

Hedging under multiple risk constraints

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Abstract Motivated by the asset–liability management problems under shortfall risk constraints, we consider in a general discrete-time framework the problem of finding the least expensive portfolio whose shortfalls with respect to a given set of stochastic benchmarks are bounded by a specific shortfall risk measure. We first show how the price of this portfolio may be computed recursively by dynamic programming for different shortfall risk measures, in complete and incomplete markets. We then focus on the specific situation where the shortfall risk constraints are imposed at each period on the next-period shortfalls, and obtain explicit results. Finally, we apply our results to a realistic asset–liability management problem of an energy company, and show how the shortfall risk constraints affect the optimal hedging of liabilities.

Keywords Multiple risk constraints · Snell envelope · Dynamic programming · Shortfall risk · Asset–liability management

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1 Introduction

In various economic contexts, institutions hold assets to cover future liabilities. Banks and insurance companies are required by the authorities to hold regulatory capital to cover the risks they take. Pension funds face random future liabilities due to longevity risk and the structure of the pension plans which may involve variable annuity-type features. The problem of managing a portfolio of assets under the condition of covering future liabilities or benchmarks, in particular in the context of pension plans, is commonly known as asset–liability management (ALM) (see e.g. [7, 14, 16]).

Often, the regulators stipulate that a specific probabilistic criterion should be used to measure the potential losses arising from not being able to cover the liabilities. For financial institutions, the Basel II framework uses the Value at Risk (VaR) to determine the regulatory capital. Insurance companies, under the European Solvency II directive, are required to evaluate the amount of capital necessary to cover their liabilities for a time horizon of one year with a probability of 99.5%. Under these frameworks, companies must therefore hold enough assets to limit the losses with respect to random future benchmarks in the sense of a probabilistic loss measure.

In the literature, the problem of hedging a single random liability under a probabilistic loss constraint has been introduced and studied in Föllmer and Leukert [9, 10]. In particular, explicit solutions were obtained for the Black–Scholes model within a continuous-time complete market setting. This was further developed in a Markovian diffusion context in Bouchard et al. [1] using stochastic control and viscosity solutions, and later extended to other classes of stochastic target problems in [2, 3]. A related strand of literature deals with portfolio management under the additional constraint of outperforming a deterministic benchmark at a given future date as in Grossman and Vila [12], or a stochastic benchmark at a given date as in Gundel and Weber [13] and Boyle and Tian [5], or a deterministic benchmark at all future dates as in El Karoui et al. [8]. However, the problem of hedging multiple stochastic benchmarks at a set of future dates has received relatively little attention in the literature.¹

Motivated by both theoretical and practical issues, in this paper, we consider the problem of an economic agent who has random liabilities payable at a finite set of future dates, with possibly path-dependent shortfall risk constraints. More precisely, the agent is looking for the cheapest discrete-time self-financing portfolio process $(V_k)_{k=0}^n$ which satisfies

$$\rho(V_1 - S_1, \dots, V_n - S_n) \leq 0,$$

where $(S_i)_{i=1}^n$ are the benchmark values at the payment dates and ρ is a specific risk measure. In the paper, we study the following three risk measures:

1. General path-dependent constraints: $\rho(X) = \mathbb{E}[L(X_1, \dots, X_n)]$ for an n -dimensional loss function L with $X = (X_1, \dots, X_n)$.

¹Very recently (while this paper was under review), the problem of hedging multiple stochastic benchmark under probabilistic loss constraints has been addressed in [4] (see Chaps. 2 and 3) using the machinery of viscosity solutions. That author works in a Markovian complete market setting, but her assumptions allow loss functions which are less regular than those in the present paper.

2. European-style constraints: $\rho(X) \leq 0$ iff $\mathbb{E}[\ell(X_k)] \leq \alpha_k$, $k = 1, \dots, n$, for a loss function ℓ and bounds $\alpha_1, \dots, \alpha_n$.
3. Next-period constraints: $\rho(X) \leq 0$ iff $\mathbb{E}[\ell(X_k)|\mathcal{F}_{k-1}] \leq \alpha_k$, $k = 1, \dots, n$.

For the three constraint types, we show how the superhedging price may be computed recursively by dynamic programming in a general non-Markovian setting, under both complete and incomplete market assumptions.

We then illustrate our results in three ways. First, in a simplified two-period setting where the market is complete and the risk-neutral probability coincides with the historical one, we derive fully explicit solutions and show how imposing the constraints in three different ways modifies the superhedging price of a random benchmark process.

Second, we consider the practically more relevant case when the market may be incomplete, but the market returns over different periods are independent, such as in multidimensional exponential Lévy models. Under the assumption that the benchmark process is deterministic, we derive fully explicit formulas for the next-period constraint with power-like or exponential loss function and for the European-style constraint with exponential loss function. The assumption of deterministic benchmark is relevant in practice and corresponds to a “portfolio insurance” point of view on the problem.

Third, we present an application of our method to the ALM problem of an energy company facing nuclear decommissioning charges. Under current regulations, future nuclear decommissioning charges are discounted using a discount rate that is higher than the risk-free rate and is supposed to reflect the rate of return of the assets in which the decommissioning funds are invested. However, this practice runs counter to the logic of financial economics [15] and creates a sizable risk of not being able to meet the future liabilities. Our methods enable us to quantify this risk.

The rest of the paper is structured as follows. In Sect. 2, we introduce three different risk constraint types and the formulations of the multi-target hedging problem according to these constraints. Section 3 presents a non-Markovian dynamic programming approach to the solution. We apply a recursive method to give a characterization of the least expensive hedging portfolio for each of the three constraint types. Section 4 compares the different constraint types on a simplified example, and provides explicit solutions for the next-period and the European style constraint for specific loss functions. Finally, Sect. 5 discusses the ALM problem of a nuclear power plant operator using real data. Technical lemmas are proved in Appendix A.

