (in Annex).

3.2 The optimization problem

The insider has the objective to maximize her expected utility function on the terminal wealth of her portfolio. Let U be a utility function defined on $(0, +\infty)$, strictly increasing, strictly concave and of class C^1 on $(0, +\infty)$, and satisfying $\lim_{x\to 0^+} U'(x) = +\infty$ and $\lim_{x\to\infty} U'(x) = 0$.

In the usual setting with no initial information, the optimization problem is formulated as

$$V_0 = \sup_{\pi\in\mathcal{A}_L} \mathbb{E}[U(X_T)]. \tag{3.4}$$

This is the optimization problem studied in [15] which has already solved the case of a standard investor whose admissible strategies are $\mathbb{G}$-predictable. Note the slight difference that [15] puts either no short-selling nor buying constraints on the investment strategies.

For the insider, the initial σ-field is non-trivial and this initial information has to be taken into consideration for the formulation of the optimization problem:

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_L} \mathbb{E}[U(X_T)|\mathcal{G}_0^M], \tag{3.5}$$

where $\mathcal{G}_0^M = \sigma(L)$. The link between those two optimization problems (3.4) and (3.5) is given by [1]: if the supremum in (3.5) is attained by some strategy in $\mathcal{A}_L$, then the ω-wise optimum is also a solution to (3.4). Although the supremum is not necessarily attained in our problem, we will see in Proposition 4.11 that there exists a sequence of admissible strategies π_n such that $\mathbb{E}[U(X_T^{\pi_n})|\mathcal{G}_0^M]$ converges in L^1 to (3.5) and we can prove that for the same sequence, $\mathbb{E}[U(X_T^{\pi_n})]$ converges to V_0.

The method to solve the initial optimization problem (3.5), similar as in [15], is to reduce the problem in an incomplete market into two problems : the after-default and before-default ones. Nevertheless, the approach we adopt here is different since the random time τ is not a totally inaccessible random time for the insider and we can no longer use the conditional default density approach, which is the key method in [15], to solve the problem.

Our approach will use the theory of initial enlargement of filtration (also called the strong information modeling in [2]) by the random default barrier L known by the insider. More precisely, we introduce a family of $\mathbb{F}$-stopping times $\tau_l = \inf\{t : \Lambda_t \ge l\}$ for all $l > 0$ which are possible realizations of L and we work under an equivalent probability measure $\mathbb{P}^L$ under which L is independent to $\mathcal{F}_T$. Thus in our framework, we shall need the Radon–Nikodým derivative process $p_t(L)$ which is the density of the historical probability measure $\mathbb{P}$ with respect to this equivalent probability measure $\mathbb{P}^L$ and it will play a similar role as the default density process in [15].

This probability density hypothesis is given below. It is a standard hypothesis in the theory of initial enlargement of filtration due to Jacod [12] and [13].

Hypothesis 3.2 *We assume that L is an $\mathfrak{A}$-measurable random variable with values in $]0,+\infty[$, which satisfies the assumption :*

$$\mathbb{P}(L \in \cdot \,|\, \mathcal{F}_t)(\omega) \sim \mathbb{P}(L \in \cdot), \quad \forall t \in [0,T], \ \mathbb{P}-a.s..$$

We denote by $P_t^L(\omega, dx)$ a regular version of the conditional law of L given $\mathcal{F}_t$ and by P^L the law of L (under the probability $\mathbb{P}$). According to [13], there exists a measurable version of the conditional density

$$p_t(x)(\omega) = \frac{dP_t^L}{dP^L}(\omega, x)$$

which is a positive $(\mathbb{F}, \mathbb{P})$-martingale. It is proved in [7] that Hypothesis 3.2 is satisfied if and only if there exists a probability measure equivalent to $\mathbb{P}$ and under which $\mathcal{F}_T$ and $\sigma(L)$ are independent. Among such equivalent probability measures, the probability $\mathbb{P}^L$ defined by the Radon–Nikodým derivative process

$$\mathbb{E}_{\mathbb{P}^L}\Big[\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}^L}\Big|\, \mathcal{G}_t^M\Big] = p_t(L)$$

is the only one that is identical to $\mathbb{P}$ on $\mathcal{F}_\infty$. In the particular case of L independent of $\mathbb{F}$ (that is the counterparty firm's manager fixes an arbitrary threshold according to the law P^L without taking into account the economic perspectives), then the process $p_t(L)$ is identically equal to 1 and the insider's optimization problem only depends on the value of the threshold but not on its conditional distribution. Otherwise, if L depends on $\mathbb{F}$, the process $p_t(L)$ reflects the anticipation of the insider on the economic situation and naturally appears in the resolution of the insider's optimization problem. For examples of L and explicit computations of corresponding $p_t(L)$, interested reader may refer to [10].

Proposition 3.3 *Under Hypothesis 3.2, we have*

$$\mathbb{E}[U(X_T)|\mathcal{G}_0^M] = \mathbb{E}\Big[p_T(l)\Big(1_{T<\tau_l}U(X_T^0(l)) + 1_{T\geq\tau_l}U(X_T^1(\tau_l))\Big)\Big]_{l=L}$$

where for $l > 0$, $\tau_l := \inf\{t : \Lambda_t \geq l\}$.

Proof We will use the change of probability to $\mathbb{P}^L$ in order to reduce to the case where L and $\mathcal{F}_T$ are independent. Firstly,

$$\begin{aligned}
\mathbb{E}[U(X_T)|\mathcal{G}_0^M] &= \mathbb{E}\Big[1_{T<\tau}U(X_T^0(L)) + 1_{T\geq\tau}U(X_T^1(\tau)) \;|L\Big] \\
&= \mathbb{E}_{\mathbb{P}^L}\Big[p_T(l)\Big(1_{T<\tau_l}U(X_T^0(l)) + 1_{T\geq\tau_l}U(X_T^1(\tau_l))\Big)\Big]_{l=L} \\
&= \mathbb{E}\Big[p_T(l)\Big(1_{T<\tau_l}U(X_T^0(l)) + 1_{T\geq\tau_l}U(X_T^1(\tau_l))\Big)\Big]_{l=L}
\end{aligned}$$

where the last two equalities follow respectively from the facts that $\mathcal{F}_T$ and $\sigma(L)$ are independent under $\mathbb{P}^L$ and that $\mathbb{P}^L$ is identical to $\mathbb{P}$ on $\mathcal{F}_T$. □

This motivates to introduce, for any $l > 0$, the set $\mathcal{A}_l$ of pairs $\pi = (\pi^0, \pi^1(\cdot))$, where π^0 and $\pi^1(\cdot)$ are respectively $\mathbb{F}$-predictable and $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable processes, such that

$$\int_0^{\tau_l \wedge T} |\pi_t^0 \sigma_t^0|^2 dt + \int_{\tau_l \wedge T}^T |\pi_t^1(\tau_l)\sigma_t^1(\tau_l)|^2 dt < \infty, \; a.s.$$

$$0 \le \pi^0 \le \delta_b \text{ and } \pi^0_{\tau_l}\gamma_{\tau_l} < 1,$$

and consider the following optimization problem

$$V_0(l) = \sup_{\pi \in \mathcal{A}_l} \mathbb{E}\left[p_T(l) \left(1_{T<\tau_l} U(X_T^0) + 1_{T \ge \tau_l} U(X_T^1(\tau_l)) \right) \right], \tag{3.6}$$

where $\tau_l = \inf\{t \ge 0 \, : \, \Lambda_t \ge l\}$ is a $\mathbb{F}$-stopping time.

The following theorem shows that the optimal value of the optimization problem (3.5) is actually equal to $V_0(L)$.

