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The generalized density approach in progressive enlargement of filtrations*

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Abstract

Motivated by credit risk modelling, we consider a type of default times whose probability law can have atoms, where standard intensity and density hypotheses in the enlargement of filtrations are not satisfied. We propose a generalized density approach in order to treat such random times in the framework of progressive enlargement of filtrations. We determine the compensator process of the random time and study the martingale and semimartingale processes in the enlarged filtration which are important for the change of probability measures and the evaluation of credit derivatives. The generalized density approach can also be applied to model simultaneous default events in the multi-default setting.

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1 Introduction

In the credit risk analysis, the theory of enlargement of filtrations, which has been developed by the French school of probability since the 1970s (see e.g. Jacod [14], Jeulin [17], Jeulin and Yor [18]), has been systematically adopted to model the default event. In the work of Elliot, Jeanblanc and Yor [10] and Bielecki and Rutkowski [2], the authors have proposed to use the progressive enlargement of filtrations to describe the market information which includes both the ambient information and the default information. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ representing the default-free market information. Let τ be a positive random variable which represents a default time. Then the global market information is modelled by the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, which is the smallest filtration containing $\mathbb{F}$ such that τ is a $\mathbb{G}$-stopping time and $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ by τ. In this framework, the reduced-form modelling approach has been widely used where one often

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supposes the existence of the $\mathbb{G}$-intensity of τ, i.e. the $\mathbb{G}$-adapted process $(\lambda_t, t \geq 0)$ such that $(1\!\!1_{\{\tau \leq t\}} - \int_0^{\tau\wedge t} \lambda_s ds, t \geq 0)$ is a $\mathbb{G}$-martingale. The process λ, also called the default intensity process, plays an important role in the default event modelling. More recently, in order to study the impact of default events, a new approach has been developed by El Karoui, Jeanblanc and Jiao [8, 9] where we suppose the density hypothesis: the $\mathbb{F}$-conditional law of τ admits a density with respect to a non-atomic measure η, i.e. for all $\theta, t \geq 0$, $\mathbb{P}(\tau \in d\theta|\mathcal{F}_t) = \alpha_t(\theta)\eta(d\theta)$ where $\alpha_t(\cdot)$ is an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function. The density hypothesis has been firstly introduced by Jacod [14] in a theoretical setting of initial enlargement of filtrations and is essential to ensure that an $\mathbb{F}$-martingale remains a semimartingale in the initially enlarged filtration. There exist explicit links between the intensity and density processes of the default time τ, which establish a relationship between the two approaches of default modelling. In particular, the density approach allows us to analyze what happens after a default event, i.e. on the set $\{\tau \leq t\}$, and has interesting applications in the study of counterparty default risks. We note that, in both intensity and density approaches, the random time τ is a totally inaccessible $\mathbb{G}$-stopping time which avoids $\mathbb{F}$-stopping times.

In this paper, we consider a type of random times which can be either accessible or totally inaccessible. The motivation comes from recent sovereign credit risks where the government of a sovereign country may default on its debt or obligations. Compared to the classical credit risk, the sovereign default is often influenced by political events. For example, the euro area members and IMF agree on a 110-billion-euro financial aid package for Greece on 02/05/2010 and another financial aid program of 109-billion-euro on 21/07/2011. The eventuality of default-or-not of the Greek government depends on the decisions made at the political meetings held at these dates. Viewed from a market investor, there are important risks that the Greek government may default at such critical dates.

From a mathematical point of view, the existence of these political events and critical dates means that the probability law of the random time τ admits atoms. Hence the sovereign default time can coincide with some pre-determined dates. In this case, the classical default modelling approaches, in particular, both intensity and density models are no longer adapted. To overcome this difficulty, we propose to generalize the density approach in [8]. More precisely, we assume that the $\mathbb{F}$-conditional law of τ contains a discontinuous part, besides the absolutely continuous part which has a density. This generalized density approach allows to consider a random time τ which has positive probability to meet a finite family of $\mathbb{F}$-stopping times.

There are related works in the credit risk modelling. In Bélanger, Shreve and Wong [1], a general framework is proposed where reduced-form models, in particular the widely-used Cox process model, can be extended to the case where default can occur at specific dates. In Gehmlich and Schmidt [12], the authors consider models where the Azéma supermartingale of τ, i.e. the process $(\mathbb{P}(\tau > t|\mathcal{F}_t))_{t\geq 0}$ contains jumps (so that the intensity does not exist) and develop the associated HJM credit term structures and no-arbitrage conditions. Carr and Linetsky [3] and Chen and Filipović [4] have studied the hybrid credit models where the default time depends on both a first-hitting time in the structural approach and an intensity-based random time in the reduced-form approach. The generalized density model that we propose can also be viewed as hybrid credit model.

In this paper, we first investigate, under the generalized density hypothesis, some classical problems in the enlargement of filtrations from a theoretical point of view. In particular, we deduce the compensator process of the random time τ, which is discontinuous in this case. This means that the intensity process does not necessarily exist. We also characterize the martingale processes in the enlarged filtration $\mathbb{G}$ and obtain the $\mathbb{G}$-semimartingale decomposition for an $\mathbb{F}$-martingale, which shows that in the generalized density setting, the (H')-hypothesis of Jacod (c.f. [14]) is satisfied, that is, any $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale. The main contribution of our work is to focus on the impact of the discontinuous part of the $\mathbb{F}$-conditional law of τ and study the impact of the critical dates on the random time.

For applications of the generalized density approach, we study the immersion property, also called the H-hypothesis in literature, i.e., any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale, which is commonly adopted in the default modelling. We give the criterion for the immersion property to hold in this context. The immersion property is in general not preserved under a change of probability measure. As one consequence of the characterization results of $\mathbb{G}$-martingales, we study the change of probability and the associated Radon-Nikodym derivatives. Another application consists of a model of two default times where the occurrence of simultaneous defaults is possible. In the literature of multiple defaults, it is often assumed that two default events do not occur at the same time. The generalized density framework provides tools to study simultaneous defaults, which is important for researches of extremal risks during a financial crisis.

The paper is organized in the following way. In section 2, we make precise the key assumption of the generalized density approach and deduce some basic results. The Section 3 is devoted to the compensator of τ and we conduct the additive and multiplicative decompositions of the Azéma supermartingale. In Section 4, we study the decomposition of $\mathbb{G}$-semimartingales in the generalized density framework by carefully dealing with the discontinuous part of the $\mathbb{F}$-conditional distributions of τ. Section 5 concludes the paper with applications to the immersion property and a model where double default is allowed.

2 Generalized density hypothesis

In this section, we present our key hypothesis, the generalized density hypothesis, and some basic properties. Let $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is a reference filtration satisfying the usual conditions, namely the filtration $\mathbb{F}$ is right continuous and $\mathcal{F}_0$ is a $\mathbb{P}$-complete σ-algebra. We use the expressions $\mathcal{O}(\mathbb{F})$ and $\mathcal{P}(\mathbb{F})$ to denote the optional and predictable σ-algebras associated to the filtration $\mathbb{F}$ respectively. Let τ be a random time on the probability space valued in $[0, +\infty]$. Denote by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ the progressive enlargement of $\mathbb{F}$ by τ, defined as $\mathcal{G}_t = \bigcap_{s>t} \big(\sigma(\{\tau \leq u\} : u \leq s)\big) \vee \mathcal{F}_t$, $t \geq 0$. Let $(\tau_i)_{i=1}^N$ be a finite family of $\mathbb{F}$-stopping times. We assume that the $\mathbb{F}$-conditional distribution of τ avoiding $(\tau_i)_{i=1}^N$ has a density with respect to a non-atomic σ-finite Borel measure η on $\mathbb{R}_+$. Namely, for any $t \geq 0$, there exists a positive $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable $(\omega, u) \mapsto \alpha_t(\omega, u)$ such that, for any bounded Borel function h on $\mathbb{R}_+$, one has

$$\mathbb{E}[\mathbb{1}_H h(\tau) \,|\, \mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}, \tag{2.1}$$

where H denotes the event

$$\{\tau < \infty\} \cap \bigcap_{i=1}^N \{\tau \neq \tau_i\}.$$

In particular, the case where the function h is constant and takes the value 1 leads to the relation

$$\mathbb{E}[\mathbb{1}_H \,|\, \mathcal{F}_t] = \int_{\mathbb{R}_+} \alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}$$

Remark 2.1. The above assumption implies that the random time τ avoids any $\mathbb{F}$-stopping time σ such that $\mathbb{P}(\sigma = \tau_i < \infty) = 0$ for all $i \in \{1, \cdots, N\}$. Namely for such $\mathbb{F}$-stopping time σ one has $\mathbb{P}(\tau = \sigma < \infty) = 0$. However, the random time τ is allowed to coincide with some of the stopping times in the family $(\tau_i)_{i=1}^N$ with a positive probability. Moreover, without loss of generality, we may assume that the family $(\tau_i)_{i=1}^N$ is increasing. In fact, if we denote by $(\tau^{(i)})_{i=1}^N$ the order statistics of $(\tau_i)_{i=1}^N$, then

$$\{\tau < \infty\} \cap \bigcap_{i=1}^N \{\tau \neq \tau_i\} = \{\tau < \infty\} \cap \bigcap_{i=1}^N \{\tau \neq \tau^{(i)}\}.$$

The following proposition shows that we can even assume that the family $(\tau_i)_{i=1}^N$ is strictly increasing until reaching infinity.

Proposition 2.2. *Let $(\tau_i)_{i=1}^N$ be an increasing family of $\mathbb{F}$-stopping times. Then there exists a family of $\mathbb{F}$-stopping times $(\sigma_i)_{i=1}^N$ which verify the following conditions:*

(a) For any $\omega \in \Omega$ and $i, j \in \{1, \cdots, N\}$, $i < j$, if $\sigma_i(\omega) < \infty$, then $\sigma_i(\omega) < \sigma_j(\omega)$; otherwise, $\sigma_j(\omega) = \infty$.

(b) For any $\omega \in \Omega$, one has $\{\sigma_1(\omega), \cdots, \sigma_N(\omega), \infty\} = \{\tau_1(\omega), \cdots, \tau_N(\omega), \infty\}$, which implies

$$\{\tau < \infty\} \cap \bigcap_{i=1}^N \{\tau \neq \tau_i\} = \{\tau < \infty\} \cap \bigcap_{i=1}^N \{\tau \neq \sigma_i\}.$$

Proof. The case where $N = 1$ is trivial. We prove the result by induction and assume $N \geqslant 2$. Let $\tau_{N+1} = \infty$ by convention. For each $k \in \{2, \cdots, N\}$, let

$$E_k = \{\tau_1 = \cdots = \tau_k < \infty\}.$$

Moreover, for $k \in \{2, \ldots, N\}$, we define

$$\tau'_k = 1\!\!1_{E_k^c} \tau_k + \sum_{i=k}^N 1\!\!1_{E_i \setminus E_{i+1}} \tau_{i+1}.$$

Note that for each $i \geqslant k$, the set E_i is $\mathcal{F}_{\tau_k}$-measurable. Therefore

$$\forall\, t \geq 0, \quad \{\tau'_k \leq t\} = \big(E_k^c \cap \{\tau_k \leq t\}\big) \cup \bigcup_{i=k}^N \Big((E_i \setminus E_{i+1}) \cap \{\tau_{i+1} \leq t\}\Big) \in \mathcal{F}_t,$$

so τ'_k is an $\mathbb{F}$-stopping time. By definition one has $\tau_1 \leq \tau'_2 \leq \cdots \leq \tau'_N \leq \tau'_{N+1}$, where $\tau'_{N+1} = \infty$. One also has, for any ω,

$$\{\tau_1(\omega), \tau_2(\omega) \cdots, \tau_{N+1}(\omega)\} = \{\tau_1(\omega), \tau'_2(\omega), \cdots, \tau'_{N+1}(\omega)\}.$$

Moreover, the strict inequality $\tau_1 < \tau'_2$ holds on $\{\tau_1 < \infty\}$. Then by the induction hypothesis on $(\tau'_2, \cdots, \tau'_{N+1})$, we obtain the required result. □

For purpose of the dynamical study of the random time τ, we need the following result which is analogous to [14, Lemme 1.8].