2 Problem formulations and risk constraints

We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T})$, where \mathcal{G}_0 is a trivial σ -field and $\mathcal{G}_T = \mathcal{F}$.

Given a finite sequence of deterministic dates $0 = t_0 < t_1 < \dots < t_n \leq T$, we study the problem of an economic agent who may invest in the financial market and is liable to make a series of payments P_1, \dots, P_n at dates t_1, \dots, t_n , where for each i , P_i is \mathcal{G}_{t_i} -measurable. This may model for example the cash flows associated to the

decommissioning and dismantling of a nuclear power plant, or to a variable annuity insurance contract. The portfolio of the agent, whose value is denoted by \tilde{V}_t , is effectively used to make the payments and therefore satisfies $\tilde{V}_t = \tilde{V}_{t-} - P_t$.

To work with self-financing portfolios, we introduce the portfolio augmented with cumulated cash flows as

$$V_t = \tilde{V}_t + \sum_{i \geq 1: t_i \leq t} P_i.$$

Given a sequence of benchmark values $(S_i)_{i=1}^n$ at the dates t_1, \dots, t_n , the agent is interested in finding the cheapest portfolio process $(V_t)_{0 \leq t \leq T}$ such that the sequence of shortfall values $V_{t_1} - S_1, \dots, V_{t_n} - S_n$ with respect to the benchmark is acceptable with respect to a specific path-dependent shortfall risk measure, to be described below. If the goal of the agent is simply to maintain a positive (modulo the risk tolerance) portfolio \tilde{V}_t after each cash flow, the benchmark values are

$$S_k = \sum_{i=1}^k P_i, \tag{2.1}$$

but other definitions, involving in particular the future cash flows, are also possible.

Since we are only interested in the values of the agent’s portfolio at the discrete dates t_0, \dots, t_n , we introduce a discrete filtration $\mathbb{F} = (\mathcal{F}_k)_{k=0,1,\dots,n}$ defined by $\mathcal{F}_k = \mathcal{G}_{t_k}$. To save space, the conditional expectation $\mathbb{E}[\cdot | \mathcal{F}_k]$ is denoted by $\mathbb{E}_k[\cdot]$. From now on, we work in the discrete-time setting and, with a slight abuse of notation, denote the value of the agent’s portfolio at time t_k by V_k .

Without loss of generality, we assume the interest rate to be zero. To define the investment opportunities for the agent, we introduce two alternative assumptions:

- (C) The set \mathcal{A} of admissible portfolio processes for the agent contains all supermartingales with respect to the probability measure \mathbb{Q} equivalent to \mathbb{P} (which plays the role of the unique martingale measure of the market). In this case, we let $\mathcal{Q} = \{\mathbb{Q}\}$.
- (NC) The set \mathcal{A} of admissible portfolio processes for the agent contains all portfolios $(V_k)_{k=0,1,\dots,n}$ of the form

$$V_t = V_0 + \sum_{k=1}^t \xi_k \cdot (X_k - X_{k-1}) - U_t,$$

where $V_0 \in \mathcal{F}_0$, U is an \mathbb{F} -adapted nondecreasing process with $U_0 = 0$, ξ is an \mathbb{F} -predictable \mathbb{R}^d -valued process corresponding to the trading strategy, and X is an \mathbb{F} -adapted \mathbb{R}^d -valued process corresponding to the prices of the risky assets. We assume that the set of equivalent martingale measures \mathcal{Q} is nonempty. It is a classical result that the portfolio process is an admissible portfolio if and only if it is \mathbb{Q} -supermartingale for any $\mathbb{Q} \in \mathcal{Q}$ (as a shorthand, we simply say that it is a \mathcal{Q} -supermartingale).

The first assumption describes a complete financial market, where at date k the agent can replicate any cash flow $H \in \mathcal{F}_{k+1}$ with $\mathbb{E}^{\mathbb{Q}}[|H|] < \infty$. If the probability

space is infinite, with discrete-time trading, one needs an infinite number of assets to ensure market completeness; however, this situation arises naturally when the agent can trade in the financial market dynamically in continuous time (that is, between the benchmark dates t_0, t_1, \dots, t_n) and the original continuous-time financial market is complete.

The second assumption describes the classical arbitrage-free multiperiod financial market model with a finite number of assets; see [11, Chap. 5].

Finally, we make the following **standing assumption** on the benchmark process:

- The benchmark process S satisfies $\sup_{Q \in \mathcal{Q}} \mathbb{E}^Q[|S_k|] < \infty$ for $k = 1, \dots, n$.

Risk constraints A general risk constraint on the sequence $V_{t_1} - S_{t_1}, \dots, V_{t_n} - S_{t_n}$ of shortfall values with respect to the given benchmark sequence can be written in the form $\rho(V_{t_1} - S_{t_1}, \dots, V_{t_n} - S_{t_n}) \leq 0$, where ρ is a risk measure, a mapping from a set of n -dimensional random vectors to \mathbb{R} . In this paper, we consider three classes of risk measures, which allow us to develop a dynamic programming methodology for computing the superhedging price.

General shortfall risk constraint The general shortfall risk constraint depends on the entire trajectory of the agent’s portfolio. Let $L : \mathbb{R}^n \rightarrow \mathbb{R}$ be an n -dimensional loss function,² which is assumed to be convex, nonincreasing in each argument and bounded from below. We choose an \mathbb{R} -valued function to simplify notation, but vector-valued loss functions can be considered in the same manner. We define the superhedging problem with general shortfall risk constraint (SR) as follows:

- (SR) Find the minimal value of $M_0 \in \mathbb{R}$ such that there exists a \mathcal{Q} -supermartingale $(M_k)_{k=0}^n$ which satisfies

$$\mathbb{E}^{\mathbb{P}}[L(M_1 - S_1, \dots, M_n - S_n)] \leq 0.$$

We denote the set of all such numbers M_0 by \mathcal{M}_{SR} and the corresponding infimum value by $V_0^{SR} := \inf \mathcal{M}_{SR}$.