Theorem 3.4 *With the above notation, we have*

$$V_0(L) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_L} \mathbb{E}[U(X_T) \,|\, \mathcal{G}_0^M] \qquad a.s.$$

Proof Assume that $(\pi^0(\cdot), \pi^1(\cdot))$ is an element in $\mathcal{A}_L$, then $(\pi^0(l), \pi^1(\cdot)) \in \mathcal{A}_l$. By Proposition 3.3 we obtain that

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_L} \mathbb{E}[U(X_T) \,|\, \mathcal{G}_0^M] \le V_0(L).$$

For the converse inequality, we shall use a measurable selection theorem. For any $\varepsilon > 0$ and any $l \in]0, \infty[$, let $F_\varepsilon(l)$ be the set of strategies $(\pi^0, \pi^1(\cdot)) \in \mathcal{A}_l$ which are ε-optimal with respect to the problem (3.6), namely such that

$$\mathbb{E}\left[p_T(l) \left(1_{T<\tau_l} U(X_T^0(l)) + 1_{T \ge \tau_l} U(X_T^1(\tau_l)) \right) \right] \ge \begin{cases} V_0(l) - \varepsilon, & \text{if } V_0(l) < +\infty, \\ 1/\varepsilon, & \text{if } V_0(l) = +\infty. \end{cases}$$

By a measurable selection theorem (cf. Benes [3, Lemma 1]), there exists a measurable (with respect to l) family $\{(\pi^0(l), \pi^1(\cdot, l))\}_{l \in \mathbb{R}_+}$ with value in $F_\varepsilon(l)$ for any $l > 0$. Finally let

$$\widetilde{\pi}^0(\cdot) := \pi^0(\cdot), \quad \widetilde{\pi}_t^1(x) := 1_{t>x}\pi_t^1(x, \Lambda_x).$$

We have $(\widetilde{\pi}^0(\cdot), \widetilde{\pi}^1(\cdot)) \in \mathcal{A}_L$ (leading to the wealth $\widetilde{X}$) and $\widetilde{\pi}_t^1(\tau_l) = \pi_t^1(\tau_l, l)$ for any $l > 0$ on $\{\tau_l < t\}$. Therefore, by Proposition 3.3,

$$\begin{aligned} \mathbb{E}[U(\widetilde{X}_T) \,|\, \mathcal{G}_0^M] &= \mathbb{E}\left[p_T(l) \left(1_{T<\tau_l} U(X_T^0(l)) + 1_{T \ge \tau_l} U(X_T^1(\tau_l)) \right) \right]_{l=L} \\ &\ge \begin{cases} V_0(L) - \varepsilon, & \text{if } V_0(L) < +\infty, \\ 1/\varepsilon, & \text{if } V_0(L) = +\infty. \end{cases} \end{aligned}$$

Since ε is arbitrary, we obtain $\mathbb{E}[U(\widetilde{X}_T) \,|\, \mathcal{G}_0^M] \ge V_0(L)$. □

3.3 The impact of the short-selling and buying constraints

In practice, there are discussions on the necessity of imposing the short-selling constraint from the regulation point of view. In this subsection, we show that without any constraint on short-selling strategies before the default, the insider may achieve a terminal wealth that is not bounded in L^1.

Proposition 3.5 *We suppose that the following conditions are satisfied:*

(1) the process Λ is a.s. strictly increasing on $[0,T]$,
(2) for any l in the support of the distribution of the law of L, $\mathbb{P}(\Lambda_T \geq l) > 0$.

Then if there is no short-selling constraint and $\gamma > 0$ (or respectively if there is no buying constraint and $\gamma < 0$), we have

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_L} \mathbb{E}[X_T \mid \mathcal{G}_0^M] = +\infty \quad a.s.$$

In addition, for any utility function U such that $\lim_{x\to+\infty} U(x) = +\infty$,

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_L} \mathbb{E}[U(X_T) \mid \mathcal{G}_0^M] = +\infty \quad a.s.$$

Proof We first do the proof for the case of no short-selling constraint and $\gamma > 0$. Let $\varphi :]0,+\infty[\to]0,+\infty[$ be an increasing function such that $\varphi(l) < l$ for any $l \in]0,+\infty[$. Let $\psi > 0$ be a constant. For each $l \in]0,+\infty[$, we define a strategy $\pi(l) = (\pi^0(l), \pi^1(\cdot)) \in \mathcal{A}_l$ as follows

$$\pi_t^0(l) = -\psi 1_{\tau_{\varphi(l)} < t}, \quad \pi^1(\cdot) \equiv 0.$$

Note that $(\pi^0(\cdot), \pi^1(\cdot))$ is an admissible strategy if the short selling constraint is removed. The value at the time τ_l of the corresponding wealth process $X^{\varphi,\psi}$ is equal to $(1+\gamma_{\tau_l}\psi)X^{\varphi,\psi}_{\tau_l -}$. By the dynamics of the wealth process (3.2) and (3.3), on $\{\tau_l \leq T\}$, we have

$$X^{\varphi,\psi}_{\tau_l} = X_0(1+\gamma_{\tau_l}\psi)\exp\left(-\int_{\tau_{\varphi(l)}}^{\tau_l} \left(\mu_t^0\psi + \frac{1}{2}(\sigma_t^0)^2\psi^2\right)dt - \int_{\tau_{\varphi(l)}}^{\tau_l} \sigma_t^0 \psi dW_t\right).$$

Moreover,

$$\begin{aligned}\mathbb{E}[X_T^{\varphi,\psi} | \mathcal{G}_0^M] &= \mathbb{E}\left[p_T(l)\left(1_{T<\tau_l} X_T^{0,\varphi,\psi}(l) + 1_{T\geq\tau_l} X_T^{1,\varphi,\psi}(\tau_l)\right)\right]_{l=L} \\ &\geq \mathbb{E}\left[1_{T\geq\tau_l} p_T(l) X^{\varphi,\psi}_{\tau_l}\right]_{l=L}\end{aligned}$$

Now fix an increasing sequence $(\varphi_n)_{n\geq 1}$ of functions such that $\varphi_n(l) < l$ for $l \in]0,+\infty[$ and $\lim_{n\to+\infty}\varphi_n(l) = l$. Thus by condition (1), $\tau_{\varphi_n(l)}$ converges a.s. to τ_l when $n \to +\infty$. The sequence of random variables $(\int_{\tau_{\varphi_n(l)}\wedge T}^{\tau_l \wedge T} \sigma_t^0 \psi\, dW_t)_{n\geq 1}$ converges a.s. to 0. Then by Fatou's lemma, we have

$$\liminf_{n\to+\infty} \mathbb{E}[1_{T\geq\tau_l} p_T(l) X^{\varphi_n,\psi}_{\tau_l}] \geqslant \mathbb{E}[1_{T\geq\tau_l} p_T(l) X_0(1+\gamma_{\tau_l}\psi)],$$

which implies the first assertion since $\mathbb{P}(\tau_l \leq T) > 0$ by condition (2) and ψ is arbitrary. We use a similar argument and the assumption on U to obtain

$$\lim_{\psi \to +\infty} \mathbb{E}[1_{T \geq \tau_l} p_T(l) U(X_0(1 + \gamma_{\tau_l} \psi))]_{l=L} = +\infty.$$

The case of no buying constraint and $\gamma < 0$ is similar by considering the symmetric strategy $\pi_t^0(l) = \psi 1_{\tau_{\varphi(l)} < t}$. □

Remark 3.6 The strategies mentioned in this proof are not arbitrage strategies because for any fixed function φ as in the proof, $\mathbb{P}(T \in [\![\tau_{\varphi(L)}, \tau[\![) > 0$ and on this event, the strategy of the insider that consists of betting on the default before maturity T turns out to be a wrong bet. Thus, on a non null probability set, the strategy of a standard investor outperforms the one of the insider.

4 Solving the optimization problem

In this section, we concentrate on solving the optimization problem (3.6)

$$\sup_{\pi \in \mathcal{A}_l} \mathbb{E}\left[p_T(l) \left(1_{T < \tau_l} U(X_T^0(l)) + 1_{T \geq \tau_l} U(X_T^1(\tau_l)) \right) \right]$$

for any fixed $l > 0$. We recall that the before-default and after-default wealth processes X^0 and X^1 are governed by two control processes π^0 and π^1 respectively, so we need to search for a couple of optimal controls $\hat{\pi} = (\hat{\pi}^0, \hat{\pi}^1)$. In the following Theorem 4.1 we explain how to decompose the optimization problem into two problems each depending only on π^0 and on π^1 respectively.

The after-default optimization problem, whose control parameter is π^1 only, will be solved firstly using the filtration $\mathbb{F}^1$:

$$\mathbb{F}^1 := (\mathcal{F}_{\tau_l \vee t})_{t \in [0,T]}.$$

Remark that the initial σ-field of the filtration $\mathbb{F}^1$ is not trivial,

$$\mathcal{F}_0^1 = \mathcal{F}_{\tau_l} = \left\{ A \in \mathfrak{A} : A \cap \{\tau_l \leq t\} \in \mathcal{F}_t, t \in [0, T] \right\}$$

and τ_l is $\mathcal{F}_{\tau_l}$-measurable. All the $\mathbb{F}^1$-adapted processes are indexed on the right-upper side by the symbol "1". In particular, we denote by $X^{1,x_l}(\tau_l)$ the solution of the SDE (3.3) defined on the stochastic interval $[\![\tau_l, T]\!]$ starting from the $\mathbb{F}$-stopping time τ_l with $\mathcal{F}_{\tau_l}$-measurable initial value x_l. We define by $\mathcal{A}_l^1$ the admissible predictable strategy set $(\pi_t^1(\tau_l), t \in]\!]\tau_l, T]\!])$ such that $\int_{\tau_l}^{\tau_l \vee T} |\pi_t^1(\tau_l)\sigma_t^1(\tau_l)|^2 dt < \infty$ a.s..