Proposition 2.3. *There exists a non-negative $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\widetilde{\alpha}(\cdot)$ such that $\widetilde{\alpha}(\theta)$ is a càdlàg $\mathbb{F}$-martingale for any $\theta \in \mathbb{R}_+$ and that*

$$\mathbb{E}[1\!\!1_H h(\tau) | \mathcal{F}_t] = \int_{\mathbb{R}_+} h(u) \widetilde{\alpha}_t(u)\, \eta(du) \qquad \mathbb{P}\text{-}a.s. \tag{2.2}$$

for any bounded Borel function h.

Proof. Let $(\alpha_t(\cdot))_{t\geq 0}$ be a family of random functions such that the relation (2.1) holds for any $t \geq 0$. We fixe a coutable dense subset D in $\mathbb{R}_+$ such as the set of all non-negative rational numbers. If s and t are two elements in D, $s < t$, there exists a positive $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\alpha_{t|s}(\cdot)$ such that

$$\forall\, \theta \in \mathbb{R}_+, \quad \alpha_{t|s}(\theta) = \mathbb{E}[\alpha_t(\theta)\,|\,\mathcal{F}_s] \qquad \mathbb{P}\text{-a.s.}$$

Note that for any bounded Borel function h, one has

$$\mathbb{E}[1\!\!1_H h(\tau)|\mathcal{F}_s] = \mathbb{E}\left[\int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du)\,\middle|\,\mathcal{F}_s\right] = \int_{\mathbb{R}_+} h(u)\alpha_{t|s}(u)\,\eta(du) \quad \mathbb{P}\text{-a.s.}$$

Hence there exists an η-negligeable set $B_{t,s}$ such that $\alpha_s(u) = \alpha_{t|s}(u)$ $\mathbb{P}$-a.s. for any $u \in \mathbb{R}_+ \setminus B_{t,s}$. Let $B = \bigcup_{(s,t)\in D^2, s<t} B_{t,s}$ and let $\widehat{\alpha}_t(\cdot) = 1\!\!1_{B^c}(\cdot)\alpha_t(\cdot)$ for any $t \in D$. We then obtain that $\widehat{\alpha}_s(u) = \mathbb{E}[\widehat{\alpha}_t(u)|\mathcal{F}_s]$, $\mathbb{P}$-a.s. for any $u \in \mathbb{R}_+$ and all elements s, t in D such that $s < t$. Moreover, since B is still η-negligeable, for any $t \in D$,

$$\mathbb{E}[1\!\!1_H h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\widehat{\alpha}_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}. \tag{2.3}$$

By [7, Theorem VI.1.2], for any $\theta \in \mathbb{R}_+$, there exists a $\mathbb{P}$-negligeable subset E_θ of Ω such that, for any $\omega \in \Omega \setminus E_\theta$, the following limits exist:

$$\widehat{\alpha}_{t+}(\omega,\theta) := \lim_{s\in D,\, s\downarrow t} \widehat{\alpha}_t(\omega,\theta), \quad \widehat{\alpha}_{t-}(\omega,\theta) := \lim_{s\in D,\, s\uparrow t} \widehat{\alpha}_t(\omega,\theta).$$

Moreover, we define

$$\widetilde{\alpha}_t(\omega,\theta) = \begin{cases} \widehat{\alpha}_t(\omega,\theta), & \text{if } \omega \notin E_\theta, \\ 0, & \text{if } \omega \in E_\theta. \end{cases}$$

Then $\widetilde{\alpha}(\theta)$ is a càdlàg $\mathbb{F}$-martingale, and therefore the random function $\alpha(\cdot)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable. We then deduce the proposition from (2.3). □

We summarize the *generalized density hypothesis* as below. In what follows, we always assume this hypothesis.

Assumption 2.4. *We assume that there exists a non-atomic σ-finite Borel measure η on $\mathbb{R}_+$, a finite family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^N$ such that $\mathbb{P}(\tau_i = \tau_j < \infty) = 0$ for any pair (i,j) of distinct indices in $\{1,\cdots,N\}$, together with an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable function $\alpha(\cdot)$ such that $\alpha(\theta)$ is a càdlàg $\mathbb{F}$-martingale for any $\theta \in \mathbb{R}_+$ and that*

$$\mathbb{E}\left[1\!\!1_{\{\tau<\infty\}} h(\tau) \prod_{i=1}^N 1\!\!1_{\{\tau\neq\tau_i\}}\,\middle|\,\mathcal{F}_t\right] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) \qquad \mathbb{P}\text{-a.s.}$$

for any bounded Borel function h.

Remark 2.5. 1) The condition $\mathbb{P}(\tau_i = \tau_j < +\infty) = 0$ is not essential in Assumption 2.4. In fact, for an arbitrary finite family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^N$, if we suppose that the random time τ has an $\mathbb{F}$-density $\alpha(\cdot)$ with respect to η avoiding $(\tau_i)_{i=1}^N$, then by Remark 2.1 and Proposition 2.2, we can always obtain another family of $\mathbb{F}$-stopping times $(\sigma_i)_{i=1}^N$ such that $\mathbb{P}(\sigma_i = \sigma_j < +\infty) = 0$ for $i \neq j$ and that τ has an $\mathbb{F}$-density avoiding the family $(\sigma_i)_{i=1}^N$. Moreover, the $\mathbb{F}$-density of τ avoiding $(\sigma_i)_{i=1}^N$ coincides with $\alpha(\cdot)$.
2) For each $i \in \{1,\cdots,N\}$, by [6, IV.81], there exists a subset $\Omega_i \in \mathcal{F}_{\tau_i}$ such that $\tau_i' := \tau_i 1\!\!1_{\Omega_i} + (+\infty)1\!\!1_{\Omega_i^c}$ is an accesible $\mathbb{F}$-stopping time and $\tau_i'' := \tau_i 1\!\!1_{\Omega_i^c} + (+\infty)1\!\!1_{\Omega_i}$ is a totally inaccessible $\mathbb{F}$-stopping time. Note that τ also admits an $\mathbb{F}$-density avoiding the family $(\tau_i', \tau_i'')_{i=1}^N$ and the $\mathbb{F}$-density is still $\alpha(\cdot)$. Therefore, without loss of generality, we may assume in addition that each $\mathbb{F}$-stopping time τ_i is either accessible or totally inaccessible.

Example 2.6. We present a simple example as below. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion and $\mathbb{F}$ be the canonical Brownian filtration. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. We denote by $\tau_1 = \inf\{t \geq 0 : B_t = a < 0\}$ and $\xi = \inf\{t \geq 0 : N_t \geq 1\}$, with the convention $\inf \emptyset = \infty$. Define a random time τ as

$$\tau = \tau_1 \wedge \xi.$$

We compute firstly the conditional distribution of τ_1. For any $0 \leq t < \theta$, one has

$$\mathbb{P}(\tau_1 > \theta|\mathcal{F}_t) = 1\!\!1_{\{\tau_1>t\}}\mathbb{P}(\min_{t\leq s\leq\theta} B_s > a\big|B_t) = 1\!\!1_{\{\tau_1>t\}}\mathrm{erf}\left(\frac{B_t - a}{\sqrt{2(\theta - t)}}\right),$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-v^2}dv$ is the Gauss error function. Next, for any $t \in \mathbb{R}_+$,

$$\mathbb{P}(\tau = \tau_1|\mathcal{F}_t) = \mathbb{P}(\tau_1 \leq \xi|\mathcal{F}_t) = 1\!\!1_{\{\tau_1\leq t\}}e^{-\lambda\tau_1} + 1\!\!1_{\{\tau_1>t\}}\left(\frac{1}{2}e^{-\lambda t} - \lambda\int_t^\infty e^{-\lambda u}\mathrm{erf}\left(\frac{B_t - a}{\sqrt{2(u - t)}}\right)du\right)$$

So τ satisfies Assumption 2.4 with the generalized density

$$\alpha_t(\theta) = \lambda e^{-\lambda\theta}\left[1\!\!1_{\{\theta\leq t\}}1\!\!1_{\{\tau_1>\theta\}} + 1\!\!1_{\{\theta>t\}}1\!\!1_{\{\tau_1>t\}}\mathrm{erf}\left(\frac{B_t - a}{\sqrt{2(\theta - t)}}\right)\right], \quad t \geq 0.$$

For each $i \in \{1, \cdots, N\}$, let p^i be a càdlàg version of the $\mathbb{F}$-martingale $(\mathbb{E}[1\!\!1_{\{\tau=\tau_i<\infty\}}|\mathcal{F}_t])_{t\geq 0}$, which is closed by $p^i_\infty = \mathbb{E}[1\!\!1_{\{\tau=\tau_i\}}|\mathcal{F}_\infty]$. We also consider the case where τ may reach infinity and denote by p^∞ a càdlàg version of the $\mathbb{F}$-martingale $(\mathbb{E}[1\!\!1_{\{\tau=\infty\}}|\mathcal{F}_t])_{t\geq 0}$, which is closed by $p^\infty_\infty = \mathbb{E}[1\!\!1_{\{\tau=\infty\}}|\mathcal{F}_\infty]$. Note that Assumption 2.4 implies that, for any $t \geq 0$,

$$\int_{\mathbb{R}_+}\alpha_t(u)\,\eta(du) + \sum_{i=1}^N p^i_t + p^\infty_t = 1 \qquad \mathbb{P}\text{-a.s.} \tag{2.4}$$

We define

$$G_t := \int_t^\infty \alpha_t(\theta)\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i>t\}}p^i_t + p^\infty_t. \tag{2.5}$$

Note that $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$, $\mathbb{P}$-a.s.. The process $G = (G_t)_{t\geq 0}$ is a càdlàg $\mathbb{F}$-supermartingale and called the Azéma supermatingale of the random time τ. Moreover, for any bounded Borel function h, one has

$$\mathbb{E}[1\!\!1_{\{\tau<\infty\}}h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)\,\eta(du) + \sum_{i=1}^N \mathbb{E}[1\!\!1_{\{\tau_i<\infty\}}h(\tau_i)p^i_{\tau_i\vee t}|\mathcal{F}_t]. \tag{2.6}$$

The following result shows that any $\mathcal{G}_t$-conditional expectation can be computed in a decomposed form, which can be viewed as a direct extension to [8, Theorem 3.1].