For the other two constraint types, we consider a one-dimensional loss function $\ell : \mathbb{R} \rightarrow \mathbb{R}$. **Throughout this paper, we always assume that the following condition on ℓ is satisfied.**

Assumption 2.1 The function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ is convex, nonincreasing and bounded from below.

The above assumptions are natural to describe a loss function. A typical example is a “put” function $\ell(x) = (-x)^+$. We take the positive part of the loss so that the situations when the liability is hedged are not penalized. In the following, we denote $\ell_\infty = \lim_{x \rightarrow +\infty} \ell(x)$ and $\ell^{-1}(x) = \inf\{y : \ell(y) \leq x\}$.

In some cases, we need stronger assumptions to get more explicit results. The following Assumption 2.2 allows us to include some widely used utility-type functions.

²In our paper, the loss function is evaluated directly on a financial position X instead of $-X$ as in [10] and [11, Chap. 4].

For example, the loss function $\ell(x) = e^{-px}$ with $p > 0$, corresponding to the entropic risk measure [11, Example 4.60], satisfies Assumption 2.2.

Assumption 2.2 The function $\ell : \mathbb{R} \rightarrow \mathbb{R}$ is convex, nonincreasing, bounded from below and of class C^1 , and there exists $x^* \in (-\infty, +\infty]$ such that ℓ is strictly convex and strictly decreasing on $(-\infty, x^*)$ and the derivative $\ell'(x)$ satisfies the Inada condition $\lim_{x \rightarrow -\infty} \ell'(x) = -\infty$ and $\lim_{x \rightarrow x^*} \ell'(x) = 0$.

The other two constraint types used in this paper are described as follows.

European-style constraint

(EU) Find the minimal value $M_0 \in \mathbb{R}$ such that there exists a \mathcal{Q} -supermartingale $(M_k)_{k=0}^n$ which satisfies, for given constants $\alpha_1, \dots, \alpha_n$,

$$\mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k)] \leq \alpha_k \quad \text{for } k = 1, \dots, n.$$

For a given family of bounds $\alpha = (\alpha_1, \dots, \alpha_n)$, we denote by $\mathcal{M}_{EU}(\alpha)$ the set of all such numbers M_0 , and by $V_0^{EU}(\alpha) := \inf \mathcal{M}_{EU}(\alpha)$ the corresponding infimum value.

Next-period constraint

(NP) Find the minimal value $M_0 \in \mathbb{R}$ such that there exists a \mathcal{Q} -supermartingale $(M_k)_{k=0}^n$ which satisfies, for given constants $\alpha_1, \dots, \alpha_n$,

$$\mathbb{E}^{\mathbb{P}}_{k-1}[\ell(M_k - S_k)] \leq \alpha_k, \quad \text{a.s. for } k = 1, \dots, n. \tag{2.2}$$

For a given family of bounds α , we denote by $\mathcal{M}_{NP}(\alpha)$ the set of all such numbers M_0 , and by $V_0^{NP}(\alpha) := \inf \mathcal{M}_{NP}(\alpha)$ the corresponding infimum value.

The European-style constraint is of course a particular case of the general shortfall risk constraint with a vector-valued loss function; however, we consider it separately in order to present more explicit results. The next-period constraint has an interesting interpretation as an ‘‘American-style’’ guarantee. By Doob’s theorem, condition (2.2) is equivalent to the following assertion: for all \mathbb{F} -stopping times τ and σ taking values in $\{0, 1, \dots, n\}$ such that $\tau \leq \sigma$,

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{i=\tau+1}^{\sigma} (\ell(M_i - S_i) - \alpha_i) \right] \leq 0.$$

This representation implies that $\mathcal{M}_{NP} \subset \mathcal{M}_{EU}$ and so $V_0^{EU} \leq V_0^{NP}$.

Remark 2.3 The tools developed in this paper could in principle be extended to solve the problem of maximizing the expected utility of terminal wealth subject to probabilistic risk constraints imposed at multiple dates. For example, in the case of general shortfall risk constraints, the problem would be formulated as follows:

Given $m \geq V_0^{SR}$ and a utility function U , find a \mathcal{Q} -supermartingale $(M_k)_{k=0}^n$ with $M_0 \leq m$ which satisfies

$$\mathbb{E}^{\mathbb{P}}[L(M_1 - S_1, \dots, M_n - S_n)] \leq 0,$$

and which maximizes $\mathbb{E}^{\mathbb{P}}[U(M_n)]$ over all such \mathcal{Q} -supermartingales.

This important extension is left to further research (see also [4, Chap. 5] for a study of a similar problem in a complete market Markovian setting).

3 Solution by dynamic programming principle

In this section, we solve the optimization problems for the three types of constraints.

3.1 Definition of the value functions

The dynamic definition of the value function is the simplest for the next-period constraint, since it involves only two successive dates. Recall that $\mathcal{M}_{NP}(\alpha)$ is the set of all $M_0 \in \mathbb{R}$ such that there exists a \mathcal{Q} -supermartingale $(M_k)_{k=0}^n$ with

$$\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(M_k - S_k)] \leq \alpha_k \quad \text{for } k = 1, \dots, n.$$

In a dynamic manner, denote by $\mathcal{M}_{NP,k}(\alpha_{k+1}, \dots, \alpha_n)$ the set of all $m \in \mathcal{F}_k$ such that there exists a \mathcal{Q} -supermartingale $(M_t)_{t=k}^n$ with $M_k = m$ satisfying

$$\mathbb{E}_{t-1}^{\mathbb{P}}[\ell(M_t - S_t)] \leq \alpha_t \quad \text{for } t = k + 1, \dots, n.$$

Since $\alpha_{k+1}, \dots, \alpha_n$ are constant, in the following, when there is no ambiguity, we omit these arguments and write $\mathcal{M}_{NP,k}(\alpha_{k+1}, \dots, \alpha_n) = \mathcal{M}_{NP,k}$. $\mathcal{M}_{NP,n}$ is the set of all integrable \mathcal{F}_n -measurable random variables and $\mathcal{M}_{NP,0}$ coincides with \mathcal{M}_{NP} . We define the dynamic value function by

$$V_k^{NP} := \text{ess inf } \mathcal{M}_{NP,k}.$$

Note that this value function, as well as the other value functions considered below, are *stochastic processes*. In this respect, they are similar to the Snell envelope process, which plays the role of the value function in non-Markovian optimal stopping problems.