The global before-default optimization problem, whose control parameter is π^0 only, involves the solution of the after-default optimization problem and will be solved in a second step, using the stopped filtration $\mathbb{F}^0$

$$\mathbb{F}^0 := (\mathcal{F}_{\tau_l \wedge t})_{t \in [0,T]}.$$

All the $\mathbb{F}^0$-adapted processes are indexed on the right-upper side by the symbol "0". The admissible predictable strategy set $\mathcal{A}_l^0$ is $(\pi_t^0(l), t \in [\![0, \tau_l \wedge T]\!])$ such that $\int_0^{\tau_l \wedge T} |\pi_t^0(l)\sigma_t^0|^2 dt < \infty, 0 \leq \pi_t^0(l) \leq \delta_b$ and $1 > \pi_{\tau_l}^0(l)\gamma_{\tau_l}$, a.s..

Theorem 4.1 *Let*

$$V^1_{\tau_l}(x_l) := \operatorname*{ess\,sup}_{\pi^1(\tau_l)\in\mathcal{A}^1_l} \mathbb{E}[p_T(l)U(X^{1,x_l}_T(\tau_l))|\mathcal{F}_{\tau_l}]. \tag{4.1}$$

Then $V_0(l)$ *defined in* (3.6) *can be written as the solution of the global optimization problem as*

$$V_0(l) = \sup_{\pi^0\in\mathcal{A}^0_l} \mathbb{E}\left[1_{T<\tau_l} p_T(l)U(X^0_T(l)) + 1_{T\geq\tau_l} V^1_{\tau_l}\big(X^0_{\tau_l}(l)(1-\pi^0_{\tau_l}(l)\gamma_{\tau_l})\big)\right]. \tag{4.2}$$

Proof Consider firstly an arbitrary admissible strategy $(\pi^0, \pi^1(\cdot)) \in \mathcal{A}_l$ for a fixed $l > 0$. By definition, $(\pi^0_t,\ t \in [\![0, \tau_l \wedge T]\!]) \in \mathcal{A}^0_l$ and $(\pi^1_t(\tau_l),\ t \in]\!]\tau_l, T]\!]) \in \mathcal{A}^1_l$. Taking the conditional expectation with respect to $\mathcal{F}_{\tau_l}$ leads to the following inequalities

$$\begin{aligned}
&\mathbb{E}\left[p_T(l)\left(1_{T<\tau_l}U(X^0_T(l)) + 1_{T\geq\tau_l}U(X^1_T(\tau_l))\right)\right]\\
&= \mathbb{E}\left[1_{T<\tau_l} p_T(l)U(X^0_T(l)) + 1_{T\geq\tau_l}\mathbb{E}\left(p_T(l)U(X^1_T(\tau_l))|\mathcal{F}_{\tau_l}\right)\right]\\
&\leq \mathbb{E}\left[1_{T<\tau_l} p_T(l)U(X^0_T(l)) + 1_{T\geq\tau_l} V^1_{\tau_l}(X^0_{\tau_l}(l)(1-\pi^0_{\tau_l}(l)\gamma_{\tau_l}))\right]\\
&\leq V_0(l).
\end{aligned}$$

For the converse inequality, let us assume for the moment that the esssup in the definition (4.1) is achieved for a given $\hat{\pi}^1(\tau_l)$ (see section 4.1 for the proof). Then for any $(\pi^0_t, t \in [\![0, \tau_l]\!])$ in $\mathcal{A}^0_l$, by a measurable selection theorem, there exists a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable process $\pi^1(\cdot)$ such that $\pi^1(\tau_l) = \hat{\pi}^1(\tau_l)$ on $]\!]\tau_l, T]\!]$ and $(\pi^0, \pi_1(\cdot)) \in \mathcal{A}_l$, where we have extended π^0 to an $\mathbb{F}$-predictable process on $\mathbb{R}_+$. Thus

$$V_0(l) \geq \mathbb{E}\left[1_{T<\tau_l} p_T(l)U(X^0_T(l)) + 1_{T\geq\tau_l} V^1_{\tau_l}(X^0_{\tau_l}(l)(1-\pi^0_{\tau_l}(l)\gamma_{\tau_l}))\right]$$

By taking the supremum over all $(\pi^0_t(l), t \in [\![0, \tau_l]\!]) \in \mathcal{A}^0_l$, we obtain the desired inequality. □

Remark 4.2 The supremum in $V_0(l)$ can be approached by a sequence of admissible strategies in $\mathcal{A}^0_l$ (see Proposition 4.11), which induces a sequence of strategies in $\mathcal{A}_L$ such that the corresponding value functions converge to $\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_L}\mathbb{E}[U(X_T)|\mathcal{G}^M_0]$.

Remark 4.3 The process $(p_t(l), t \in [0, T])$ is essential in our approach of initial information. From a technical point of view, it plays a similar role to the default density process in [15] $(\alpha_t(\theta), t \in [0, T])$ defined as $\alpha_t(\theta)d\theta = \mathbb{P}(\tau \in d\theta|\mathcal{F}_t)$. From the modeling point of view, it is useful to compare the two processes.

- In the particular case where the $\mathcal{F}_t$-conditional law of L admits a density $g_t(l)$ with respect to the Lebesgue measure, the default density can be completely deduced (see [6, Proposition 3]) in this framework as $\alpha_t(\theta) = \lambda_\theta g_t(\Lambda_\theta)$ for $t \geq \theta$ and $\alpha_t(\theta) = \mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]$ for $t < \theta$ where λ is the process given in Sect. 2.
- In the general case, the law of L can have atoms, then the default density does not exist and the approach in [15] is no longer valid, whereas the insider's optimization problem can be solved with the process $p_.(l)$.

4.1 The after-default optimization

In this section, we focus on the optimization problem (4.1) in the filtration $\mathbb{F}^1 = (\mathcal{F}_{\tau_l \vee t})_{t\in[0,T]}$

$$V^1_{\tau_l}(x_l) = \operatorname*{ess\,sup}_{\pi^1(\tau_l)\in\mathcal{A}^1_l} \mathbb{E}[p_T(l)U(X^{1,x_l}_T(\tau_l))|\mathcal{F}_{\tau_l}]$$

where τ_l is an $\mathbb{F}$-stopping time and the initial after-default wealth x_l is $\mathcal{F}_{\tau_l}$-measurable. In the following, we call this problem the after-default optimization problem because it involves the strategy process π^1 and the wealth processes X^1 only after default. However, we note that it is an intermediary optimization problem of the global one which is formulated at the initial date. Therefore, it is not surprising that it depends on the conditional density process $p_.(l)$ of the default threshold.

This problem is similar to a standard optimization problem, we will extend the results in our framework where the initial time τ_l is a random time (and is an $\mathbb{F}$-stopping time). We define the process

$$Z_t(\tau_l) = \exp\Big(-\int_{\tau_l}^{\tau_l\vee t} \frac{\mu^1_u(\tau_l)}{\sigma^1_u(\tau_l)} dW_u - \frac{1}{2}\int_{\tau_l}^{\tau_l\vee t} \left|\frac{\mu^1_u(\tau_l)}{\sigma^1_u(\tau_l)}\right|^2 du\Big), \quad t\in[0,T].$$

This process is an $\mathbb{F}^1$-local martingale (cf. [18, page 20]), we assume that the coefficients $\mu^1(\tau_l)$ and $\sigma^1(\tau_l)$ satisfy a Novikov criterion (see Theorem 4.4 below) so that $(Z_t(\tau_l))_{t\in[0,T]}$ is an $\mathbb{F}^1$-martingale.