Proposition 2.7. *Let $Y_T(\cdot)$ be $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable such that*
1) $1\!\!1_{\cap_{i=1}^N\{\tau_i\neq\theta\}}Y_T(\theta)\alpha_T(\theta)$ is integrable for any $\theta \in \mathbb{R}_+$ and $\int_{\mathbb{R}_+}\big|\mathbb{E}[Y_T(\theta)\alpha_T(\theta)]\big|\eta(d\theta) < +\infty$,
2) $1\!\!1_{\{\tau_i<\infty\}}Y_T(\tau_i)p^i_{\tau_i\vee T}$ is integrable for any $i \in \{1, \cdots, N\}$.
Then the random variable $1\!\!1_{\{\tau<\infty\}}Y_T(\tau)$ is integrable, and for any $t \leq T$,

$$\mathbb{E}[1\!\!1_{\{\tau<\infty\}}Y_T(\tau)|\mathcal{G}_t] = 1\!\!1_{\{\tau>t\}}\tilde{Y}_t + 1\!\!1_{\{\tau\leq t\}}\hat{Y}_t(\tau) \quad \mathbb{P}\text{-a.s.} \tag{2.7}$$

where

$$\tilde{Y}_t = \frac{1\!\!1_{\{G_t>0\}}}{G_t}\left[\int_t^{+\infty}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i>t\}}\mathbb{E}[1\!\!1_{\{\tau_i<\infty\}}Y_T(\tau_i)p^i_{\tau_i\vee T}|\mathcal{F}_t]\right] \tag{2.8}$$

and

$$\hat{Y}_t(\theta) = 1\!\!1_{\cap_{i=1}^N\{\theta\neq\tau_i\}}\frac{1\!\!1_{\{\alpha_t(\theta)>0\}}}{\alpha_t(\theta)}\mathbb{E}[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]+\sum_{i=1}^N 1\!\!1_{\{\theta=\tau_i\}}\frac{1\!\!1_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}[Y_T(\tau_i)p_T^i|\mathcal{F}_t], \quad \theta\leq t. \tag{2.9}$$

Proof. We may assume that $Y_T(\cdot)$ is non-negative without loss of generality so that the following proof works without discussing the integrability (as a byproduct, we can prove the case where $Y_T(\cdot)$ is non-negative without any integrability condition). The integrability of $Y_T(\tau)$ results from the finiteness of each term in the following formulas. The first term on the right-hand side of (2.7) is obtained as a consequence of the so-called key lemma in the progressive enlargement of filtration ([10, Lemma 3.1]):

$$1\!\!1_{\{\tau>t\}}\mathbb{E}[1\!\!1_{\{\tau<\infty\}}Y_T(\tau)|\mathcal{G}_t] = 1\!\!1_{\{\tau>t\}}\frac{1\!\!1_{\{G_t>0\}}}{G_t}\mathbb{E}[1\!\!1_{\{t<\tau<\infty\}}Y_T(\tau)|\mathcal{F}_t].$$

Note that

$$\begin{aligned}\mathbb{E}[1\!\!1_{\{t<\tau<\infty\}}Y_T(\tau)|\mathcal{F}_T] &= \int_t^{+\infty} Y_T(u)\alpha_T(u)\eta(du) + \sum_{i=1}^N \mathbb{E}[1\!\!1_{\{t<\tau=\tau_i<\infty\}}Y_T(\tau_i)|\mathcal{F}_T]\\ &= \int_t^{+\infty} Y_T(u)\alpha_T(u)\eta(du) + \sum_{i=1}^N \mathbb{E}[1\!\!1_{\{t<\tau_i<\infty\}}Y_T(\tau_i)p^i_{\tau_i\vee T}|\mathcal{F}_T]\end{aligned}$$

which implies (2.8). For the second term in (2.7), we shall prove by verification. Let $Z_t(\cdot)$ be a bounded $\mathcal{F}_t\otimes\mathcal{B}(\mathbb{R}_+)$-measurable random variable, one has

$$\begin{aligned}\mathbb{E}[\hat{Y}_t(\tau)Z_t(\tau)1\!\!1_{\{\tau\leq t\}}] &= \mathbb{E}\Big[1\!\!1_{H\cap\{\tau\leq t\}}\frac{1\!\!1_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)|\mathcal{F}_t]_{\theta=\tau}\Big]\\ &\quad+\sum_{i=1}^N\mathbb{E}\Big[1\!\!1_{\{\tau=\tau_i\leq t\}}\frac{1\!\!1_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}[Y_T(\tau_i)Z_t(\theta)p_T^i|\mathcal{F}_t]_{\theta=\tau}\Big].\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}\Big[1\!\!1_{H\cap\{\tau\leq t\}}\frac{1\!\!1_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)|\mathcal{F}_t]_{\theta=\tau}\Big] &= \mathbb{E}\Big[\int_0^t\mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)|\mathcal{F}_t]\eta(d\theta)\Big]\\ = \int_0^t\mathbb{E}[Y_T(\theta)Z_t(\theta)\alpha_T(\theta)]\eta(d\theta) &= \mathbb{E}\big[1\!\!1_{H\cap\{\tau\leq t\}}Y_T(\tau)Z_t(\tau)\big].\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\Big[1\!\!1_{\{\tau=\tau_i\leq t\}}\frac{1\!\!1_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}[Y_T(\tau_i)Z_t(\theta)p_T^i|\mathcal{F}_t]_{\theta=\tau}\Big] &= \mathbb{E}[1\!\!1_{\{\tau_i\leq t\}}Y_T(\tau_i)Z_t(\tau_i)p_T^i]\\ &= \mathbb{E}[1\!\!1_{\{\tau=\tau_i\leq t\}}Y_T(\tau)Z_t(\tau)].\end{aligned}$$

Therefore we obtain

$$\mathbb{E}[1\!\!1_{\{\tau\leq t\}}Y_T(\tau)|\mathcal{G}_t] = 1\!\!1_{\{\tau\leq t\}}\hat{Y}_t(\tau) \quad \mathbb{P}\text{-a.s.}$$

since $\hat{Y}_t(\cdot)$ is $\mathcal{F}_t\otimes\mathcal{B}([0,t])$-measurable. The proposition is thus proved. □

Remark 2.8. (1) For any integrable $\mathcal{G}_T$-measurable random variable Z, one can always find a $\mathcal{F}_T\otimes\mathcal{B}(\mathbb{R}_+)$-measurable function $Y_T(\cdot)$ such that $1\!\!1_{\{\tau<\infty\}}Z = 1\!\!1_{\{\tau<\infty\}}Y_T(\tau)$, $\mathbb{P}$-a.s. and verifies the integrability conditions in the previous proposition. Without loss of generality, we can assume that Z is non-negative. We begin with an arbitrary $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable non-negative random function $Z_T(\cdot)$ such that $1\!\!1_{\{\tau<\infty\}} Z = 1\!\!1_{\{\tau<\infty\}} Z_T(\tau)$. Then by Proposition 2.7 in the non-negative case (where the integrability conditions are not necessary), one has $\int_0^\infty \mathbb{E}[Z_T(\theta)\alpha_T(\theta)]\eta(d\theta) < \infty$. Therefore, the set K of $\theta \in \mathbb{R}_+$ such that $\mathbb{E}[Z_T(\theta)\alpha_T(\theta)] = +\infty$ is η-negligeable. By replacing $Z_T(\cdot)$ by zero on the set

$$(\Omega \times K) \cap \bigcap_{i=1}^N \{(\omega, \theta) \in \Omega \times \mathbb{R}_+ \mid \tau_i(\omega) \neq \theta\},$$

we find another random function $Y_T(\cdot)$ such that $Y_T(\tau) = Z_T(\tau)$, $\mathbb{P}$-a.s. Moreover, $Y_T(\cdot)$ satisfies the integrability conditions as in the proposition.

(2) As a direct consequence, for any $t \leq T$, one has

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \frac{1}{G_t} \left[\int_T^\infty \alpha_t(\theta)\eta(d\theta) + \sum_{i=1}^N \mathbb{E}[1\!\!1_{\{\tau_i>T\}} p_T^i | \mathcal{F}_t] + p_t^\infty \right], \quad \mathbb{P}\text{-a.s.}. \tag{2.10}$$

3 Compensator process

In the credit risk literature, the compensator and the intensity processes of τ play an important role in the default event modelling. The general method for computing the compensator is given in [18] by using the Doob-Meyer decomposition of the Azéma supermartingale G. In [8], an explicit result is obtained under the density hypothesis (see also [11] and [20]) where the compensator is absolutely continuous and the intensity exists. In this section, we focus on the compensator process under the generalized density hypothesis.

We introduce the following notations. For any $i \in \{1, \cdots, N\}$, denote by D^i the process $(1\!\!1_{\{\tau_i \leq t\}})_{t\geq 0}$. We use the expression Λ^i to denote the $\mathbb{F}$-compensator process of D^i, that is, Λ^i is an increasing $\mathbb{F}$-predictable process such that $M^i := D^i - \Lambda^i$ is an $\mathbb{F}$-martingale with $M_0^i = 0$. Note that, if τ_i is a predictable $\mathbb{F}$-stopping time, then $\Lambda^i = D^i$ and $M^i = 0$. The following result generalizes [8, Proposition 4.1 (1)]. Here the Azéma supermartingale G is a process with jumps and needs to be treated with care.

Proposition 3.1. *The Doob-Meyer decomposition of the Azéma's supermartingale G is given by $G_t = G_0 + M_t - A_t$, where A is an $\mathbb{F}$-predictable increasing process given by*

$$A_t = \int_0^t \alpha_\theta(\theta)\eta(d\theta) + \sum_{i=1}^N \int_{]0,t]} p_{s-}^i d\Lambda_s^i + \sum_{i=1}^N \langle M^i, p^i \rangle_t, \tag{3.1}$$

Proof. For any $t \geq 0$, let

$$C_t = \int_0^t \alpha_\theta(\theta)\eta(d\theta).$$

The process C is $\mathbb{F}$-adapted and increasing. It is moreover continuous since η is assumed to be non-atomic. Note that by (2.5),

$$G_t = \mathbb{E}\left[\int_t^\infty \alpha_\theta(\theta)\eta(d\theta) \,\middle|\, \mathcal{F}_t \right] + \sum_{i=1}^N 1\!\!1_{\{\tau_i>t\}} p_t^i + p_t^\infty.$$

The process

$$C_t + \int_t^\infty \alpha_t(\theta)\eta(d\theta) = \mathbb{E}\left[\int_0^\infty \alpha_\theta(\theta)\eta(d\theta) \,\middle|\, \mathcal{F}_t \right], \quad t \geq 0$$

is a square integrable $\mathbb{F}$-martingale since

$$\mathbb{E}\bigg[\bigg(\int_0^\infty \alpha_\theta(\theta)\eta(d\theta)\bigg)^2\bigg] = 2\mathbb{E}\bigg[\int_0^\infty \eta(d\theta)\alpha_\theta(\theta)\int_\theta^\infty \eta(du)\alpha_u(u)\bigg]$$
$$= 2\mathbb{E}\bigg[\int_0^\infty \alpha_\theta(\theta)\mathbb{E}[1\!\!1_{A\cap\{\tau>\theta\}}|\mathcal{F}_\theta]\eta(d\theta)\bigg] \leq 2.$$

Moreover, one has

$$\begin{aligned}1\!\!1_{\{\tau_i>t\}}p^i_t &= 1\!\!1_{\{\tau_i>0\}}p^i_0 + \int_{]0,t]} 1\!\!1_{\{\tau_i\geq s\}}dp^i_s - \int_{]0,t]} p^i_{s-}dD^i_s - [D^i,p^i]_t,\\ &= 1\!\!1_{\{\tau_i>0\}}p^i_0 + \int_{]0,t]} 1\!\!1_{\{\tau_i\geq s\}}dp^i_s - \int_{]0,t]} p^i_{s-}dM^i_s - \int_{]0,t]} p^i_{s-}d\Lambda^i_s - [D^i,p^i]_t,\end{aligned}$$

where

$$[D^i,p^i]_t = \sum_{0<s\leq t}\Delta D^i_s\Delta p^i_s = 1\!\!1_{\{\tau_i\leq t\}}\Delta p^i_{\tau_i}.$$

One can also rewrite $[D^i,p^i]$ as

$$[D^i,p^i] = [\Lambda^i,p^i] + [M^i,p^i] = [\Lambda^i,p^i] + ([M^i,p^i] - \langle M^i,p^i\rangle) + \langle M^i,p^i\rangle.$$

Note that $[\Lambda^i,p^i]$ is an $\mathbb{F}$-martingale since Λ^i is $\mathbb{F}$-predictible and p^i is an $\mathbb{F}$-martingale (see [7, VIII.19]). Moreover $\langle M^i,p^i\rangle$ is an $\mathbb{F}$-predictable process such that $[M^i,p^i] - \langle M^i,p^i\rangle$ is an $\mathbb{F}$-martingale. Therefore we obtain that

$$A_t = C_t + \int_{]0,t]} p^i_{s-}d\Lambda^i_s + \langle M^i,p^i\rangle_t, \quad t\geq 0$$

is a predictable process, and $G+A$ is an $\mathbb{F}$-martingale. □

In the following, we denote by $\Lambda^{\mathbb{F}}$ the process

$$\Lambda^{\mathbb{F}}_t := \int_{]0,t]}\frac{1\!\!1_{\{G_{s-}>0\}}}{G_{s-}}dA_s, \quad t\geq 0 \tag{3.2}$$

which is an $\mathbb{F}$-predictable process. It is well known that the $\mathbb{G}$-compensator of τ is $\Lambda^{\mathbb{G}} = (\Lambda^{\mathbb{F}}_{\tau\wedge t})_{t\geq 0}$ (c.f. [18, Proposition 2]). We observe from Proposition 3.1 that the compensator $\Lambda^{\mathbb{F}}$ is in general a discontinuous process and may have jump at the stopping times $(\tau_i)_{i=1}^N$, so that the intensity does not exist in this case. A similar phenomenon appears in the generalized Cox process model proposed in [1] where the default can occur at specific dates. A general model where the Azéma supermartingale is discontinuous has also been studied in [12].