To define the dynamic value function for the European constraint, we first give a dynamic characterization of the constraints involved in the definition of $V_0^{EU}(\alpha_1, \dots, \alpha_n)$ using a family of \mathbb{P} -supermartingales.

Lemma 3.1 $M_0 \in \mathcal{M}_{EU}$ if and only if there exist a \mathcal{Q} -supermartingale $(M_t)_{t=0}^n$ and a family of \mathbb{P} -supermartingales $(N_t^k)_{t=0}^k$, $k \in \{1, \dots, n\}$, with $N_0^k = \alpha_k$ and $N_k^k \geq \ell(M_k - S_k)$.

Proof Let $(M_t)_{t=0}^n$ and $(N_t^k)_{t=0}^k, k = 1, \dots, n$, satisfy the conditions of the lemma. Then

$$\alpha_k = N_0^k \geq \mathbb{E}^{\mathbb{P}}[N_k^k] \geq \mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k)].$$

Therefore M_0 belongs to \mathcal{M}_{EU} . Conversely, given a \mathcal{Q} -supermartingale $(M_k)_{k=0}^n$ such that M_0 belongs to \mathcal{M}_{EU} , define the processes $(N_t^k)_{t=0}^k$ by $N_0^k = \alpha_k$ and $N_t^k = \mathbb{E}_t[\ell(M_k - S_k)]$ for $t \in \{1, \dots, k\}$. The condition $\mathbb{E}^{\mathbb{P}}[\ell(M_k - S_k)] \leq \alpha_k$ implies that $\mathbb{E}[N_1^k] \leq N_0^k$ and therefore $(N_t^k)_{t=0}^k$ is a \mathbb{P} -supermartingale. \square

From the above proposition, it becomes clear that the dynamic value function V_k^{EU} , where $k \in \{1, \dots, n\}$, takes as parameters a family of random variables in order to describe the dynamic evolution of successive bounds on the expected value.

Let $k < n$. For \mathcal{F}_k -measurable random variables N^{k+1}, \dots, N^n , we denote by $\mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$ the set of all $m \in \mathcal{F}_k$ such that there exists a \mathcal{Q} -supermartingale $(M_i)_{i=k}^n$ with $M_k = m$ satisfying

$$\mathbb{E}^{\mathbb{P}}[\ell(M_t - S_t)] \leq N^t$$

for $t = k + 1, \dots, n$. By convention, $\mathcal{M}_{EU,n}$ coincides with the set of all \mathcal{F}_n -measurable random variables. We define

$$V_k^{EU}(N^{k+1}, \dots, N^n) := \text{ess inf } \mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n),$$

so that in particular $V_n^{EU} = -\infty$.

The general shortfall risk constraint involves the values of the superhedging portfolio since the beginning of trading. Therefore, the dynamic value function V_k^{SR} depends on M_1, \dots, M_k . For random variables (M_1, \dots, M_k) such that M_i is \mathcal{F}_i -measurable for $i = 1, \dots, k$, and for $N_k \in \mathcal{F}_k$, let $\mathcal{M}_{SR,k}(N_k, M_1, \dots, M_k)$ be the set of all random variables $m \in \mathcal{F}_k$ such that there exists a \mathcal{Q} -supermartingale $(\tilde{M}_t)_{t=k}^n$ with $\tilde{M}_k = m$ satisfying

$$\mathbb{E}^{\mathbb{P}}[L(M_1 - S_1, \dots, M_k - S_k, \tilde{M}_{k+1} - S_{k+1}, \dots, \tilde{M}_n - S_n)] \leq N_k. \tag{3.1}$$

We then define the value function at date k by

$$V_k^{SR}(N_k, M_1, \dots, M_k) := \text{ess inf } \mathcal{M}_{SR,k}(N_k, M_1, \dots, M_k)$$

for $k \in \{0, 1, \dots, n - 1\}$.

We now show that the value functions V^{NP} and V^{EU} are well defined. For the general shortfall risk constraint, the finiteness of the value function is difficult to establish in full generality and will therefore be an assumption of our main theorem.

Lemma 3.2 *Let $k \in \{0, \dots, n - 1\}$.*

- *If $\alpha_j > \ell_\infty$ for each $j \in \{k + 1, \dots, n\}$, then $V_k^{NP} < +\infty$ a.s.*
- *If $N^j > \ell_\infty$ a.s. and $\mathbb{E}^{\mathbb{Q}}[|\ell^{-1}(N^j)|] < \infty$ for each $j \in \{k + 1, \dots, n\}$ and every $\mathbb{Q} \in \mathcal{Q}$, then $V_k^{EU}(N^{k+1}, \dots, N^n) < +\infty$ a.s.*

Proof Let us start with the European constraint and define, for $t \geq k$,

$$M_t = \max_{s=k+1, \dots, n} \ell^{-1}(N^s) + \sum_{s=k+1}^t |S_s| + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=t+1}^n |S_s| \right].$$

Then $(M_t)_{t=k}^n$ is a \mathcal{Q} -supermartingale (under the assumption (NC), this follows from [11, Corollary 7.4]) and

$$\mathbb{E}_k^{\mathbb{P}}[\ell(M_t - S_t)] \leq \ell \left(\max_{s=k+1, \dots, n} \ell^{-1}(N^s) \right) = \min_{s=k+1, \dots, n} N^s \leq N^t,$$

which means that $M_k \in \mathcal{M}_{EU,k}$ and so $V_k^{EU}(N^{k+1}, \dots, N^n) < M_k < +\infty$ a.s.