Theorem 4.4 *We assume that for any $l>0$, the coefficients $\mu^1_u(\tau_l)$ and $\sigma^1_u(\tau_l)$ satisfy the Novikov criterion*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{\tau_l}^{\tau_l\vee T}\left|\frac{\mu^1_u(\tau_l)}{\sigma^1_u(\tau_l)}\right|^2 du\right)\right] < \infty.$$

Then the value function process to problem (4.1) *is a.s. finite and is given by*

$$\hat{V}^1_{\tau_l}(x_l) = \mathbb{E}\left[p_T(l)U\left(I\big(\hat{y}_{\tau_l}(x_l)\frac{Z_T(\tau_l)}{p_T(l)}\big)\right)\Big|\mathcal{F}^1_0\right]$$

where $I=(U')^{-1}$ and the Lagrange multiplier $\hat{y}_{\tau_l}(\cdot)$ is the unique $\mathcal{F}_{\tau_l}\otimes\mathcal{B}(\mathbb{R}_+)$-measurable solution of the equation

$$\frac{1}{Z_t(\tau_l)}\mathbb{E}\left[Z_T(\tau_l)I\big(\hat{y}_{\tau_l}(x_l)\frac{Z_T(\tau_l)}{p_T(l)}\big)\Big|\mathcal{F}^1_0\right] = x_l.$$

The corresponding optimal wealth is equal to

$$\hat{X}^{1,x_l}_t(\tau_l) = \frac{1}{Z_t(\tau_l)}\mathbb{E}\left[Z_T(\tau_l)I\big(\hat{y}_{\tau_l}(x_l)\frac{Z_T(\tau_l)}{p_T(l)}\big)\Big|\mathcal{F}^1_t\right], \quad t\in[\![\tau_l,T]\!]. \tag{4.3}$$

Proof Note that after the default, the market is complete, characterized by the state price density process $Z(\tau_l)$. The process $Z(\tau_l)X^{1,x_l}(\tau_l)$ is a positive local $\mathbb{F}^1$-martingale, and thus a supermartingale, leading to the following budget constraint

$$\mathbb{E}\left(Z_T(\tau_l)X_T^{1,x_l}(\tau_l)\,\Big|\,\mathcal{F}_0^1\right) \le x_l.$$

Conversely, the martingale representation theorem on the Brownian filtration implies that for any $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable $X_T(\cdot)$, there exists a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$- measurable process $\phi(\cdot)$ such that

$$X_T(\tau_l)1_{\tau_l<T} = \mathbb{E}(X_T(\tau_l)1_{\tau_l<T}|\mathcal{F}_{\tau_l}) + \int_{\tau_l}^{\tau_l\vee T} \phi_u(\tau_l)du$$

Therefore the after default optimization problem is solved by means of the Lagrange multiplier

$$V_{\tau_l}^1(x_l) = \mathbb{E}\Big[p_T(l)U\Big(I\big(\hat{y}_{\tau_l}(x_l)\frac{Z_T(\tau_l)}{p_T(l)}\big)\Big)|\mathcal{F}_0^1\Big]$$

and the optimal wealth is given by

$$\hat{X}_t^{1,x_l}(\tau_l) = \frac{1}{Z_t(\tau_l)}\mathbb{E}\left[Z_T(\tau_l)I\big(\hat{y}_{\tau_l}(x_l)\frac{Z_T(\tau_l)}{p_T(l)}\big)|\mathcal{F}_t^1\right]$$

where $I = (U')^{-1}$ and the Lagrange multiplier $\hat{y}_{\tau_l}(x_l)$ is $\mathcal{F}_{\tau_l}\otimes\mathcal{B}(\mathbb{R}_+)$-measurable and satisfies $\hat{X}_{\tau_l}^{1,x_l}(\tau_l) = x_l$. The existence, uniqueness and measurability of the Lagrange multiplier $\hat{y}_{\tau_l}(x_l)$ in the case of a non trivial initial σ-field is proved in Proposition 4.5 of Hillairet [8]. □

Remark 4.5 Concerning the optimal wealth $\hat{X}^{1,x_l}(\tau_l)$ in (4.3), we can prove[2] that starting from a same wealth x_l at the default time τ_l, the optimal wealth process of the after-default optimization problem is the same for the initial (insider) and the progressive (standard investor) information. This result is natural, since after the default, the two information flows coincide. But the input wealth of the after-default optimization problem will not be the same for the two information flows since they are different before τ_l.

We will quantify numerically the gain of an insider compared to a standard investor for the global optimization problem. We now consider, as in [15], Constant Relative Risk Aversion (CRRA) utility functions

$$U(x) = \frac{x^p}{p}, \quad 0 < p < 1,\ x > 0$$

and $I(x) = x^{\frac{1}{p-1}}$. Direct computations from the previous theorem yield the optimal wealth

[2] The proof is based on the relationship between the two key processes $p(l)$ and $\alpha(\tau_l)$. More precisely, the coefficient $\frac{p_T(l)}{\alpha_T(\tau_l)}$ coincides with the ratio between the Lagrange multipliers for respectively the insider and the standard investor.

$$\hat{X}_t^{1,x_l}(\tau_l) = \frac{x_l}{Z_t(\tau_l)} \frac{\mathbb{E}\left[p_T(l)\left(\frac{Z_T(\tau_l)}{p_T(l)}\right)^{\frac{p}{p-1}} \Big| \mathcal{F}_t^1\right]}{\mathbb{E}\left[p_T(l)\left(\frac{Z_T(\tau_l)}{p_T(l)}\right)^{\frac{p}{p-1}} \Big| \mathcal{F}_{\tau_l}\right]}, \quad t \in [\![\tau_l, T]\!]$$

and the optimal value function

$$\hat{V}_{\tau_l}^1(x_l) = \frac{x_l^p}{p}\left(\mathbb{E}\left[p_T(l)\left(\frac{Z_T(\tau_l)}{p_T(l)}\right)^{\frac{p}{p-1}} \Big| \mathcal{F}_{\tau_l}\right]\right)^{1-p} =: \frac{x_l^p}{p} K_{\tau_l} \qquad (4.4)$$

where $K_{\tau_l} = (\mathbb{E}[p_T(l)(\frac{Z_T(\tau_l)}{p_T(l)})^{\frac{p}{p-1}} \mid \mathcal{F}_{\tau_l}])^{1-p}$ is $\mathcal{F}_{\tau_l}$-measurable and only depends on the stopping time τ_l and on market parameters.

4.2 The global before-default optimization

We now consider the optimization problem (4.2) with CRRA utility functions. Using (4.4), we have to solve :

$$V_0(l) = \sup_{\pi^0 \in \mathcal{A}_l^0} \mathbb{E}\Big[1_{T<\tau_l} p_T(l) U(X_T^0(l)) + 1_{T \geq \tau_l} K_{\tau_l} U\big(X_{\tau_l}^0(l)(1 - \pi_{\tau_l}^0(l)\gamma_{\tau_l})\big)\Big]$$

where the $\mathcal{F}_{\tau_l}$-measurable random variable K_{τ_l} does not depend on the control process $\pi^0 \in \mathcal{A}_l^0$. We will use a dynamic programming approach. Recall that $\mathbb{F}^0 = (\mathcal{F}_{\tau_l \wedge t})_{t \in [0,T]}$ is the stopped filtration at the default time. Since $1_{t<\tau_l}$ is $\mathcal{F}_{\tau_l \wedge t}$-measurable, we have

$$\begin{aligned}
&\mathbb{E}\big[1_{T<\tau_l} p_T(l) U(X_T^0(l)) + 1_{T \geq \tau_l} K_{\tau_l} U\big(X_{\tau_l}^0(l)(1 - \pi_{\tau_l}^0(l)\gamma_{\tau_l})\big) | \mathcal{F}_{\tau_l \wedge t}\big] \\
&= 1_{t \geq \tau_l} K_{\tau_l} U\left(X_{\tau_l}^0(l)(1 - \pi_{\tau_l}^0(l)\gamma_{\tau_l})\right) \\
&\quad + \mathbb{E}\big[1_{T<\tau_l} p_T(l) U(X_T^0(l)) + 1_{t<\tau_l \leq T} K_{\tau_l} U\big(X_{\tau_l}^0(l)(1 - \pi_{\tau_l}^0(l)\gamma_{\tau_l})\big) | \mathcal{F}_{\tau_l \wedge t}\big]
\end{aligned}$$

For any $\nu \in \mathcal{A}_l^0$, we introduce the family of $\mathbb{F}^0$-adapted processes

$$\mathcal{X}_t(\nu) := \operatorname*{ess\,sup}_{\pi^0 \in \mathcal{A}_l^0(t,\nu)} \mathbb{E}\Big[1_{T<\tau_l} p_T(l) U(X_T^0(l)) + 1_{t<\tau_l \leq T} K_{\tau_l} U\big(X_{\tau_l}^0(l)(1 - \pi_{\tau_l}^0(l)\gamma_{\tau_l})\big) | \mathcal{F}_{\tau_l \wedge t}\Big]$$

where $\mathcal{A}_l^0(t,\nu)$ is the set of controls coinciding with ν until time t: for any $t \in [0, T]$, $\nu \in \mathcal{A}_l^0$, $\mathcal{A}_l^0(t,\nu) = \{\pi^0 \in \mathcal{A}_l^0 : \pi_{.\wedge t}^0 = \nu_{.\wedge t}\}$. We have $V_0(l) = \mathcal{X}_0(\nu)$ for any $\nu \in \mathcal{A}_l^0$.