We can treat general $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^N$, (see Remark 2.5). In case they are predictable $\mathbb{F}$-stopping times, $\Lambda^i_t = 1\!\!1_{\{\tau_i\leq t\}}$ and $M^i_t = 0$, so the last term on the right-hand side of (3.1) vanishes and we obtain

$$A_t = \int_0^t \alpha_\theta(\theta)\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\leq t\}}p^i_{\tau_i-}$$

In case where $\{\tau_i\}_{i=1}^N$ are totally inaccessible $\mathbb{F}$-stopping times, then τ is a totally inaccessible $\mathbb{G}$-stopping time. In this case, the compensator process of τ is continuous. A similar result can be found in Coculescu [5].

Proposition 3.2. *If $(\tau_i)_{i=1}^N$ are totally inaccessible $\mathbb{F}$-stopping times, then τ is a totally inaccessible $\mathbb{G}$-stopping time.*

Proof. Since τ_i is totally inaccessible, the $\mathbb{F}$-compensator process Λ^i is continuous. Moreover, $\langle M^i, p^i\rangle$ is the compensator of the process $[D^i, p^i] = (\mathbb{1}_{\{\tau_i \le t\}}\Delta p^i_{\tau_i})_{t\ge 0}$ and hence is continuous (see [7, VI.78] and the second part of its proof for details). Therefore the process A in the Doob-Meyer decomposition of G is continuous since η is non-atomic. This implies that the $\mathbb{F}$-compensator $\Lambda^{\mathbb{F}}$ of τ is continous. Thus the process $(\mathbb{1}_{\{\tau>t\}} + \Lambda^{\mathbb{F}}_{\tau\wedge t})_{t\ge 0}$ is a uniformly integrable $\mathbb{G}$-martingale, which is continuous outside the graphe of τ, and has jump size 1 at τ. Still by [7, VI.78], τ is a totally inaccessible $\mathbb{G}$-stopping time.

□

There exists a multiplicative decomposition of the Azéma supermartingale. By [13, Corollary 6.35], $G\exp(\Lambda^{\mathbb{F}})$ is an $\mathbb{F}$-martingale, which is the Doléans-Dade exponential of the $\mathbb{F}$-martingale $\tilde{M}$ such that

$$d\tilde{M}_t = \frac{\mathbb{1}_{\{G_{t-}>0\}}}{G_{t-}} dM_t.$$

In the following, we give the explicit multiplicative decomposition under the generalized density hypothesis as a general case of [8, Proposition 4.1 (2)].

Proposition 3.3. *Let $\xi := \inf\{t > 0 : G_t = 0\}$ and denote by $\Lambda^{\mathbb{F},c}$ the continuous part of $\Lambda^{\mathbb{F}}$. The multiplicative decomposition of the Azéma supermartingale G is given by*

$$G_t = L_t e^{-\Lambda^{\mathbb{F},c}_t} \prod_{0<u\le t} (1 - \Delta\Lambda^{\mathbb{F}}_u), \quad t \ge 0, \tag{3.3}$$

where L is an $\mathbb{F}$-martingale solution of the stochastic differential equation

$$L_t = 1 + \int_{]0,t\wedge\xi]} \frac{L_{s-}}{(1-\Delta\Lambda^{\mathbb{F}}_s)G_{s-}} dM_s, \quad t \ge 0. \tag{3.4}$$

Proof. On the one hand, for any $t \ge 0$, if there exists $u \in]0,t]$ such that $\Delta\Lambda^{\mathbb{F}}_u = 1$, making the right-hand side of (3.3) vanish, then we have ${}^p(\mathbb{1}_{[\![0,\tau[\![})_u = 0$, which implies that $G_u = 0$. It is a classic result that G is a non-negative supermartingale which sticks at 0 (c.f. [21, page 379]), then $G_t = 0$. On the other hand, if $\Delta\Lambda^{\mathbb{F}} \ne 1$, we denote by $M^{\mathbb{F}}$ the $\mathbb{F}$-martingale defined as

$$dM^{\mathbb{F}}_t = \frac{\mathbb{1}_{\{G_{t-}>0\}}}{G_{t-}} dM_t.$$

Let $S = M^{\mathbb{F}} - \Lambda^{\mathbb{F}}$. Then one has $G_t = 1 + \int_{]0,t]} G_{u-} dS_u$ for all $t \in \mathbb{R}_+$. By [13, Corollaire 6.35], $G = \mathcal{E}(S) = L\mathcal{E}(-\Lambda^{\mathbb{F}})$, where $L = \mathcal{E}(\tilde{M}^{\mathbb{F}})$ such that

$$d\tilde{M}^{\mathbb{F}}_t = \frac{\mathbb{1}_{\{0<t\le\xi\}}}{1-\Delta\Lambda^{\mathbb{F}}_t} dM^{\mathbb{F}}_t$$

(here we use the fact that $\xi = \inf\{t > 0 : \Delta S_t = -1\}$ and $-\Delta\Lambda^{\mathbb{F}} \ne -1$ on $]\!]0,\xi]\!]$). Then, L is the solution of

$$L_t = 1 + \int_{]0,t]} L_{s-} d\tilde{M}^{\mathbb{F}}_s, \quad t \ge 0.$$

□

4 Martingales and semimartingales in $\mathbb{G}$

In this section, we are interested in the $\mathbb{G}$-martingales. We first characterize the $\mathbb{G}$-martingales by using $\mathbb{F}$-martingale conditions, as done in [8, Proposition 5.6]. However, under the generalized density hypothesis, we shall distinguish necessary and sufficient conditions although they have similar forms at the first sight. In fact, the decomposition of a $\mathbb{G}$-adapted process is not unique, and the martingale property can not hold true for all modifications. This makes the necessary and sufficient conditions subtly different.

Proposition 4.1. *Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-adapted process, which is written in the decomposed form $Y_t^{\mathbb{G}} = 1\!\!1_{\{\tau>t\}}Y_t + 1\!\!1_{\{\tau\le t\}}Y_t(\tau), t\ge 0$, $\mathbb{P}$-a.s. where Y is an $\mathbb{F}$-adapted process and $Y(\cdot)$ is an $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted process. Then $Y^{\mathbb{G}}$ is a $\mathbb{G}$-(local) martingale if the following conditions are verified:*

(a) $1\!\!1_{\cap_{i=1}^N\{\tau_i\neq\theta\}}Y(\theta)\alpha(\theta)$ *is an $\mathbb{F}$-(local) martingale on $[\theta,\infty[$ for any $\theta\in\mathbb{R}_+$;*

(b) $Y(\tau_i)p^i$ *is an $\mathbb{F}$-(local) martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,N\}$;*

(c) the process $Y_tG_t+\int_0^t Y_u(u)\alpha_u(u)\eta(du)+\sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}$, $t\ge 0$ is an $\mathbb{F}$-(local) martingale.

Proof. We first treat the martingale case. By Proposition 2.7, the conditional expectation $\mathbb{E}[Y_T^{\mathbb{G}}|\mathcal{G}_t]$ can be written as the sum of

$$1\!\!1_{\{\tau>t\}}\frac{1\!\!1_{\{G_t>0\}}}{G_t}\mathbb{E}\Big[1\!\!1_{\{\tau>T\}}Y_T+1\!\!1_{\{t<\tau\le T\}}Y_T(\tau)\Big|\mathcal{F}_t\Big]=1\!\!1_{\{\tau>t\}}\frac{1\!\!1_{\{G_t>0\}}}{G_t}\bigg(\mathbb{E}\left[Y_TG_T|\mathcal{F}_t\right] + \int_t^T\mathbb{E}\left[Y_T(u)\alpha_T(u)|\mathcal{F}_t\right]\eta(du)+\sum_{i=1}^N\mathbb{E}\left[1\!\!1_{\{t<\tau_i\le T\}}Y_T(\tau_i)p_T^i|\mathcal{F}_t\right]\bigg)$$

and

$$1\!\!1_{\{\tau\le t\}}\left(1\!\!1_{\cap_{i=1}^N\{\tau\neq\tau_i\}}\frac{1\!\!1_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t\right]_{\theta=\tau}+\sum_{i=1}^N 1\!\!1_{\{\tau=\tau_i\}}\frac{1\!\!1_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}\left[Y_T(\tau_i)p_T^i|\mathcal{F}_t\right]\right).$$

Hence, $\mathbb{E}[Y_T^{\mathbb{G}}|\mathcal{G}_t]-Y_t^{\mathbb{G}}$ equals the sum of the following terms

$$1\!\!1_{\{\tau>t\}}\frac{1\!\!1_{\{G_t>0\}}}{G_t}\bigg(\mathbb{E}\left[Y_TG_T-Y_tG_t|\mathcal{F}_t\right]+\int_t^T\mathbb{E}\left[Y_T(u)\alpha_T(u)|\mathcal{F}_t\right]\eta(du) + \sum_{i=1}^N\mathbb{E}\left[1\!\!1_{\{t<\tau_i\le T\}}Y_T(\tau_i)p_T^i|\mathcal{F}_t\right]\bigg) \tag{4.1}$$

and

$$1\!\!1_{\{\tau\le t\}}\bigg(-Y_t(\tau)+1\!\!1_{\cap_{i=1}^N\{\tau\neq\tau_i\}}\frac{1\!\!1_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}\mathbb{E}\left[Y_T(\theta)\alpha_T(\theta)|\mathcal{F}_t\right]_{\theta=\tau} + \sum_{i=1}^N 1\!\!1_{\{\tau=\tau_i\}}\frac{1\!\!1_{\{p_t^i>0\}}}{p_t^i}\mathbb{E}\left[Y_T(\tau_i)p_T^i|\mathcal{F}_t\right]\bigg). \tag{4.2}$$

Since the measure η is non-atomic, one has

$$\int_t^T\mathbb{E}\left[Y_T(u)\alpha_T(u)|\mathcal{F}_t\right]\eta(du)=\mathbb{E}\left[\int_t^T 1\!\!1_{\cap_{i=1}^N\{\tau_i\neq u\}}Y_T(u)\alpha_T(u)\eta(du)\,\middle|\,\mathcal{F}_t\right].$$

By the condition (a), it is equal to

$$\mathbb{E}\left[\int_t^T 1\!\!1_{\cap_{i=1}^N\{\tau_i\neq u\}}Y_u(u)\alpha_u(u)\eta(du)\,\middle|\,\mathcal{F}_t\right]=\mathbb{E}\left[\int_t^T Y_u(u)\alpha_u(u)\eta(du)\,\middle|\,\mathcal{F}_t\right],$$

where we use again the fact that η is non-atomic. Therefore, by the condition (b), one can rewrite the term (4.1) as

$$\begin{aligned}&1\!\!1_{\{\tau>t\}}\frac{1\!\!1_{\{G_t>0\}}}{G_t}\bigg(\mathbb{E}\left[Y_TG_T-Y_tG_t|\mathcal{F}_t\right]+\mathbb{E}\left[\int_t^T Y_u(u)\alpha_u(u)\eta(du)\Big|\mathcal{F}_t\right]\\&\qquad+\sum_{i=1}^N\mathbb{E}\left[1\!\!1_{\{t<\tau_i\le T\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}|\mathcal{F}_t\right]\bigg),\end{aligned}\tag{4.3}$$

which vanishes thanks to the condition (c). Moreover, by condition (a) and (b), we can rewrite (4.2) as

$$1\!\!1_{\{\tau\le t\}}\left(-Y_t(\tau)+1\!\!1_{\cap_{i=1}^N\{\tau\neq\tau_i\}}\frac{1\!\!1_{\{\alpha_t(\tau)>0\}}}{\alpha_t(\tau)}Y_t(\tau)\alpha_t(\tau)+\sum_{i=1}^N1\!\!1_{\{\tau=\tau_i\}}\frac{1\!\!1_{\{p^i_t>0\}}}{p^i_t}Y_t(\tau_i)p^i_t\right),$$

which also vanishes.