For the next-period constraint, we let, for $t \geq k$,

$$M_t = \max_{s=k+1, \dots, n} \ell^{-1}(\alpha_s) + \sum_{s=k+1}^t |S_s| + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_t^{\mathbb{Q}} \left[\sum_{s=k+1}^n |S_s| \right],$$

which leads to

$$\mathbb{E}_{t-1}^{\mathbb{P}}[\ell(M_t - S_t)] \leq \ell \left(\max_{s=k+1, \dots, n} \ell^{-1}(\alpha_s) \right) \leq \alpha_t. \quad \square$$

To finish this section, we prove the following technical lemma, which will be necessary for the proof of the dynamic programming principles below.

Lemma 3.3

- Let m, m' belong to $\mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$. Then $m \wedge m'$ belongs to $\mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$.
- Let m, m' belong to $\mathcal{M}_{NP,k}$. Then $m \wedge m'$ belongs to $\mathcal{M}_{NP,k}$.
- Let m, m' belong to $\mathcal{M}_{SR,k}(N_k, M_1, \dots, M_k)$. Then $m \wedge m'$ also belongs to $\mathcal{M}_{SR,k}(N_k, M_1, \dots, M_k)$.

Proof We provide the proof for the European-style constraints; for the next-period or general shortfall risk constraints, one can use a similar argument. By definition, there exist \mathcal{Q} -supermartingales $(M_t)_{t=k}^n$ and $(M'_t)_{t=k}^n$ satisfying $M_k = m$ and $M'_k = m'$ such that for any $t \in \{k+1, \dots, n\}$, it holds that $\mathbb{E}_k^{\mathbb{P}}[\ell(M_t - S_t)] \leq N^t$ and $\mathbb{E}_k^{\mathbb{P}}[\ell(M'_t - S_t)] \leq N^t$. Let A be the set $\{m \geq m'\}$, which is \mathcal{F}_k -measurable. For any $t \in \{k, \dots, n\}$, we define $M''_t = \mathbb{1}_A M'_t + \mathbb{1}_{A^c} M_t$. Since $A \in \mathcal{F}_k$ and since $(M_t)_{t=k}^n$ and $(M'_t)_{t=k}^n$ are both \mathcal{Q} -supermartingales, we obtain that $(M''_t)_{t=k}^n$ is a \mathcal{Q} -supermartingale, which satisfies $M''_k = \min\{m, m'\}$ and

$$\mathbb{E}_k^{\mathbb{P}}[\ell(M''_t - S_t)] = \mathbb{1}_A \mathbb{E}_k^{\mathbb{P}}[\ell(M'_t - S_t)] + \mathbb{1}_{A^c} \mathbb{E}_k^{\mathbb{P}}[\ell(M_t - S_t)] \leq N^t$$

for all t in $\{k+1, \dots, n\}$. Therefore, $\min\{m, m'\} \in \mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$. □

3.2 Dynamic programming principles

In this section, we characterize the value-function processes in a backward and recursive form.

For the European-style constraint, we define $\widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$ to be the set of all random variables $m \in \mathcal{F}_k$ such that there exist random variables $M_{k+1} \in \mathcal{F}_{k+1}$, $N_{k+1}^{k+1} \in \mathcal{F}_{k+1}$, $N_{k+1}^{k+2} \in \mathcal{F}_{k+1}, \dots, N_{k+1}^n \in \mathcal{F}_{k+1}$ satisfying almost surely

$$\mathbb{E}^{\mathbb{Q}}[|M_{k+1}|] < \infty \quad \text{and} \quad \mathbb{E}_k^{\mathbb{Q}}[M_{k+1}] \leq m \quad \text{for all } \mathbb{Q} \in \mathcal{Q}, \tag{3.2}$$

$$\mathbb{E}^{\mathbb{P}}[|N_{k+1}^i|] < \infty \quad \text{and} \quad \mathbb{E}_k^{\mathbb{P}}[N_{k+1}^i] \leq N_k^i \quad \text{for all } i = k + 1, \dots, n, \tag{3.3}$$

$$N_{k+1}^{k+1} \geq \ell(M_{k+1} - S_{k+1}), \tag{3.4}$$

$$M_{k+1} \geq V_{k+1}^{EU}(N_{k+1}^{k+2}, \dots, N_{k+1}^n). \tag{3.5}$$

Similarly, for the next-period constraint, we define $\widetilde{\mathcal{M}}_{NP,k}$ to be the set of all random variables $m \in \mathcal{F}_k$ such that exists a random variable $M_{k+1} \in \mathcal{F}_{k+1}$ satisfying almost surely

$$\mathbb{E}^{\mathbb{Q}}[|M_{k+1}|] < \infty \quad \text{and} \quad \mathbb{E}_k^{\mathbb{Q}}[M_{k+1}] \leq m \quad \text{for all } \mathbb{Q} \in \mathcal{Q}, \tag{3.6}$$

$$\mathbb{E}_k^{\mathbb{P}}[\ell(M_{k+1} - S_{k+1})] \leq \alpha_{k+1}, \tag{3.7}$$

$$M_{k+1} \geq V_{k+1}^{NP}. \tag{3.8}$$

For the general shortfall risk constraint, we define $\widetilde{\mathcal{M}}_{SR,k}(N_k, M_1, \dots, M_k)$ for $k = 0, \dots, n - 2$ to be the set of random variables $m \in \mathcal{F}_k$ such that there exist random variables $M_{k+1} \in \mathcal{F}_{k+1}$ and $N_{k+1} \in \mathcal{F}_{k+1}$ satisfying almost surely