In the following result, we show that the strategy constraints play a crucial role in the optimization: the optimal strategy at the default time is the short-selling (resp. buying) constraint bound if γ is positive (resp. negative).

Proposition 4.6 *For any $\pi^0 \in \mathcal{A}_l^0$, there exists a sequence of strategies $(\pi_n^0 \in \mathcal{A}_l^0)_{n \in \mathbb{N}^*}$ such that $\pi_{n,\tau_l}^0 = \delta_b 1_{\{\gamma<0\}}$ and*

$$\lim_{n \to +\infty} \mathcal{X}_0(\pi_n^0) \geq \mathcal{X}_0(\pi^0).$$

Proof Let $(\tau_n)_{n\in\mathbb{N}^*}$, with $\tau_n < \tau_l$, be an increasing sequence of $\mathbb{F}$-stopping times that converge to τ_l. Starting from a strategy $\pi^0 \in \mathcal{A}_l^{0,\delta}$, we define another strategy $\pi_n^0 = 1_{[\![0,\tau_n]\!]}\pi^0 + 1_{]\!]\tau_n,\tau_l]\!]}\delta_b 1_{\{\gamma<0\}}$ that remains in $\mathcal{A}_l^{0,\delta}$. We denote as $X^0(l)$ and $X_n^0(l)$ the corresponding wealth before default, and as $\mathcal{X}_0(\pi^0)$ and $\mathcal{X}_0(\pi_n^0)$ the corresponding value function for those strategies of the before-default global optimization problem. On the one hand, by dominated convergence theorem, it is easy to check that

$$\lim_{n\to+\infty} \mathbb{E}\big[1_{T<\tau_l} p_T(l)|U(X_T^0(l)) - U(X_{n,T}^0(l))|\big] = 0.$$

On the other hand, on the event $\{T \geq \tau_l\}$

$$\frac{X_{n,\tau_l}^0(l)(1-\pi_{n,\tau_l}^0(l)\gamma_{\tau_l})}{X_{\tau_l}^0(l)(1-\pi_{\tau_l}^0(l)\gamma_{\tau_l})} = \frac{(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l})}{(1-\pi_{\tau_l}^0(l)\gamma_{\tau_l})}$$
$$\times \exp\left(\int_{\tau_n}^{\tau_l}(-(\pi_s^0-\delta_b 1_{\{\gamma<0\}})\mu_s^0 + \frac{1}{2}(\sigma_s^0)^2((\pi_s^0)^2 - 1_{\{\gamma<0\}}\delta_b^2))ds - \int_{\tau_n}^{\tau_l}(\pi_s^0 - \delta_b 1_{\{\gamma<0\}})\sigma_s^0 dW_s\right)$$

$\pi^0 \in \mathcal{A}_l^0$ implies that $0 \leq \pi_{\tau_l}^0(l) \leq \delta_b$ and $\frac{(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l})}{(1-\pi_{\tau_l}^0(l)\gamma_{\tau_l})} \geq 1$.
$\tau_n \to \tau_l$ implies that the exponential term tends to 1 a.s.. Thus

$$\lim_{n\to+\infty} \frac{X_{n,\tau_l}^0(l)(1-\pi_{n,\tau_l}^0(l)\gamma_{\tau_l})}{X_{\tau_l}^0(l)(1-\pi_{\tau_l}^0(l)\gamma_{\tau_l})} \geq 1 \quad \text{and}$$

$$\lim_{n\to+\infty} \mathbb{E}\left[1_{T\geq\tau_l} K_{\tau_l} U(X_{n,\tau_l}^0(l)(1-\pi_{n,\tau_l}^0(l)\gamma_{\tau_l}))\right]$$
$$\geq \mathbb{E}\left[1_{T\geq\tau_l} K_{\tau_l} U(X_{\tau_l}^0(l)(1-\pi_{\tau_l}^0(l)\gamma_{\tau_l}))\right].$$

Consequently,

$$\lim_{n\to+\infty} \mathcal{X}_0(\pi_n^0) \geq \mathcal{X}_0(\pi^0).$$

□

We now characterize the optimal strategy process. Let $X^{\nu,0}$ denote the wealth process derived from the control $\nu \in \mathcal{A}_l^0$. From the dynamic programming principle, the following result holds:

Lemma 4.7 *For any* $\nu \in \mathcal{A}_l^0$, *the process*

$$\xi_t^\nu := \mathcal{X}_t(\nu) + 1_{t\geq\tau_l} K_{\tau_l} U(X_{\tau_l}^{\nu,0}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l})), \quad 0 \leq t \leq T$$

is an $\mathbb{F}^0$*-supermartingale. Furthermore, the optimal strategy* $\widehat{\pi}^0$ *is characterized by the martingale property :* $(\xi_t^{\widehat{\pi}^0})_{0\leq t\leq T}$ *is an* $\mathbb{F}^0$*-martingale.*

Proof Let s, t be two times such that $s \leq t \leq T$.

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{X}_t(\nu) + 1_{t\geq\tau_l} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge s}\Big] \\
&= \mathbb{E}\Big[\mathcal{X}_t(\nu) + 1_{s<\tau_l\leq t} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge s}\Big] \\
&\quad + 1_{s\geq\tau_l} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))
\end{aligned}$$

We make explicit the conditional expectation :

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{X}_t(\nu) + 1_{s<\tau_l\leq t} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge s}\Big] \\
&= \mathbb{E}\Big[\operatorname*{ess\,sup}_{\pi^0\in\mathcal{A}^0_l(t,\nu)} \mathbb{E}\big[1_{T<\tau_l} p_T(l) U(X^0_T(l)) + 1_{t<\tau_l\leq T} K_{\tau_l} U(X^0_{\tau_l}(l)(1-\pi^0_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge t}\big] \\
&\quad + 1_{s<\tau_l\leq t} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge s}\Big] \\
&\leq \operatorname*{ess\,sup}_{\pi^0\in\mathcal{A}^0_l(s,\nu)} \mathbb{E}\big[1_{T<\tau_l} p_T(l) U(X^0_T(l)) + 1_{s<\tau_l\leq T} K_{\tau_l} U(X^0_{\tau_l}(l)(1-\pi^0_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge s}\big]
\end{aligned} \tag{4.5}$$

the last inequality following from the fact that in the last esssup, the optimal control is taken from the date $s \leq t$. Thus

$$\begin{aligned}
&\mathbb{E}\Big[\mathcal{X}_t(\nu) + 1_{t\geq\tau_l} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))|\mathcal{F}_{\tau_l\wedge s}\Big] \\
&\leq \mathcal{X}_s(\nu) + 1_{s\geq\tau_l} K_{\tau_l} U(X^{\nu,0}_{\tau_l}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l}))
\end{aligned}$$

and $(\xi^\nu_t)_{0\leq t\leq T}$ is an $\mathbb{F}^0$-supermartingale. It is an $\mathbb{F}^0$-martingale if and only if the inequality (4.5) is an equality for all $t \in [0, T]$, meaning that ν is the optimal control on $[0, t]$, for all $t \leq T$. This characterizes the optimal strategy. □

Remark that the $\mathbb{F}^0$-adapted process defined for $0 \leq t \leq T$ as

$$\begin{aligned}
Y_t &:= \frac{\mathcal{X}_t(\nu)}{U(X^{\nu,0}_t(l))} \\
&= \operatorname*{ess\,sup}_{\pi^0\in\mathcal{A}^0_l(t,\nu)} \mathbb{E}\Big[1_{T<\tau_l} p_T(l)\Big(\frac{X^0_T(l)}{X^{\nu,0}_t(l)}\Big)^p + 1_{t<\tau_l\leq T} K_{\tau_l}\Big(\frac{X^0_{\tau_l}(l)}{X^{\nu,0}_t(l)}\Big)^p (1-\pi^0_{\tau_l}(l)\gamma_{\tau_l})|\mathcal{F}_{\tau_l\wedge t}\Big]
\end{aligned} \tag{4.6}$$

does not depend on $\nu \in \mathcal{A}^0_l$, and is constant after τ_l. We will give a characterization of the process Y in terms of a backward stochastic differential equation (BSDE) and of the optimal strategy. Before this, we give a characterization of $\mathbb{F}^0$-martingale.