In the following, we treat the local martingale case. Assume that the processes in (a)-(c) are local $\mathbb{F}$-martingales, then there exists a common sequence of $\mathbb{F}$-stopping times which localizes the processes (a)-(c) simultaneously. Thus it remains to prove the following claim: assume that σ is an $\mathbb{F}$-stopping time such that

(1) $1\!\!1_{\cap_{i=1}^n\{\tau_i\neq\theta\}}1\!\!1_{\{\sigma>0\}}Y^\sigma(\theta)\alpha^\sigma(\theta)$ is an $\mathbb{F}$-martingale on $[\theta,\infty[$ for $\theta\in\mathbb{R}_+$,

(2) $1\!\!1_{\{\sigma>0\}}Y^\sigma(\tau_i)p^{i,\sigma}$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$,

(3) the process $1\!\!1_{\{\sigma>0\}}\Big(Y^\sigma_tG^\sigma_t+\int_0^{\sigma\wedge t}Y_u(u)\alpha_u(u)\eta(du)+\sum_{i=1}^N1\!\!1_{\{\tau_i\le\sigma\wedge t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}\Big)$, $t\ge0$ is an $\mathbb{F}$-martingale,

then the process $1\!\!1_{\{\sigma>0\}}Y^{\mathbb{G},\sigma}$ is a $\mathbb{G}$-martingale.

Note that the processes $\alpha(\theta)$ and p^i are all $\mathbb{F}$-martingales for $\theta\ge0$, $i\in\{1,\dots,N\}$. Therefore, the conditions (1) and (2) imply the corresponding conditions in replacing $\alpha^\sigma(\theta)$ and $p^{i,\sigma}$ by $\alpha(\theta)$ and p^i respectively. We then deduce the following conditions

(1') $1\!\!1_{\cap_{i=1}^N\{\tau_i\neq\theta\}}1\!\!1_{\{\sigma>0\}}\big(1\!\!1_{\{\sigma<\theta\}}Y^\sigma+1\!\!1_{\{\sigma\ge\theta\}}Y^\sigma(\theta)\big)\alpha(\theta)$ is an $\mathbb{F}$-martingale on $[\theta,\infty[$ for any $\theta\ge0$,

(2') $1\!\!1_{\{\sigma>0\}}\big(1\!\!1_{\{\tau_i>\sigma\}}Y^\sigma+1\!\!1_{\{\tau_i\le\sigma\}}Y^\sigma(\tau_i)\big)p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$ for $i\in\{1,\cdots,N\}$,

(3') $1\!\!1_{\{\sigma>0\}}\Big(Y^\sigma_tG_t+\int_0^t(1\!\!1_{\{\sigma<u\}}Y^\sigma_u+1\!\!1_{\{\sigma\ge u\}}Y^\sigma_u(u))\alpha_u(u)\eta(du)+\sum_{i=1}^N1\!\!1_{\{\tau_i\le t\}}(1\!\!1_{\{\tau_i>\sigma\}}Y^\sigma_{\tau_i}+1\!\!1_{\{\tau_i\le\sigma\}}Y^\sigma_{\tau_i}(\tau_i))p^i_{\tau_i}\Big)$, $t\ge0$ is an $\mathbb{F}$-martingale.

One has $1\!\!1_{\{\sigma<\theta\}}Y^\sigma_t=1\!\!1_{\{\sigma<\theta\}}Y_\sigma$ on $[\theta,\infty[$ and hence

$$\begin{aligned}&\big(1\!\!1_{\{\sigma<\theta\}}Y^\sigma_t+1\!\!1_{\{\sigma\ge\theta\}}Y^\sigma_t(\theta)\big)\alpha_t(\theta)-Y^\sigma_t(\theta)\alpha^\sigma_t(\theta)\\&=1\!\!1_{\{\sigma<\theta\}}\big(Y_\sigma\alpha_t(\theta)-Y_\sigma(\theta)\alpha_\sigma(\theta)\big)+1\!\!1_{\{\sigma\ge\theta\}}Y_{t\wedge\sigma}(\theta)\big(\alpha_t(\theta)-\alpha_{t\wedge\sigma}(\theta)\big)\\&=1\!\!1_{\{\sigma<\theta\}}\big(Y_\sigma\alpha_t(\theta)-Y_\sigma(\theta)\alpha_\sigma(\theta)\big)+1\!\!1_{\{\sigma\ge\theta\}}Y_\sigma(\theta)\big(\alpha_t(\theta)-\alpha_{t\wedge\sigma}(\theta)\big),\quad t\ge\theta\end{aligned}$$

is an $\mathbb{F}$-martingale, which implies that (1) leads to (1'). Similarly, one has $1\!\!1_{\{\tau_i>\sigma\}}Y^\sigma_t=1\!\!1_{\{\tau_i>\sigma\}}Y_\sigma$ on $[\![\tau_i,\infty[\![$ and hence (2) leads to (2'). Finally, by (2.5), we obtain that

$$G_t+\int_0^t\alpha_u(u)\eta(du)+\sum_{i=1}^N1\!\!1_{\{\tau_i\le t\}}p^i_{\tau_i},\quad t\ge0$$

is an $\mathbb{F}$-martingale and hence

$$\mathbb{1}_{\{\sigma>0\}}Y_\sigma\Big(G_t - G_t^\sigma + \int_{\sigma\wedge t}^t \alpha_u(u)\eta(du) + \sum_{i=1}^N \mathbb{1}_{\{\sigma\wedge t<\tau_i\leq t\}}p_{\tau_i}^i\Big), \quad t\geq 0$$

is also an $\mathbb{F}$-martingale. Hence the condition (3) leads to (3'). By the martingale case of the proposition proved above, applied to the process

$$\mathbb{1}_{\{\sigma>0\}}Y_t^{\mathbb{G},\sigma} = \mathbb{1}_{\{\tau>t\}}\mathbb{1}_{\{\sigma>0\}}Y_t^\sigma + \mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\{\sigma>0\}}\big(\mathbb{1}_{\{\tau>\sigma\}}Y_t^\sigma + \mathbb{1}_{\{\tau\leq\sigma\}}Y_t^\sigma(\tau)\big),$$

we obtain that $\mathbb{1}_{\{\sigma>0\}}Y^{\mathbb{G},\sigma}$ is a $\mathbb{G}$-martingale. In fact, if we replace in the conditions (a)–(c) the process Y by $\mathbb{1}_{\{\sigma>0\}}Y^\sigma$, and $Y_t(\theta)$ by $\mathbb{1}_{\{\sigma>0\}}\big(\mathbb{1}_{\{\theta>\sigma\}}Y_t^\sigma + \mathbb{1}_{\{\theta\leq\sigma\}}Y_t^\sigma(\theta)\big)$, then the conditions (a)–(c) become (1')–(3'). The proposition is thus proved. □

In view of Proposition 4.1, it is natural to examine whether the converse is true. However, given a $\mathbb{G}$-adapted process $Y^{\mathbb{G}}$, the decomposition $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\leq t\}}Y_t(\tau)$, $\mathbb{P}$-a.s. is not unique. For example, if one modifies arbitrarily the value of $Y(\theta)$ on $\bigcap_{i=1}^n\{\tau_i\neq\theta\}$ for θ in an η-negligiable set, the decomposition equality remains valid. However, the $\mathbb{F}$-martingale property of $\mathbb{1}_{\cap_{i=1}^N\{\tau_i\neq\theta\}}Y(\theta)\alpha(\theta)$ cannot hold for all such modifications. In the following, we prove that, if $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale, then one can find at least one decomposition of $Y^{\mathbb{G}}$ such that Y and $Y(.)$ satisfy the $\mathbb{F}$-martingale conditions in Proposition 4.1.

Proposition 4.2. *Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-martingale. There exist a càdlàg $\mathbb{F}$-adapted process Y and an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable processes $Y(\cdot)$ which verify the following conditions :*

(a) $\mathbb{1}_{\cap_{i=1}^N\{\tau_i\neq\theta\}}Y(\theta)\alpha(\theta)$ is an $\mathbb{F}$-martingale on $[\theta,\infty[$;

(b) $Y(\tau_i)p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$ for any $i\in\{1,\cdots,N\}$;

(c) the process $Y_tG_t + \int_0^t Y_u(u)\alpha_u(u)\eta(du) + \sum_{i=1}^N \mathbb{1}_{\{\tau_i\leq t\}}Y_{\tau_i}(\tau_i)p_{\tau_i}^i, t\geq 0$ is an $\mathbb{F}$-martingale;

and such that, for any $t\geq 0$ one has $Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}Y_t + \mathbb{1}_{\{\tau\leq t\}}Y_t(\tau), t\geq 0$, $\mathbb{P}$-a.s.

Proof. The process $Y^{\mathbb{G}}$ can be written in the following decomposition form

$$Y_t^{\mathbb{G}} = \mathbb{1}_{\{\tau>t\}}\tilde{Y}_t + \mathbb{1}_{\{\tau\leq t\}}\hat{Y}_t(\tau), \tag{4.4}$$

where $\tilde{Y}$ and $\hat{Y}(\cdot)$ are respectively $\mathbb{F}$-adpated and $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted processes. Since $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale, for $i\in\{1,\cdots,N\}$ and $0\leq t\leq T$, one has

$$\mathbb{E}[Y_T^{\mathbb{G}}\mathbb{1}_{\{\tau=\tau_i\leq t\}}|\mathcal{F}_t] = \mathbb{E}[Y_t^{\mathbb{G}}\mathbb{1}_{\{\tau=\tau_i\leq t\}}|\mathcal{F}_t],$$

which implies

$$\mathbb{1}_{\{\tau_i\leq t\}}\mathbb{E}[\hat{Y}_T(\tau_i)p_T^i|\mathcal{F}_t] = \mathbb{1}_{\{\tau_i\leq t\}}\hat{Y}_t(\tau_i)p_t^i.$$

This equality shows that $\hat{Y}(\tau_i)p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty[\![$. We take a càdlàg version of this martingale and replace $\hat{Y}(\tau_i)$ on $[\![\tau_i,\infty[\![$ by the càdlàg version of this martingale multiplied by $\mathbb{1}_{\{p^i>0\}}(p^i)^{-1}$. This gives an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}_+)$-measurable version of $\hat{Y}(\cdot)$ and the equality (4.4) remains true $\mathbb{P}$-almost surely.