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}[|M_{k+1}|] < \infty \quad \text{and} \quad \mathbb{E}_k^{\mathbb{Q}}[M_{k+1}] \leq m \quad \text{for all } \mathbb{Q} \in \mathcal{Q}, \\ \mathbb{E}^{\mathbb{P}}[|N_{k+1}|] < \infty \quad \text{and} \quad \mathbb{E}_k^{\mathbb{P}}[N_{k+1}] \leq N_k, \\ \widetilde{M}_{k+1} \geq V_{k+1}^{SR}(N_{k+1}, M_1, \dots, M_k, \widetilde{M}_{k+1}). \end{cases} \tag{3.9}$$

Theorem 3.4 *Let either assumption (C) or assumption (NC) be satisfied.*

– Let $k \in \{0, \dots, n - 1\}$ and assume that for each $j \in \{k + 1, \dots, n\}$, $N^j > \ell_\infty$ almost surely, $\mathbb{E}^{\mathbb{P}}[|N^j|] < \infty$ and $\mathbb{E}^{\mathbb{Q}}[|\ell^{-1}(N^j)|] < \infty$ for every $\mathbb{Q} \in \mathcal{Q}$. Then the value function $V_k^{EU}(N^{k+1}, \dots, N^n)$ is given by

$$V_k^{EU}(N^{k+1}, \dots, N^n) = \text{ess inf } \widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n).$$

– Let $k \in \{0, \dots, n - 1\}$ and assume that $\alpha_{k+1} > \ell_\infty, \dots, \alpha_n > \ell_\infty$. Then the value function V_k^{NP} is given by

$$V_k^{NP} = \text{ess inf } \widetilde{\mathcal{M}}_k.$$

– Let $k = 0, 1, \dots, n - 2$ and assume that both $\widetilde{\mathcal{M}}_{SR,k}(N_k, M_1, \dots, M_k)$ and $\mathcal{M}_{SR,k}(N_k, M_1, \dots, M_k)$ are not empty. Then

$$V_k^{SR}(N_k, M_1, \dots, M_k) = \text{ess inf } \widetilde{\mathcal{M}}_{SR,k}(N_k, M_1, \dots, M_k).$$

Proof In this proof we assume that $k < n - 1$, the case $k = n - 1$ being trivial.

European constraint We first show that $\widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$ is not empty. Let $N_{k+1}^{k+1} := N^{k+1}, \dots, N_{k+1}^n := N^n$. These variables satisfy condition (3.3), and using the argument given at the beginning of the proof of Lemma 3.2 with k replaced with $k + 1$, it follows that

$$V_{k+1}^{EU}(N_{k+1}^{k+2}, \dots, N_{k+1}^n) \leq \max_{s=k+2, \dots, n} \ell^{-1}(N^s) + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{k+1}^{\mathbb{Q}} \left[\sum_{s=k+2}^n |S_s| \right].$$

Then, choosing

$$m = \max_{s=k+1, \dots, n} \ell^{-1}(N^s) + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_k^{\mathbb{Q}} \left[\sum_{s=k+1}^n |S_s| \right],$$

$$M_{k+1} = \max_{s=k+1, \dots, n} \ell^{-1}(N^s) + |S_{k+1}| + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{k+1}^{\mathbb{Q}} \left[\sum_{s=k+2}^n |S_s| \right],$$

it is easy to see that $m \in \widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$ (once again, under (NC), the supermartingale property follows from [11, Corollary 7.4]).

Now let m be an arbitrary element of $\widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$, and let M_{k+1} and $N_{k+1}^{k+1}, \dots, N_{k+1}^n$ satisfy the conditions (3.2)–(3.5). By definition of the essential infimum, there exists a sequence of random variables $(\widehat{M}_{k+1}^{(j)})_{j \in \mathbb{N}}$ belonging to $\mathcal{M}_{EU,k+1}(N_{k+1}^{k+2}, \dots, N_{k+1}^n)$ such that $M_{k+1} \geq \inf_{j \in \mathbb{N}} \widehat{M}_{k+1}^{(j)}$. Moreover, the previous lemma shows that the sequence $(\widehat{M}_{k+1}^{(j)})_{j \in \mathbb{N}}$ can be chosen decreasing without loss of generality. Then there exist \mathcal{Q} -supermartingales $(\widehat{M}_t^{(j)})_{t=k+1}^n$ such that

$$N_{k+1}^i \geq \mathbb{E}_{k+1}^{\mathbb{P}}[\ell(\widehat{M}_i^{(j)} - S_i)] \quad \text{for } i \in \{k+2, \dots, n\}.$$

Let $m^{(j)} = \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_k^{\mathbb{Q}}[\widehat{M}_{k+1}^{(j)} \vee M_{k+1}]$ for any $j \in \mathbb{N}$ and define

$$M_t^{(j)} = \widehat{M}_t^{(j)} + (M_{k+1} - \widehat{M}_{k+1}^{(j)})^+ \quad \text{for } t \geq k+1 \quad \text{and} \quad M_k^{(j)} = m^{(j)}.$$

We claim that $m^{(j)} \in \mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$ for all j , with associated \mathcal{Q} -supermartingales $(M_t^{(j)})_{t=k}^n$. Indeed, for $t \geq k+2$,

$$\mathbb{E}_k^{\mathbb{P}}[\ell(M_t^{(j)} - S_t)] \leq \mathbb{E}_k^{\mathbb{P}}[\mathbb{E}_{k+1}^{\mathbb{P}}[\ell(\widehat{M}_t^{(j)} - S_t)]] \leq \mathbb{E}_k^{\mathbb{P}}[N_{k+1}^i] \leq N_k^i$$

and

$$\mathbb{E}_k^{\mathbb{P}}[\ell(M_{k+1}^{(j)} - S_{k+1})] \leq \mathbb{E}_k^{\mathbb{P}}[\ell(M_{k+1} - S_{k+1})] \leq N^{k+1}.$$

Therefore $V_k^{EU}(N^{k+1}, \dots, N^n) \leq m^{(j)}$ for any $j \in \mathbb{N}$.