Lemma 4.8 *Let $(M_t)_{t\in[0,T]}$ be an $\mathbb{F}^0$-martingale. Then there exists an $\mathbb{F}$-predictable process ϕ in $L^2_{loc}(W)$ such that $M_t = M_0 + \int_0^t \phi_s 1_{s\leq\tau_l} dW_s,\ t \in [0, T]$.*

Proof We first prove that $(M_t)_{t\in[0,T]}$ is also an $\mathbb{F}$-martingale. Indeed, for $s \le t \le T$

$$M_s = \mathbb{E}(M_t|\mathcal{F}_{\tau_l\wedge s}) = \mathbb{E}(\mathbb{E}(M_t|\mathcal{F}_{\tau_l})|\mathcal{F}_s) = \mathbb{E}(M_t|\mathcal{F}_s)$$

because M_t is $\mathcal{F}_{\tau_l}$-measurable. Thus, by representation theorem for the $\mathbb{F}$-martingale, and since $(M_t)_{t\in[0,T]}$ is stopped at time τ_l, there exists ϕ an $\mathbb{F}$-predictable process such that $M_t = M_0 + \int_0^t \phi_s 1_{s\le\tau_l} dW_s$. □

We are now ready to characterize the optimal strategy. Remark that $Y_t = \frac{\mathcal{X}_t(\nu)}{U(X_t^{\nu,0}(l))}$ is positive on $[\![0,\tau_l[\![$ (and zero after τ_l) thus $Y \in L_l^+(\mathbb{F}^0)$ where $L_l^+(\mathbb{F}^0)$ is the set of $\mathbb{F}^0$-adapted processes $\tilde{Y}$ such that $\tilde{Y}_t > 0$ for $t\in[\![0,\tau_l[\![$ and $\tilde{Y}_t = 0$ for $t\in[\![\tau_l,+\infty[\![$.

Theorem 4.9 *The process Y defined in (4.6) is the smallest solution in $L_l^+(\mathbb{F}^0)$ to the BSDE: for any $t\in[\![0,T\wedge\tau_l]\!]$,*

$$Y_t = 1_{T<\tau_l}\, p_T(l) + 1_{t<\tau_l\le T} K_{\tau_l}\frac{(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l})^p}{p} + \int_t^{T\wedge\tau_l} f(\theta, Y_\theta, \phi_\theta)d\theta - \int_t^{T\wedge\tau_l}\phi_\theta dW_\theta, \tag{4.7}$$

for some $\phi\in L^2_{loc}(W)$, and where

$$f(s, Y_s, \phi_s) = p \operatorname*{ess\,sup}_{\nu\in\mathcal{A}_l^0, s.t.\ \nu_{\tau_l}=\delta_b 1_{\{\gamma<0\}}} \Big[(\mu_s^0 Y_s + \sigma_s^0\phi_s)\nu_s - \frac{1-p}{2}Y_s|\nu_s\sigma_s^0|^2\Big].$$

Remark 4.10 As in Theorem 4.2 in [15], the optimal strategy before default is characterized through the optimization of the driver of a BSDE. However, the main difference relies in the fact that in our case, the driver has a jump at the default time τ_l. Nevertheless, since the jump occurs (if it occurs) only at the terminal date of the BSDE, standard theory on BSDE still applies.

Proof By Lemma 4.7, for any $\nu\in\mathcal{A}_l^0$

$$\xi_t^\nu = U(X_t^{\nu,0}(l))Y_t + 1_{t\ge\tau_l}K_{\tau_l}U(X_{\tau_l}^{\nu,0}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l})\quad 0\le t\le T$$

is an $\mathbb{F}^0$-supermartingale. In particular, by taking $\nu = 0$, we see that the processes $(Y_t + K_{\tau_l}1_{t\ge\tau_l})_{0\le t\le T}$, and thus $(Y_t)_{0\le t\le T}$ are $\mathbb{F}^0$-supermartingales. By the Doob-Meyer decomposition and Lemma 4.8, there exists $\phi\in L^2_{loc}(W)$, and a finite variation increasing $\mathbb{F}^0$-predictable process A such that:

$$dY_t = \phi_t dW_t - dA_t, \quad t\in[\![0,T\wedge\tau_l]\!].$$

From Itô's formula, we deduce that the finite variation process in the decomposition of the $\mathbb{F}^0$-supermartingale ξ^ν, $\nu\in\mathcal{A}_l^0$, is given by $-A^\nu$ with

$$dA_t^\nu = (X_t^{\nu,0}(l))^p\Big\{\frac{1}{p}dA_t - (\mu_t^0 Y_t + \sigma_t^0\phi_t 1_{t\le\tau_l})\nu_t dt + \frac{1-p}{2}Y_t|\nu_t\sigma_t^0|^2 dt - K_t\frac{(1-\nu_t\gamma_t)^p}{p}d1_{t\ge\tau_l}\Big\}.$$

A^ν is nondecreasing and the martingale property of $\xi^{\hat\pi^0}$ implies that

$$dA_t = p\Big[(\mu_t^0 Y_t + \sigma_t^0\phi_t 1_{t\le\tau_l}))\hat\pi_t^0 dt - \frac{1-p}{2}Y_t|\hat\pi_t^0\sigma_t^0|^2 dt + K_t\frac{(1-\hat\pi_t^0\gamma_t)^p}{p}d1_{t\ge\tau_l}\Big]$$

and

$$A_t = A_0 + p\operatorname*{ess\,sup}_{\nu\in\mathcal{A}_l^0}\Big[\int_0^t \big((\mu_s^0 Y_s + \sigma_s^0\phi_s)\nu_s - \frac{1-p}{2}Y_s|\nu_s\sigma_s^0|^2\big)ds + 1_{t\ge\tau_l}K_{\tau_l}\frac{(1-\nu_{\tau_l}\gamma_{\tau_l})^p}{p}\Big].$$

Maximizing at τ_l leads to $\nu_{\tau_l} = \delta_b 1_{\{\gamma<0\}}$ (see Proposition 4.6) and

$$A_t = A_0 + p\operatorname*{ess\,sup}_{\nu\in\mathcal{A}_l^0 s.t.\ \nu_{\tau_l}=\delta_b 1_{\{\gamma<0\}}}\Big[\int_0^t \big(\mu_s^0 Y_s + \sigma_s^0\phi_s)\nu_s - \frac{1-p}{2}Y_s|\nu_s\sigma_s^0|^2\big)ds\Big] + 1_{t\ge\tau_l}K_{\tau_l}\frac{(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l})^p}{p}.$$

Furthermore, $Y_T = 1_{T<\tau_l}p_T(l)$ and $(Y_t)_{0\le t\le T}$ is constant after τ_l , thus (Y,ϕ) solves the BSDE (4.7). Note that Y is not a continuous process, it may jump at time τ_l.

We now prove that Y is smallest solution in the $L_l^+(\mathbb{F}^0)$ to the BSDE (4.7). Let $\tilde Y\in L_l^+(\mathbb{F}^0)$ be another solution, and we define the family of nonnegative $\mathbb{F}^0$-adapted processes $\xi^\nu(\tilde Y)$, $\nu\in\mathcal{A}_l^0$, as

$$\xi_t^\nu(\tilde Y) = U(X_t^{\nu,0}(l))\tilde Y_t + 1_{t\ge\tau_l}K_{\tau_l}U(X_{\tau_l}^{\nu,0}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l})),\quad t\in[0,T].$$

By similar calculations as above, $d\xi_t^\nu(\tilde Y) = d\tilde M_t^\nu - d\tilde A_t^\nu$, where $\tilde A^\nu$ is a nondecreasing $\mathbb{F}^0$-adapted process, and $\tilde M^\nu$ is an $\mathbb{F}^0$-local martingale. By Fatou's lemma, this implies that the process $\xi^\nu(\tilde Y)$ is an $\mathbb{F}^0$-supermartingale, for any $\nu\in\mathcal{A}_l^0$. Since $\tilde Y_T = 1_{T<\tau_l}p_T(l)$, we deduce that for all $\nu\in\mathcal{A}_l^0$, for all $t\in[0,T]$

$$\mathbb{E}\Big[U(X_T^{\nu,0})1_{T<\tau_l}p_T(l) + 1_{t\ge\tau_l}K_{\tau_l}U(X_{\tau_l}^{\nu,0}(l)(1-\nu_{\tau_l}(l)\gamma_{\tau_l})\Big|\mathcal{F}_t^0\Big] \le U(X_t^{\nu,0})\tilde Y_t.$$

Since $p>0$, $U(X_t^{\nu,0})$ is positive. By dividing the above inequalities by $U(X_t^{\nu,0})$, we deduce by definition of Y (see (4.6)), and arbitrariness of $\nu\in\mathcal{A}_l^0$, that $Y_t\le\tilde Y_t$, $0\le t\le T$. This shows that Y is the smallest solution to the BSDE (4.7). □