Similarly, for $0\leq t\leq T$, one has

$$\mathbb{E}[Y_T^{\mathbb{G}}\mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\cap_{i=1}^N\{\tau\neq\tau_i\}}|\mathcal{F}_t] = \mathbb{E}[Y_t^{\mathbb{G}}\mathbb{1}_{\{\tau\leq t\}}\mathbb{1}_{\cap_{i=1}^N\{\tau\neq\tau_i\}}|\mathcal{F}_t],$$

which implies

$$\int_0^t \mathbb{E}[\hat{Y}_T(\theta)\alpha_T(\theta)|\mathcal{F}_t]\,\eta(d\theta) = \int_0^t \hat{Y}_t(\theta)\alpha_t(\theta)\eta(d\theta). \tag{4.5}$$

Let D be a countable dense subset of $\mathbb{R}_+$. For any $\theta \in \mathbb{R}_+$ and all $s, t \in D$ such that $\theta \le s \le t$, let

$$\hat{Y}_{t|s}(\theta) = \frac{1\!\!1_{\{\alpha_s(\theta)>0\}}}{\alpha_s(\theta)}\mathbb{E}[\hat{Y}_t(\theta)\alpha_t(\theta)|\mathcal{F}_s].$$

The equality (4.5) shows that there exists an η-negligeable Borel subset B of $\mathbb{R}_+$ such that $\hat{Y}_{t|s}(\theta)\alpha_s(\theta) = \mathbb{E}[\hat{Y}_t(\theta)\alpha_t(\theta)|\mathcal{F}_s]$ provided that $\theta \notin B$. By the same arguments as in the proof of Proposition 2.3, we obtain a càdlàg $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted process $Y(\cdot)$ verifying the conditions (a) and (b), and such that $Y_t^{\mathbb{G}} = 1\!\!1_{\{\tau>t\}}\tilde{Y}_t + 1\!\!1_{\{\tau\le t\}}Y_t(\tau)$, $\mathbb{P}$-a.s..

For the last condition (c), for any $t \ge 0$, let

$$Y_t^{\mathbb{F}} = \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = \tilde{Y}_t G_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}Y_t(\tau_i)p_t^i.$$

The process $Y^{\mathbb{F}}$ is an $\mathbb{F}$-martingale. Since $Y(\tau_i)p^i$ is an $\mathbb{F}$-martingale on $[\![\tau_i, +\infty[\![$ for all $i = 1, \cdots, N$, we obtain that the process

$$\tilde{Y}_t G_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}, \quad t \ge 0$$

is also an $\mathbb{F}$-martingale. Let Z be a càdlàg version of this $\mathbb{F}$-martingale and let

$$Y_t = \frac{1\!\!1_{\{G_t>0\}}}{G_t}\left(Z_t - \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) - \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}Y_{\tau_i}(\tau_i)p^i_{\tau_i}\right), \quad t \ge 0$$

which is a càdlàg version of the process $\tilde{Y}$. The equality $Y_t^{\mathbb{G}} = 1\!\!1_{\{\tau>t\}}Y_t + 1\!\!1_{\{\tau\le t\}}Y_t(\tau)$, $\mathbb{P}$-a.s. still holds. The result is thus proved. □

In the theory of enlargement of filtrations, it is a classical problem to study whether an $\mathbb{F}$-martingale remains a $\mathbb{G}$-semimartingale. The standard hypothesis under which this assertion holds true is the density hypothesis (c.f. [14, Section 2] in the initial enlargement and [8, Proposition 5.9], [15, Theorem 3.1] in the progress enlargement of filtrations). We now give an affirmative answer to this question under the generalized density hypothesis, which provides a weaker condition.

Proposition 4.3. *Any $\mathbb{F}$-local martingale $U^{\mathbb{F}}$ is a $\mathbb{G}$-semimartingale which has the following decomposition:*

$$\begin{aligned} U_t^{\mathbb{F}} = U_t^{\mathbb{G}} &+ \int_{]0,t\wedge\tau]} \frac{d\langle U^{\mathbb{F}}, \bar{M}\rangle_s}{G_{s-}} \\ &+ 1\!\!1_{\cap_{i=1}^N\{\tau\neq\tau_i\}}\int_{]\tau,t\vee\tau]}\frac{d\langle U^{\mathbb{F}}, \alpha(u)\rangle_s}{\alpha_{s-}(u)}\Big|_{u=\tau} + \sum_{i=1}^N 1\!\!1_{\{\tau=\tau_i\}}\int_{]\tau,t\vee\tau]}\frac{d\langle U^{\mathbb{F}}, p^i\rangle_s}{p^i_{s-}}, \end{aligned} \tag{4.6}$$

where $U^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale and $\bar{M}$ is the $\mathbb{F}$-martingale defined as

$$\bar{M}_t = \mathbb{E}\left[\int_0^\infty \alpha_u(u)\eta(du)\Big|\mathcal{F}_t\right] + \sum_{i=1}^N p^i_{t\wedge\tau_i} + p_t^\infty, \quad t \ge 0. \tag{4.7}$$

Proof. Let

$$\bar{A}_t = \int_0^t \alpha_u(u)\eta(u) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}p^i_{\tau_i}.$$

One has $G = \bar{M} - \bar{A}$. We denote by

$$K_t = \int_{]0,t]}\frac{d\langle U^{\mathbb{F}}, \bar{M}\rangle_s}{G_{s-}}$$

and for $\theta \le t$,

$$K_t(\theta) = \mathbb{1}_{\cap_{i=1}^N \{\theta \neq \tau_i\}} \int_{]\theta,t]} \frac{d\langle U^{\mathbb{F}}, \alpha(\theta)\rangle_s}{\alpha_{s-}(\theta)} + \sum_{i=1}^N \mathbb{1}_{\{\theta=\tau_i\}} \int_{]\theta,t]} \frac{d\langle U^{\mathbb{F}}, p^i\rangle_s}{p^i_{s-}}$$

We define the process $U^{\mathbb{G}}$ as

$$U^{\mathbb{G}}_t = \mathbb{1}_{\{\tau>t\}} \left(U^{\mathbb{F}}_t - K_t\right) + \mathbb{1}_{\{\tau\le t\}} \left(U^{\mathbb{F}}_t - K_\tau - K_t(\tau)\right) = \mathbb{1}_{\{\tau>t\}}\tilde{U}_t + \mathbb{1}_{\{\tau\le t\}}\hat{U}_t(\tau),$$

where $\tilde{U}_t = U^{\mathbb{F}}_t - K_t$ and $\hat{U}_t(\theta) = U^{\mathbb{F}}_t - K_\theta - K_t(\theta)$. We check firstly that $\tilde{U}$ and $\hat{U}(\cdot)$ verify the condition (c) in Proposition 4.1. Let $Z = (Z_t)_{t\ge 0}$ be a process defined as

$$Z_t = \tilde{U}_t G_t + \int_0^t \hat{U}_u(u) d\bar{A}_u, \quad t \ge 0.$$

Then

$$\begin{aligned} dZ_t &= d(\tilde{U}_t G_t) + \hat{U}_t(t) d\bar{A}_t = d(U^{\mathbb{F}}_t G_t) - d(K_t G_t) + (U^{\mathbb{F}}_t - K_t) d\bar{A}_t \\ &= U^{\mathbb{F}}_{t-} dG_t + G_{t-} dU^{\mathbb{F}}_t + d[U^{\mathbb{F}}, G]_t - K_t dG_t - G_{t-} dK_t + U^{\mathbb{F}}_{t-} d\bar{A}_t - K_t d\bar{A}_t + d[U^{\mathbb{F}}, \bar{A}]_t \\ &= (U^{\mathbb{F}}_{t-} - K_t) d\bar{M}_t + G_{t-} dU^{\mathbb{F}}_t + d[U^{\mathbb{F}}, \bar{M}]_t - d\langle U^{\mathbb{F}}, \bar{M}\rangle_t. \end{aligned}$$

Therefore Z is an $\mathbb{F}$-local martingale.

We check now the conditions (a) and (b) in Proposition 4.1. On the set $\{\theta \neq \tau_1\} \cap \ldots \cap \{\theta \neq \tau_N\} \cap \{\alpha_t(\theta) > 0\}$, one has

$$d\left(\hat{U}_t(\theta)\alpha_t(\theta)\right) = \left(U^{\mathbb{F}}_{t-} - K_\theta - K_t(\theta)\right) d\alpha_t(\theta) + \alpha_{t-}(\theta) dU^{\mathbb{F}}_t + d[U^{\mathbb{F}}, \alpha(\theta)]_t - d\langle U^{\mathbb{F}}, \alpha(\theta)\rangle_t, \quad \theta \le t$$

and on the set $\{\tau_i \le t\} \cap \{p^i_t > 0\}$ for all $i = 1, \ldots, N$,

$$d\left(\hat{U}_t(\tau_i) p^i_t\right) = \left(U^{\mathbb{F}}_{t-} - K_{\tau_i} - K_t(\tau_i)\right) dp^i_t + p^i_{t-} dU^{\mathbb{F}}_t + d[U^{\mathbb{F}}, p^i]_t - d\langle U^{\mathbb{F}}, p^i\rangle_t.$$

Therefore the process $\mathbb{1}_{\cap_{i=1}^N \{\theta \neq \tau_i\}} \hat{U}(\theta)\alpha(\theta)$ is an $\mathbb{F}$-local martingale on $[\theta, \infty[$, and the process $\hat{U}(\tau_i)p^i$ is an $\mathbb{F}$-local martingale on $[\![\tau_i, \infty[\![$ for all $i = 1, \ldots, N$. By Proposition 4.1, we obtain that $U^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. □

Remark 4.4. We note that the decomposition $G = \bar{M} - \bar{A}$ in the proof of the above proposition is different from the Doob-Meyer decomposition of G since $\bar{A}$ is an $\mathbb{F}$-optional process. However, if $\mathbb{F}$ is quasi left continuous, this decomposition coincides with the Doob-Meyer decomposition. A general discussion concerning the optional decomposition can be found in Song [22].

5 Applications

In this section, as applications of previous results in the generalized density approach, we first discuss about the immersion property which is widely adopted in the credit risk models and then study a two-name model with simultaneous defaults.

5.1 Immersion property

The pair of filtrations $(\mathbb{F}, \mathbb{G})$ is said to verify the immersion property if any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. In the literature of default modelling, the immersion property is often supposed for the pricing of credit derivatives at times before default. We give below a criterion under the generalized density hypothesis for the immersion property to hold true.