Under the assumption (C), by the monotone convergence theorem, this implies that

$$V_k^{EU}(N^{k+1}, \dots, N^n) \leq \inf_{j \in \mathbb{N}} \mathbb{E}_k^{\mathbb{Q}}[\widehat{M}_{k+1}^{(j)} \vee M_{k+1}] = \mathbb{E}_k^{\mathbb{Q}}[\lim_j \widehat{M}_{k+1}^{(j)} \vee M_{k+1}] \leq m.$$

Hence we obtain $V_k^{EU}(N^{k+1}, \dots, N^n) \leq \text{ess inf } \widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$.

Alternatively, let (NC) be satisfied. Then

$$\widehat{M}_{k+1}^{(j)} \vee M_{k+1} = m^{(j)} + \xi_{k+1}^{(j)} \cdot (X_{k+1} - X_k) - U_{k+1}^{(j)},$$

where for every j , $\xi_{k+1}^{(j)} \in \mathcal{F}_k$ and $U_{k+1}^{(j)} \in \mathcal{F}_{k+1}$ with $U_{k+1}^{(j)} \geq 0$. Since $m^{(j)}$ is decreasing and bounded from below by $\text{ess sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_k^{\mathbb{Q}}[M_{k+1}]$, it converges a.s. to a limit m^* . Therefore, $\xi_{k+1}^{(j)} \cdot (X_{k+1} - X_k) - U_{k+1}^{(j)}$ converges a.s. to $M_{k+1} - m^*$. By Lemma 1.68 in [11], we then conclude that there exist ξ^* and U^* such that

$$M_{k+1} = m^* + \xi_{k+1}^* \cdot (X_{k+1} - X_k) - U_{k+1}^*.$$

This in turn implies that

$$m^* \leq \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_k^{\mathbb{Q}}[M_{k+1}] \leq m.$$

Since on the other hand

$$m^* \geq V_k^{EU}(N^{k+1}, \dots, N^n),$$

we conclude that $V_k^{EU}(N^{k+1}, \dots, N^n) \leq \text{ess inf } \widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$.

To prove the converse inequality, we take any $M_k \in \mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$ and let (M_k, \dots, M_n) be the associated \mathcal{Q} -supermartingale. For $t = k + 1, \dots, n$, let $N_{k+1}^t = \mathbb{E}_{k+1}^{\mathbb{P}}[\ell(M_t - S_t)]$. Then (3.2) is satisfied by the supermartingale property, (3.3) follows from the definition of $\mathcal{M}_{EU,k}(N^{k+1}, \dots, N^n)$, and (3.4) is a trivial equality. Further, remark that

$$M_{k+1} \in \mathcal{M}_{EU,k+1}(N_{k+1}^{k+2}, \dots, N_{k+1}^n),$$

since by construction for $t = k + 2, \dots, n$,

$$\mathbb{E}_{k+1}^{\mathbb{P}}[\ell(M_t - S_t)] = N_{k+1}^t.$$

Therefore, $M_k \in \widetilde{\mathcal{M}}_{EU,k}(N^{k+1}, \dots, N^n)$, which implies the converse inequality.

Next-period constraint We first show that $\widetilde{\mathcal{M}}_{NP,k}$ is not empty. Similarly as above,

$$V_{k+1}^{NP} \leq \max_{s=k+2, \dots, n} \ell^{-1}(\alpha_s) + \text{ess sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{k+1}^{\mathbb{Q}} \left[\sum_{s=k+2}^n |S_s| \right],$$

and choosing

$$m = \max_{s=k+1, \dots, n} \ell^{-1}(\alpha_s) + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_k^{\mathbb{Q}} \left[\sum_{s=k+1}^n |S_s| \right],$$

$$M_{k+1} = \max_{s=k+1, \dots, n} \ell^{-1}(\alpha_s) + |S_{k+1}| + \operatorname{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{k+1}^{\mathbb{Q}} \left[\sum_{s=k+2}^n |S_s| \right],$$

we conclude that $m \in \widetilde{\mathcal{M}}_{NP,k}$.

Let $m \in \widetilde{\mathcal{M}}_{NP,k}$ and let $M_{k+1} \in \mathcal{F}_{k+1}$ satisfy the conditions (3.6)–(3.8). By Lemma 3.3, there exist a decreasing sequence of random variables $(\widehat{M}_{k+1}^{(j)})_{j \in \mathbb{N}}$ belonging to $\mathcal{M}_{NP,k+1}$ with $M_{k+1} \geq \inf_{j \in \mathbb{N}} \widehat{M}_{k+1}^{(j)}$, and \mathcal{Q} -supermartingales $(\widehat{M}_t^{(j)})_{t=k+1}^n$ such that

$$\mathbb{E}_{t-1}^{\mathbb{P}}[\ell(\widehat{M}_t^{(j)} - S_t)] \leq \alpha_t, \quad t = k + 2, \dots, n.$$

Let $m^{(j)}$ and $M_t^{(j)}$ for $t \geq k + 1$ be defined as in the proof of the first part. Then $m^{(j)} \in \mathcal{M}_{NP,k}$ for all j , with associated \mathcal{Q} -supermartingales $(M_t^{(j)})_{t=k}^n$. Indeed, for $t \geq k + 2$,

$$\mathbb{E}_{t-1}^{\mathbb{P}}[\ell(M_t^{(j)} - S_t)] = \mathbb{E}_{t-1}^{\mathbb{P}}[\ell(\widehat{M}_t^{(j)} + (M_{k+1} - \widehat{M}_{k+1}^{(j)})^+ - S_t)] \leq \alpha_t$$

and

$$\mathbb{E}_k^{\mathbb{P}}[\ell(M_{k+1}^{(j)} - S_{k+1})] \leq \mathbb{E}_k^{\mathbb{P}}[\ell(M_{k+1} - S_{k+1})] \leq \alpha_{k+1}.$$

Therefore $V_k^{NP} \leq m^{(j)}$ for any $j \in \mathbb{N}$. We can then finish the proof under the alternative assumptions (C) or (NC) using exactly the same argument as in the proof of the first part.