For optimizing (4.6) via the BSDE (4.7), a naive approach would consist in optimizing π^0 at time τ_l, leading to an $\pi_{\tau_l}^0 = \delta_b 1_{\{\gamma<0\}}$, and then optimizing for $s<\tau_l$ the driver

$$f^0(s,Y_s^0,\phi_s^0) = \operatorname*{ess\,sup}_{0\le\nu_s\le\delta_b} p[\big(\mu_s^0 Y_s^0 + \sigma_s^0\phi_s^0)\nu_s - \frac{1-p}{2}Y_s^0|\nu_s\sigma_s^0|^2],$$

where Y^0 is solution to the BSDE: for $t \in [\![0, T \wedge \tau_l]\!]$

$$Y_t^0 = 1_{T<\tau_l} p_T(l) + 1_{t<\tau_l \le T} K_{\tau_l} \frac{(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l})^p}{p} + \int_t^{T\wedge\tau_l} f^0(\theta, Y_\theta^0, \phi_\theta^0) d\theta - \int_t^{T\wedge\tau_l} \phi_\theta^0 dW_\theta,$$

leading to the optimal portfolio $\hat{\pi}_s^0$. Thus, the natural candidate to be the optimal strategy before default is

$$\pi^{\rm np} := 1_{[\![0,\tau_l[\![}\hat{\pi}^0 + \delta_b 1_{\{\gamma<0\}} 1_{\{[\![\tau_l]\!]\}}, \tag{4.8}$$

but unfortunately $\pi^{\rm np}$ is not a predictable process. Nevertheless, we will prove the existence of a sequence of predictable strategies in $\mathcal{A}_l^0$ such that the corresponding value function tends to the value function relative to this non predictable strategy. To do this, for any strategy $\pi^0 \in \mathcal{A}_l^0$, we recall the corresponding value function of the before default global optimization problem

$$\mathcal{X}_0(\pi^0) = \mathbb{E}\Big[1_{T<\tau_l} p_T(l) U(X_T^{\pi^0}(l)) + 1_{T\ge\tau_l} K_{\tau_l} U(X_{\tau_l}^{\pi^0}(l)(1-\pi_{\tau_l}^0(l)\gamma_{\tau_l}))\Big]. \tag{4.9}$$

Note that (4.9) can also be defined for a strategy π that is predictable only on $[\![0, \tau_l[\![$ (and not necessarly on $[\![0, \tau_l]\!]$). Using Proposition 4.6, we have the following result:

Proposition 4.11 *Let $(\tau_n)_{n\in\mathbb{N}^*}$ be an increasing sequence of $\mathbb{F}$-predictable stopping times that converge to τ_l. We consider the strategies $(\pi_n^0 = 1_{[\![0,\tau_n]\!]}\hat{\pi}^0 + 1_{]\!]\tau_n,\tau_l]\!]}\delta_b 1_{\{\gamma<0\}})$ where $\hat{\pi}^0$ is the optimal process for the driver of the following BSDE: for $t \in [\![0, T\wedge\tau_l[\![$*

$$Y_t^0 = 1_{T<\tau_l} p_T(l) + 1_{t<\tau_l \le T} K_{\tau_l} \frac{(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l})^p}{p} + \int_t^{T\wedge\tau_l} f^0(\theta, Y_\theta^0, \phi_\theta^0) d\theta - \int_t^{T\wedge\tau_l} \phi_\theta^0 dW_\theta,$$
$$f^0(s, y, \phi) = p \sup_{0\le\nu\le\delta_b} \Big[(\mu_s^0 y + \sigma_s^0 \phi)\nu - \frac{1-p}{2} y |\nu\sigma_s^0|^2\Big].$$

Those strategies are in $\mathcal{A}_l^0$ and satisfy

$$\lim_{n\to+\infty} \mathcal{X}_0(\pi_n^0) = V_0(l) = \mathbb{E}\Big[1_{T<\tau_l} p_T(l) U(X_T^{\hat{\pi}^0}(l)) + 1_{T\ge\tau_l} K_{\tau_l} U(X_{\tau_l}^{\hat{\pi}^0}(l)(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l}))\Big].$$

Proof For any $n \in \mathbb{N}^*$, the strategy $\pi_n^0 := 1_{[\![0,\tau_n]\!]}\hat{\pi}^0 + 1_{]\!]\tau_n,\tau_l]\!]}\delta_b 1_{\{\gamma<0\}}$ is in $\mathcal{A}_l^0$, and π_n^0 converges to the non-predictable optimal strategy $\pi^{\rm np}$ defined in (4.8) when $n \to \infty$. Moreover, for any $n \in \mathbb{N}^*$, $\mathcal{X}_0(\pi_n^0) \le \mathcal{X}_0(\pi^{\rm np})$ and by Proposition 4.6

$$\mathcal{X}_0(\pi^{\rm np}) \ge \lim_{n\to+\infty} \mathcal{X}_0(\pi_n^0) \ge \mathcal{X}_0(\hat{\pi}^0).$$

But the proof of Proposition 4.6 still holds if we change the value at time τ_l of the portfolio π^0, thus the converse inequality $\mathcal{X}_0(\pi^{\mathrm{np}}) \leq \lim_{n\to+\infty} \mathcal{X}_0(\pi_n^0)$ holds and

$$\begin{aligned}
&\mathbb{E}\Big[1_{T<\tau_l} p_T(l) U(X_T^{\hat{\pi}^0}(l)) + 1_{T\geq\tau_l} K_{\tau_l} U(X_{\tau_l}^{\hat{\pi}^0}(l)(1-\delta_b 1_{\{\gamma<0\}}\gamma_{\tau_l}))\Big] \\
&= \mathcal{X}_0(\pi^{\mathrm{np}}) = \mathcal{X}_0(1_{[\![0,\tau_l[\![}\hat{\pi}^0 + \delta_b 1_{\{\gamma<0\}} 1_{\tau_l}) \\
&= \lim_{n\to+\infty} \mathcal{X}_0(1_{[\![0,\tau_n]\!]}\hat{\pi}^0 + 1_{]\!]\tau_n,\tau_l]\!]}\delta_b 1_{\{\gamma<0\}}) \\
&= \lim_{n\to+\infty} \mathcal{X}_0(\pi_n^0)
\end{aligned}$$

□

5 Numerical illustrations

We now illustrate our previous results by explicit models and we aim to compare the optimization results for an insider, a standard investor and a Merton investor. We recall that all investors start with an initial wealth X_0. For the purpose of comparison, we choose a similar model as the one studied in [15]. More precisely, we let the parameters μ^0, σ^0, γ be constant, and $\mu^1(\theta), \sigma^1(\theta)$ are deterministic functions of θ given by

$$\mu^1(\theta) \ = \ \mu^0\frac{\theta}{T}, \ \ \sigma^1(\theta) \ = \ \sigma^0(2-\frac{\theta}{T}), \quad \theta \in [0, T],$$

which means that the ratio of the after-default and before-default for the return rate of the asset is smaller than 1 and for the volatility is larger than 1. Moreover, these ratios increase or decrease linearly with the default time respectively: the after-default rate of return drops to zero, when the default time occurs near the initial date, and converges to the before-default rate of return, when the default time occurs near the finite investment horizon. For the volatility, this ratio converges to the double (resp. initial) value of the before-default volatility, when the default time goes to the initial (resp. terminal horizon) time. Moreover, in order to satisfy the hypothesis in the simulation part of [15], we have to assume that the default barrier L has no atoms (to ensure the density hypothesis, see Remark 4.3) and that L is independent of the filtration $\mathbb{F}$ (so that the default density is a deterministic function). In this case, $p_T(L) = 1$. For the default event, we consider a simple case where the process $(\lambda_t, t \geq 0)$ is a constant $\lambda > 0$ and the default threshold L follows the uni-exponential law. Hence, $\mathbb{P}(\tau > t) = e^{-\lambda t}$ and the default density is a deterministic function $\alpha(\theta) = \lambda e^{-\lambda\theta}$ for all $\theta \geq 0$.