Proposition 5.1. *The immersion property holds for* $(\mathbb{F},\mathbb{G})$ *under the following conditions:*

(a) $\alpha_t(\theta)=\alpha_\theta(\theta)$ *for* $0\le\theta\le t$ *on* $\bigcap_{i=1}^N\{\tau_i\neq\theta\}$*;*

(b) $p_t^i=p_{\tau_i\wedge t}^i$ *for any* $i\in\{1,\cdots,N\}$*.*

Proof. Let Y be an $\mathbb{F}$-martingale. It can be considered as a $\mathbb{G}$-adapted process and admits the following decomposition $Y_t=1\!\!1_{\{\tau>t\}}Y_t+1\!\!1_{\{\tau\le t\}}Y_t, t\ge 0$. The condition (a) implies that the process $1\!\!1_{\cap_{i=1}^n\{\tau_i\neq\theta\}}\alpha(\theta)Y$ is an $\mathbb{F}$-martingale on $[\theta,\infty[$ for any $\theta>0$. The condition (b) implies that Yp^i is an $\mathbb{F}$-martingale on $[\![\tau_i,\infty]\!]$ for any $i\in\{1,\cdots,N\}$. For the last condition in Proposition 4.1, we have

$$\begin{aligned}&Y_tG_t+\int_0^t Y_u\alpha_u(u)\eta(du)+\sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}Y_{\tau_i}p_{\tau_i}^i\\&=Y_t\int_0^\infty\alpha_t(u)\eta(du)+\sum_{i=1}^N Y_{\tau_i\wedge t}p_{\tau_i\wedge t}^i+Y_tp_t^\infty\\&=Y_t\bigg(\int_0^\infty\alpha_t(u)\eta(du)+\sum_{i=1}^N p_t^i+p_t^\infty\bigg)+\sum_{i=1}^N(Y_{\tau_i\wedge t}-Y_t)p_{\tau_i\wedge t}^i\\&=Y_t+\sum_{i=1}^N(Y_{\tau_i\wedge t}-Y_t)p_{\tau_i\wedge t}^i\end{aligned}$$

where the second equality comes from the fact $p_{\tau_i\wedge t}^i=p_t^i$ and the third equality comes from (2.4). Since Y is an $\mathbb{F}$-martingale, $\big((Y_{\tau_i\wedge t}-Y_t)p_{\tau_i\wedge t}^i\big)_{t\ge 0}$ is an $\mathbb{F}$-martingale for any $i=1,\cdots,N$. Hence we obtain the result. □

Conversely, if the immersion property holds, then

(a) we can choose suitable conditional density process $\alpha(\cdot)$ such that $\alpha_t(\theta)=\alpha_\theta(\theta)$ for $0\le\theta\le t$ on $\bigcap_{i=1}^n\{\tau_i\neq\theta\}$

(b) for any $i\in\{1,\ldots,N\}$, the $\mathbb{F}$-martingale p^i is stopped at τ_i.

However, the condition (a) may not hold in general since we are allowed to change the value of $\alpha_t(\theta)$ for θ in a η-negligible set without changing the $\mathbb{F}$-conditional law of τ.

The immersion property is not necessarily preserved under a change of probability measure. In the following, we study the change of probability measures based on the previous results of $\mathbb{G}$-martingale characterization, similar as in [8, Section 6.1]. Firstly, we deduce relevant processes under a change of probability measure, as a generalization of [8, Theorem 6.1]. Secondly, we show that to begin from an arbitrary probability measure (where the immersion is not necessarily satisfied), we can always find a change of probability which is invariant on $\mathbb{F}$, and the immersion property holds under the new probability measure.

Proposition 5.2. *Let* $Y^{\mathbb{G}}$ *be a positive* $\mathbb{G}$*-martingale of expectation* 1*, which is written in the decomposed form as* $Y_t^{\mathbb{G}}=1\!\!1_{\{\tau>t\}}Y_t+1\!\!1_{\{\tau\le t\}}Y_t(\tau)$ *where* Y *and* $Y(\cdot)$ *are positive processes which are respectively* $\mathbb{F}$*-adapted and* $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$*-adapted. Let* $\mathbb{Q}$ *be the probability measure such that* $d\mathbb{Q}/d\mathbb{P}=Y_t^{\mathbb{G}}$ *on* $\mathcal{G}_t$ *for any* $t\ge 0$*. Then the random time* τ *satisfies Assumption 2.4 under the probability* $\mathbb{Q}$*, and the* $(\mathbb{F},\mathbb{Q})$*-conditional density avoiding* $(\tau_i)_{i=1}^N$ *and the* $(\mathbb{F},\mathbb{Q})$*-conditional probabilty of* $\tau=\tau_i<\infty$ *can be written in the following form*

$$\alpha_t^{\mathbb{Q}}(\theta)=1\!\!1_{\{\theta\le t\}}\frac{Y_t(\theta)}{Y_t^{\mathbb{F}}}\alpha_t(\theta)+1\!\!1_{\{\theta>t\}}\frac{\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t]}{Y_t^{\mathbb{F}}},\quad p_t^{i,\mathbb{Q}}=\frac{Y_t(\tau_i)p_t^i}{Y_t^{\mathbb{F}}}\tag{5.1}$$

where

$$Y_t^{\mathbb{F}} := \mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = G_t Y_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}} Y_t(\tau_i) p_t^i.$$

Proof. Let $Y^{\mathbb{G}}$ be as in the statement of the proposition. Let h be a bounded Borel function, then

$$\mathbb{E}^{\mathbb{Q}}[1\!\!1_{\{\tau<\infty\}}h(\tau)|\mathcal{F}_t] = \lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}}[1\!\!1_{\{\tau\le n\}}h(\tau)|\mathcal{F}_t] = \lim_{n\to+\infty}\frac{\mathbb{E}[1\!\!1_{\{\tau\le n\}}Y^{\mathbb{G}}_{\tau\vee t}h(\tau)|\mathcal{F}_t]}{\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t]} \tag{5.2}$$

where we use the optional stopping theorem of Doob for the second equality. Note that

$$\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = G_t Y_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\,\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}} Y_t(\tau_i) p_t^i$$

and for any $n\in\mathbb{N}$,

$$\begin{aligned}\mathbb{E}[1\!\!1_{\{\tau\le n\}}Y^{\mathbb{G}}_{\tau\vee t}h(\tau)|\mathcal{F}_t] = \int_0^n &\Big(1\!\!1_{\{\theta\le t\}}Y_t(\theta)\alpha_t(\theta) + 1\!\!1_{\{\theta>t\}}\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t]\Big)h(\theta)\,\eta(d\theta)\\ &+\sum_{i=1}^N\Big(1\!\!1_{\{\tau_i\le t\wedge n\}}Y_t(\tau_i)p_t^i h(\tau_i) + 1\!\!1_{\{\tau_i>t\}}\mathbb{E}[Y_{\tau_i}(\tau_i)p^i_{\tau_i}h(\tau_i)1\!\!1_{\{\tau_i\le n\}}|\mathcal{F}_t]\Big).\end{aligned}$$

Hence

$$\begin{aligned}\lim_{n\to+\infty}\mathbb{E}[1\!\!1_{\{\tau\le n\}}Y^{\mathbb{G}}_{\tau\vee t}h(\tau)|\mathcal{F}_t] = \int_0^\infty &\Big(1\!\!1_{\{\theta\le t\}}Y_t(\theta)\alpha_t(\theta) + 1\!\!1_{\{\theta>t\}}\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t]\Big)h(\theta)\,\eta(d\theta)\\ &+\sum_{i=1}^N\Big(1\!\!1_{\{\tau_i\le t\}}Y_t(\tau_i)p_t^i h(\tau_i) + 1\!\!1_{\{\tau_i>t\}}\mathbb{E}[Y_{\tau_i}(\tau_i)p^i_{\tau_i}h(\tau_i)1\!\!1_{\{\tau_i<\infty\}}|\mathcal{F}_t]\Big),\end{aligned}$$

which implies the required result together with (5.2). □

Proposition 5.3. *We assume that the processes $\alpha(\cdot)$ and p^i, $i\in\{1,\cdots,N\}$ are strictly positive. Let Y and $Y(\cdot)$ be respectively $\mathbb{F}$-adapted and $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted processes such that*

$$Y_t = \frac{1}{G_t}\Big(1 - \int_0^t \alpha_\theta(\theta)\,\eta(d\theta) - \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}p^i_{\tau_i}\Big), \tag{5.3}$$

$$Y_t(\theta) = 1\!\!1_{\cap_{i=1}^N\{\tau_i\neq\theta\}}\frac{\alpha_\theta(\theta)}{\alpha_t(\theta)} + \sum_{i=1}^N 1\!\!1_{\{\tau_i=\theta\}}\frac{p^i_\theta}{p^i_t},\quad 0\le\theta\le t. \tag{5.4}$$

Then the $\mathbb{G}$-adapted process $Y^{\mathbb{G}}$ defined by $Y_t^{\mathbb{G}} = 1\!\!1_{\{\tau>t\}}Y_t + 1\!\!1_{\{\tau\le t\}}Y_t(\tau)$ is a non-negative $\mathbb{G}$-martingale with expectation 1. Moreover, if we denote by $\mathbb{Q}$ the probability measure such that $d\mathbb{Q}/d\mathbb{P} = Y_t^{\mathbb{G}}$ on $\mathcal{G}_t$, then the restriction of $\mathbb{Q}$ on $\mathcal{F}_\infty$ coincides with $\mathbb{P}$ and $(\mathbb{F},\mathbb{G})$ verifies the immersion property under the probability $\mathbb{Q}$. Moreover, one has $\alpha^{\mathbb{Q}}_\theta(\theta) = \alpha_\theta(\theta)$ on $\bigcap_{i=1}^N\{\tau_i\neq\theta\}$ and $p^{i,\mathbb{Q}}_{\tau_i} = p^i_{\tau_i}$.

Proof. The assertion that $Y^{\mathbb{G}}$ is a $\mathbb{G}$-martingale results from Proposition 4.1. Moreover, one has

$$\mathbb{E}[Y_t^{\mathbb{G}}|\mathcal{F}_t] = G_t Y_t + \int_0^t Y_t(\theta)\alpha_t(\theta)\eta(d\theta) + \sum_{i=1}^N 1\!\!1_{\{\tau_i\le t\}}Y_t(\tau_i)p_t^i = 1.$$

Therefore the expectation of $Y_t^{\mathbb{G}}$ is 1, and the restriction of $\mathbb{Q}$ to $\mathcal{F}_\infty$ coincides with $\mathbb{P}$. Il remains to verify that $(\mathbb{F},\mathbb{G})$ satisfies to the immersion property under the probability

$\mathbb{Q}$ and the invariance of the values of $\alpha_\theta(\theta)$ and $p^i_{\tau_i}$. By the previous proposition, on $\bigcap_{i=1}^n\{\tau_i \neq \theta\}$ one has

$$\alpha_t^{\mathbb{Q}}(\theta) = 1\!\!1_{\{\theta\le t\}} Y_t(\theta)\alpha_t(\theta) + 1\!\!1_{\{\theta>t\}}\mathbb{E}[Y_\theta(\theta)\alpha_\theta(\theta)|\mathcal{F}_t] = 1\!\!1_{\{\theta\le t\}}\alpha_\theta(\theta) + 1\!\!1_{\{\theta>t\}}\mathbb{E}[\alpha_\theta(\theta)|\mathcal{F}_t]$$

and

$$p_t^{i,\mathbb{Q}} = Y_t(\tau_i)p_t^i = p^i_{\tau_i} \text{ on } \{\tau_i \le t\}.$$

In particular, one has $\alpha_\theta^{\mathbb{Q}}(\theta) = \alpha_\theta(\theta)$ on $\bigcap_{i=1}^N\{\tau_i \neq \theta\}$ and $p_t^{i,\mathbb{Q}} = p^i_{\tau_i}$. Moreover, by Proposition 5.1 we obtain that $(\mathbb{F},\mathbb{G})$ satisfies to the immersion property under the probability $\mathbb{Q}$. The result is thus proved. □

5.2 A two-name model with simultaneous default

The density approach has been adopted to study multiple random times in [9], [16] and [19]. In the classical literature of multi-default modelling, one often supposes that there is no simultaneous defaults, notably in the classical intensity and density models. For example, if we suppose that the conditional joint $\mathbb{F}$-density exists for two default times, then the probability that the two defaults coincide equals to zero (see [9]). However, during the financial crisis where the risk of contagious defaults is high, it is important to study simultaneous defaults whose occurrence is rare but will have significant impact on financial market. The generalized density approach provides mathematical tools to study simultaneous defaults. The idea consists of using a recurrence method.

In the following, we consider two random times σ_1 and σ_2 defined on the probability space $(\Omega,\mathcal{A},\mathbb{F},\mathbb{P})$ and we assume that

$$\mathbb{P}(\sigma_1 \in d\theta_1, \sigma_2 \in d\theta_2|\mathcal{F}_t) = \beta_t(\theta_1,\theta_2)d\theta_1 d\theta_2 + \Delta_*(q_t(\theta)d\theta), \tag{5.5}$$

where $\beta(\cdot,\cdot)$ and $q(\cdot)$ are respectively positive càdlàg $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+^2)$ and $\mathbb{F}\otimes\mathcal{B}(\mathbb{R}_+)$-adapted processes, and $\Delta : \mathbb{R}_+ \to \mathbb{R}_+^2$ denotes the diagonal embedding which sends $x \in \mathbb{R}_+$ to $(x,x) \in \mathbb{R}^2$, and $\Delta_*(q_t(\theta)d\theta)$ is the direct image of the Borel measure $q_t(\theta)d\theta$ by the map Δ. Namely for any bounded Borel function $h(\cdot)$ on $\mathbb{R}_+^2$, one has

$$\mathbb{E}[h(\sigma_1,\sigma_2)|\mathcal{F}_t] = \int_{\mathbb{R}_+^2}\beta_t(\theta_1,\theta_2)h(\theta_1,\theta_2)d\theta_1 d\theta_2 + \int_{\mathbb{R}_+} q_t(\theta)h(\theta,\theta)d\theta.$$

In particular, the $\mathbb{F}$-conditional probability of simultaneous default is given by

$$\mathbb{P}[\sigma_1 = \sigma_2|\mathcal{F}_t] = \int_{\mathbb{R}_+} q_t(\theta)\,d\theta.$$

We shall apply previous results to this two-default model. Let $\mathbb{F}^1$ be the progressive enlargement of $\mathbb{F}$ by the random time σ_1. Then σ_1 is an $\mathbb{F}^1$-stopping time. The filtration $\mathbb{F}^1$ will play the role of the reference filtration in the previous sections.

Proposition 5.4. *The random time σ_2 satisfies the generalized density hypothesis with respect to the filtration $\mathbb{F}^1$. The $\mathbb{F}^1$-conditional density of σ_2 avoiding σ_1 is given by*

$$\alpha_t^{2|1}(\theta) = 1\!\!1_{\{\sigma_1>t\}}\frac{\int_t^\infty \beta_t(s,\theta)ds}{G_t^1} + 1\!\!1_{\{\sigma_1\le t\}}\frac{\beta_t(\sigma_1,\theta)}{\alpha_t^1(\sigma_1)},\quad t\ge 0 \tag{5.6}$$

where $\alpha_t^1(\cdot)$ is the $\mathcal{F}_t$-density of σ_1 and $G_t^1 = \mathbb{P} = (\sigma_1 > t|\mathcal{F}_t)$. In addition, the $\mathbb{F}^1$-conditional probability of simultaneous default is given by

$$p_t := \mathbb{P}(\sigma_2 = \sigma_1|\mathcal{F}_t^1) = 1\!\!1_{\{\sigma_1>t\}}\frac{\int_t^\infty q_t(\theta)d\theta}{G_t^1} + 1\!\!1_{\{\sigma_1\le t\}}\frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)}. \tag{5.7}$$

Proof. The hypothesis (5.5) implies that

$$\mathbb{P}(\sigma_1 \in d\theta|\mathcal{F}_t) = \left(\int_{\mathbb{R}_+} \beta_t(\theta, \theta_2)d\theta_2 + q_t(\theta) \right) d\theta$$

So the random time σ_1 admits $\mathbb{F}$-conditional density which is given by

$$\alpha_t^1(\theta) = \int_{\mathbb{R}_+} \beta_t(\theta, \theta_2)d\theta_2 + q_t(\theta). \tag{5.8}$$

Let $G_t^1 = \mathbb{P}(\sigma_1 > t|\mathcal{F}_t) = \int_t^\infty \alpha_t^1(\theta)d\theta$. Direct computations yields

$$\mathbb{P}(\sigma_2 = \sigma_1|\mathcal{F}_t^1) = 1\!\!1_{\{\sigma_1 > t\}} \frac{\int_t^\infty q_t(\theta)d\theta}{G_t^1} + 1\!\!1_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)}. \tag{5.9}$$

In fact, the term on the set $\{\sigma_1 > t\}$ is classical. For the term on the set $\{\sigma_1 \le t\}$ in (5.9), consider a bounded test function $Y_t(\cdot)$ which is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable, by (5.5) one has

$$\mathbb{E}[1\!\!1_{\{\sigma_1 = \sigma_2 \le t\}} Y_t(\sigma_1)] = \int_0^t \mathbb{E}[q_t(\theta)Y_t(\theta)]d\theta.$$

Since

$$\mathbb{E}\Big[1\!\!1_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)} Y_t(\sigma_1)\Big] = \int_0^t \mathbb{E}\Big[\frac{q_t(\theta)}{\alpha_t^1(\theta)} Y_t(\theta)\alpha_t^1(\theta)\Big]d\theta = \int_0^t \mathbb{E}[q_t(\theta)Y_t(\theta)]d\theta,$$

then

$$1\!\!1_{\{\sigma_1 \le t\}}\mathbb{P}(\sigma_2 = \sigma_1|\mathcal{F}_t^1) = 1\!\!1_{\{\sigma_1 \le t\}} \frac{q_t(\sigma_1)}{\alpha_t^1(\sigma_1)}.$$

In a similar way, we obtain (5.6). □

Remark 5.5. By the symmetry between σ_1 and σ_2, the generalized density hypothesis is also satisfied by σ_1 with respect to the filtration $\mathbb{F}^2$.

We are interested in the compensator process of σ_2 in the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \ge 0}$ which is the progressive enlargement of $\mathbb{F}^1$ by the random time σ_2. The random time σ_1 admits $\mathbb{F}$-density, so σ_1 is a totally inaccessible $\mathbb{F}^1$-stopping time. By Proposition 3.2, we know that σ_2 is a totally inaccessible $\mathbb{G}$-stopping time and the intensity exists.

Proposition 5.6. *The random time σ_2 has a $\mathbb{G}$-intensity given by*

$$\lambda_t^{2,\mathbb{G}} = 1\!\!1_{\{\sigma_2 > t\}} \left(1\!\!1_{\{\sigma_1 > t\}} \frac{\int_t^\infty \beta_t(\theta_1, t)d\theta_1 + q_t(t)}{\int_t^\infty \int_t^\infty \beta_t(\theta_1, \theta_2)d\theta_1 d\theta_2 + \int_t^\infty q_t(\theta)d\theta} + 1\!\!1_{\{\sigma_1 \le t\}} \frac{\beta_t(\sigma_1, t)}{\int_t^{+\infty} \beta_t(\sigma_1, \theta)d\theta} \right).$$

Similarly, the $\mathbb{G}$-intensity of σ_1 is given by

$$\lambda_t^{1,\mathbb{G}} = 1\!\!1_{\{\sigma_1 > t\}} \left(1\!\!1_{\{\sigma_2 > t\}} \frac{\int_t^\infty \beta_t(t, \theta_2)d\theta_2 + q_t(t)}{\int_t^\infty \int_t^\infty \beta_t(\theta_1, \theta_2)d\theta_1 d\theta_2 + \int_t^\infty q_t(\theta)d\theta} + 1\!\!1_{\{\sigma_2 \le t\}} \frac{\beta_t(t, \sigma_2)}{\int_t^{+\infty} \beta_t(\theta, \sigma_2)d\theta} \right).$$

Proof. It suffices to prove the first assertion. The $\mathbb{G}$-compensator of σ_2 is given by

$$\Lambda_t^{2,\mathbb{G}} = \int_0^{\sigma_2 \wedge t} \frac{dA_s^{2|1}}{G_s^{2|1}}, \quad t \ge 0.$$

where

$$G_t^{2|1} = \mathbb{P}(\sigma_2 > t|\mathcal{F}_t^1) = \frac{1\!\!1_{\{\sigma_1 > t\}}}{G_t^1} \left(\int_t^\infty \int_t^\infty \beta_t(s, \theta)dsd\theta + \int_t^\infty q_t(\theta)d\theta \right) + \frac{1\!\!1_{\{\sigma_1 \le t\}}}{\alpha_t^1(\sigma_1)} \int_t^{+\infty} \beta_t(\sigma_1, \theta)d\theta$$

and $A^{2|1}$ is the compensator of the $\mathbb{F}^1$-conditional survival process $G^{2|1}$ of σ_2. By Proposition 3.1,

$$A_t^{2|1} = \int_0^t \alpha_\theta^{2|1}(\theta)d\theta + \int_0^t p_{s-}d\Lambda_s^1 + \langle M^1, p\rangle_t, \quad t \geq 0,$$

where $\alpha^{2|1}$ and p are given as in Proposition 5.4, Λ^1 is the $\mathbb{F}^1$-compensator of σ_1 given by

$$\Lambda_t^1 = \int_0^{\sigma_1\wedge t} \frac{\alpha_s^1(s)}{G_s^1}\, ds, \quad t \geq 0.$$

and $M_t^1 = 1\!\!1_{\{\sigma_1 \leq t\}} - \Lambda_t^1$, $t \geq 0$ is $\mathbb{F}^1$-martingale. Note that $\langle M^1, p\rangle$ is the $\mathbb{F}^1$-compensator of the process

$$1\!\!1_{\{\sigma_1\leq t\}}\Delta p_{\sigma_1} = 1\!\!1_{\{\sigma_1\leq t\}}\Big(\frac{q_{\sigma_1}(\sigma_1)}{\alpha_{\sigma_1}^1(\sigma_1)} - \frac{\int_{\sigma_1}^\infty q_{\sigma_1}(\theta)d\theta}{G_{\sigma_1}^1}\Big), \quad t \geq 0,$$

which equals

$$\int_0^{\sigma_1\wedge t} \frac{\alpha_s^1(s)H_s}{G_s^1}ds, \quad t \geq 0$$

where (cf. [8, Corollary 4.6])

$$H_t = \frac{q_t(t)}{\alpha_t^1(t)} - \frac{\int_t^\infty q_t(\theta)d\theta}{G_t^1}.$$

Hence we obtain that

$$A_t^{2|1} = \int_0^t \alpha_\theta^{2|1}(\theta)d\theta + \int_0^{\sigma_1\wedge t} \frac{q_\theta(\theta)}{G_\theta^1}d\theta$$

which implies that the random time σ_2 has a $\mathbb{G}$-intensity given as in the proposition. □

Remark 5.7. The equality

$$\mathbb{P}(\sigma_1\wedge\sigma_2 > t|\mathcal{F}_t) = \int_t^\infty\int_t^\infty \beta_t(\theta_1,\theta_2)d\theta_1 d\theta_2 + \int_t^{+\infty} q_t(\theta)d\theta$$

shows that $\mathbb{F}$-intensity process of $\sigma_1\wedge\sigma_2$ is

$$\lambda_t^{\min} := \frac{\int_t^\infty \beta_t(\theta,t) + \beta_t(t,\theta)d\theta + q_t(t)}{\int_t^\infty\int_t^\infty \beta_t(\theta,\theta_2)d\theta_1 d\theta_2 + \int_t^\infty q_t(\theta)d\theta}.$$

Note that the relation

$$1\!\!1_{\{\sigma_1\wedge\sigma_2>t\}}\lambda^{\min} = 1\!\!1_{\{\sigma_1\wedge\sigma_2>t\}}(\lambda_t^{1,\mathbb{G}} + \lambda_t^{2,\mathbb{G}})$$

does not hold in general under the generalized density hypothesis.

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