Consider the CRRA utility $U(x) = \frac{x^p}{p}, 0 < p < 1$, the after-default value function is given from (4.4) by

$$V_{\tau_l}^1(x) = K_{\tau_l} U(x)$$

where

$$K_{\tau_l} = \Big(\mathbb{E}[Z_T(\tau_l)^{\frac{p}{p-1}}]\Big)^{1-p} = \exp\Big(\frac{1}{2}\frac{p}{1-p}\Big(\frac{\mu^1(\tau_l)}{\sigma^1(\tau_l)}\Big)^2(\tau_l \vee T - \tau_l)\Big)$$

Furthermore, the solution of the before-default optimization problem is given by

$$V_0(l) = Y_0 U(X_0)$$

where Y is the solution of the BSDE (4.7) when letting $\phi = 0$, i.e.,

$$Y_t = 1_{T<\tau_l} + 1_{T\geq\tau_l>t} K_{\tau_l} \frac{(1-\delta_b \gamma 1_{\{\gamma<0\}})^p}{p} + \int_t^{\tau_l \wedge T} f(\theta, Y_\theta) d\theta \tag{5.1}$$

where

$$f(t, y) = p \operatorname*{ess\,sup}_{\nu \in \mathcal{A}_l^0, \nu_{\tau_l} = \delta_b 1_{\{\gamma<0\}}} \{\mu^0 \nu_t - \frac{1-p}{2} (\nu_t \sigma^0)^2\} y.$$

We notice that in the case where the default time τ_l occurs after the maturity T, the optimal strategy coincides with the classical Merton strategy with constraint $\pi \in [0, \frac{1}{\gamma} 1_{\{\gamma>0\}} + \delta_b 1_{\{\gamma<0\}}[$ (Merton strategy does not take into account the eventuallity of the default). In the case where τ_l occurs before T, the process Y is stopped at τ_l, with the terminal value depending on the quantity K_{τ_l}, and the strategy at τ_l is equal to the trading constraint. We use an iterative Howard algorithm [11] to solve the equation (5.1). The following results are based on the model parameters described below: $\mu^0 = 0.03$, $\sigma^0 = 0.2$, $T = 1$, the risk aversion parameter $p = 0.7$ and the buying constraint $\delta_b = 1$. In addition, we fix the default intensity $\lambda = 0.3$. This corresponds to a relatively high default risk.

Figure 5.1 compares, for the insider, standard and Merton investors, the optimal value function and the wealth process for one given trajectory in the case of positive values for γ (that is a loss of the risky asset at the default time). The loss given default is $\gamma = 0.3$. At the default time which occurs before the maturity, the value function and the wealth process suffer a brutal loss for all the three strategies. For the value function, the insider outperforms the other two investors before and after the default occurs. Before the default, the value function for the standard investor is smaller than the Merton one because the latter does not consider at all the potential default risk. However, when the default occurs, the investor outperforms the Merton strategy since the default risk is taken into account from the beginning. For the wealth process, we observe that the insider's wealth coincides with the one of the Merton investor during a long time before the default occurs. However, due to her information on the default event, she can adjust the strategy just before the default in order not to be impacted by the loss of the risky asset. On the contrary, both the standard and Merton investors' wealth suffer a loss at the default time, the loss of the standard investor being less than for the Merton investor. Therefore the insider obtains the largest wealth after default.

Figure 5.2 and 5.3 consider the case of negative γ (that is a gain of the risk asset at the default time) with the other parameters being unchanged. We observe a similar phenomenon for the optimal value function with a loss at the default time for all investors. However, for all the three types of investors, the wealth process has a gain since the jump of the risky asset is positive. Besides, the profit of the insider is more important as the jump size $|\gamma|$ is larger.

Fig. 5.1 Value function and wealth process for insider, investor and Merton: $\gamma = 0.3$.

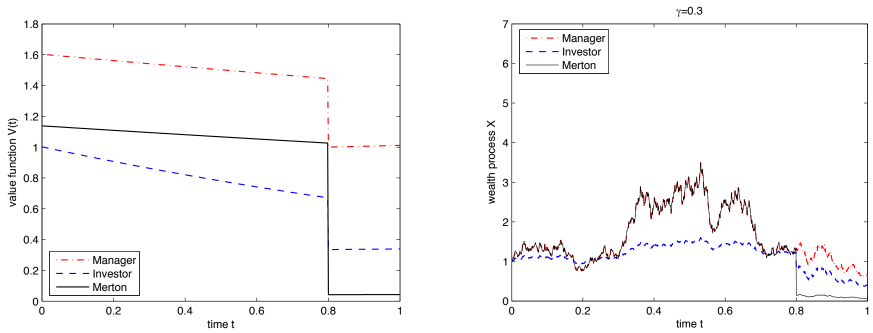
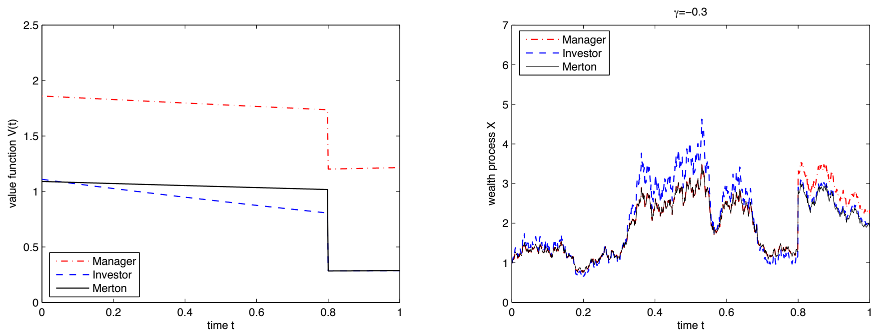


Fig. 5.2 Value function and wealth process for insider, investor and Merton: $\gamma = -0.3$.



Finally, we discuss by numerical tests the possibility to relax the short-selling constraint, that is, instead of prohibiting completely the short-selling strategy, we suppose that the investors can effectuate short-selling tradings under a given constraint δ_s. The results for different values of δ_s are illustrated in Figure 5.4: not surprisingly, for the case of a loss at default, the optimal value function of both the insider and the standard investor is an increasing function of δ_s. Moreover, the gain is more significant for the insider.

6 Annex

We recall the canonical decomposition of $\mathbb{G}^M$-adapted (respectively $\mathbb{G}^M$-predictable) processes (see Jeulin [14] Lemma 3.13 and 4.4).

Fig. 5.3 Value function and wealth process for insider, investor and Merton: $\gamma = -0.7$.

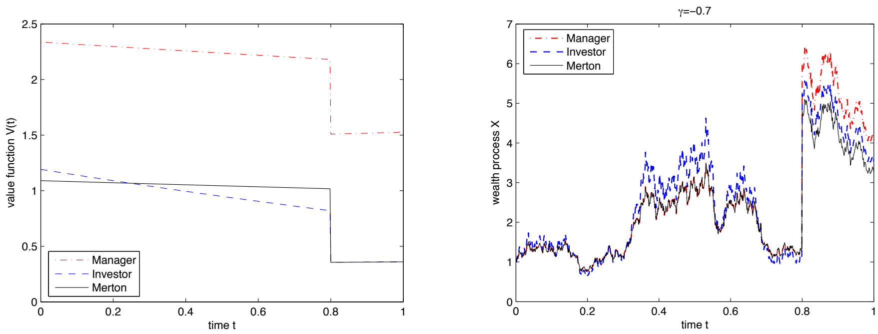
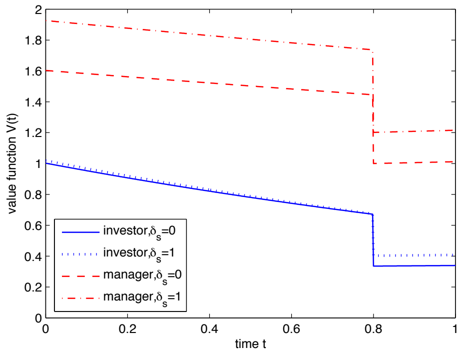


Fig. 5.4 The impact of the short-selling constraint $\lambda = 0.3$ and $\gamma = 0.3$.



Lemma 6.1 *1. For $t \geq 0$, any $\mathcal{G}_t^M$-measurable random variable can be written in the form $Y_t = 1_{\tau>t}Y_t^0(L) + 1_{\tau\leq t}Y_t^1(\tau)$ where $Y_t^0(\cdot)$ and $Y_t^1(\cdot)$ are $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.*

2. Any $\mathbb{G}^M$-adapted process Y admits the decomposition form $Y_t = 1_{\tau>t}Y_t^0(L) + 1_{\tau\leq t}Y_t^1(\tau)$ where $Y^0(\cdot)$ and $Y^1(\cdot)$ are $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted.[3]

3. Any $\mathbb{G}^M$-predictable process Y admits the decomposition form $Y_t = 1_{\tau\geq t}Y_t^0(L) + 1_{\tau<t}Y_t^1(\tau)$ where $Y^0(\cdot)$ and $Y^1(\cdot)$ are $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable, $\mathcal{P}(\mathbb{F})$ being the predictable σ-algebra associated with the filtration $\mathbb{F}$.

Remark 6.2 To compare with the case of a standard investor, we recall that any $\mathcal{G}_t$-measurable random variable Z_t can be written as $Z_t = 1_{\tau>t}Z_t^0 + 1_{\tau\leq t}Z_t^1(\tau)$ where Z_t^0 and $Z_t^1(\cdot)$ are respectively $\mathcal{F}_t$-measurable and $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

[3] Namely for any $t \geq 0$, the function $Y_t^i(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

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