For the converse inequality, let $m \in \mathcal{M}_{NP,k}$. There then exists a \mathcal{Q} -supermartingale $(M_t)_{t=k}^n$ with $M_k = m$ such that

$$\mathbb{E}_{t-1}^{\mathbb{P}}[\ell(M_t - S_t)] \leq \alpha_t \quad \text{for } t = k + 1, \dots, n,$$

which implies that $M_{k+1} \in \mathcal{M}_{NP,k+1}$ and therefore $M_{k+1} \geq V_{k+1}^{NP}$. Combining this inequality with the condition $\mathbb{E}_k^{\mathbb{P}}[\ell(M_{k+1} - S_{k+1})] \leq \alpha_{k+1}$, we obtain that $m \in \widetilde{\mathcal{M}}_{NP,k}$, and so $V_k^{NP} \geq \operatorname{ess\,inf} \widetilde{\mathcal{M}}_k^{NP}$.

General shortfall risk constraint Let $m \in \widetilde{\mathcal{M}}_{SR,k}(N, M_1, \dots, M_k)$ and let M_{k+1} and N_{k+1} be random variables which satisfy (3.9). By Lemma 3.3, there exist a decreasing sequence of random variables $(\widehat{M}_{k+1}^{(j)})_{j \in \mathbb{N}}$ in $\mathcal{M}_{SR,k+1}(N_{k+1}, M_1, \dots, M_k, M_{k+1})$ such that $M_{k+1} \geq \inf_{j \in \mathbb{N}} \widehat{M}_{k+1}^{(j)}$ and \mathcal{Q} -supermartingales $(\widehat{M}_t^{(j)})_{t=k+1}^n$ such that

$$\mathbb{E}_{k+1}^{\mathbb{P}}[L(M_1 - S_1, \dots, M_k - S_k, \widehat{M}_{k+1}^{(j)} - S_{k+1}, \dots, \widehat{M}_n^{(j)} - S_n)] \leq N_{k+1}.$$

Let $m^{(j)}$ and $M_t^{(j)}$ for $t \geq k + 1$ be defined as in the proof of the first part. Then $m^{(j)} \in \mathcal{M}_{SR,k}$ for all j , with associated \mathcal{Q} -supermartingales $(M_t^{(j)})_{t=k}^n$. Indeed, since L is decreasing in each argument,

$$\begin{aligned} & \mathbb{E}_k^{\mathbb{P}}[L(M_1 - S_1, \dots, M_k - S_k, M_{k+1}^{(j)} - S_{k+1}, \dots, M_n^{(j)} - S_n)] \\ & \leq \mathbb{E}_k^{\mathbb{P}}[L(M_1 - S_1, \dots, M_{k+1} - S_{k+1}, \widehat{M}_{k+2}^{(j)} - S_{k+2}, \dots, \widehat{M}_n^{(j)} - S_n)] \leq N_k. \end{aligned}$$

Therefore, $m^{(j)} \geq V_k^{SR}(N_k, M_1, \dots, M_k)$ for any $j \in \mathbb{N}$, and using exactly the same argument as in the proof the first part, we may conclude that

$$\text{ess inf } \widetilde{\mathcal{M}}_{SR,k}(N_k, M_1, \dots, M_k) \geq V_k^{SR}(N_k, M_1, \dots, M_k).$$

Conversely, let $m \in \mathcal{M}_{SR,k}(N, M_1, \dots, M_k)$ and let $(\widetilde{M}_t)_{t=k}^n$ be the associated \mathcal{Q} -supermartingale satisfying (3.1). Define N_{k+1} by

$$N_{k+1} = \mathbb{E}_{k+1}^{\mathbb{P}}[L(M_1 - S_1, \dots, M_k - S_k, \widetilde{M}_{k+1} - S_{k+1}, \dots, \widetilde{M}_n - S_n)].$$

By the definition of V_{k+1}^{SR} , this means that

$$\widetilde{M}_{k+1} \geq V_{k+1}^{SR}(N_{k+1}, M_1, \dots, M_k, \widetilde{M}_{k+1}),$$

which shows that m , \widetilde{M}_{k+1} and N_{k+1} verify the conditions in (3.9). This in turn implies that $m \in \widetilde{\mathcal{M}}_{SR,k}(N_k, M_1, \dots, M_k)$ and so

$$\text{ess inf } \widetilde{\mathcal{M}}_{SR,k}(N_k, M_1, \dots, M_k) \leq V_k^{SR}(N_k, M_1, \dots, M_k). \quad \square$$

3.3 Explicit solutions in the complete market case

With an extra regularity condition on the loss function ℓ , we obtain a more explicit solution for the superhedging problem under the next-period constraint in the complete market case.

Proposition 3.5 *Let the assumption (C) be satisfied. Let ℓ be a loss function satisfying Assumption 2.2 and let $I : (-\infty, 0) \rightarrow \mathbb{R}$ be the inverse function of ℓ' . Suppose in addition that $\alpha_k > \ell_\infty$ for any $k \in \{1, \dots, n\}$ and that there exists a strictly negative random variable $Y_{k-1} \in \mathcal{F}_{k-1}$ such that*

$$\mathbb{E}^{\mathbb{P}}[\ell(I(Y_{k-1}Z_k/Z_{k-1}))] < +\infty, \quad \text{where } Z_k = \mathbb{E}_k^{\mathbb{P}}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right]. \quad (3.10)$$

Then the value function V^{NP} satisfies $V_t^{NP} = \widehat{V}_t$ a.s. for $t = 0, \dots, n - 1$, where \widehat{V}_t is given by

$$\widehat{V}_{n-1} = \mathbb{E}_{n-1}^{\mathbb{Q}}[S_n + I(\lambda_{n-1}Z_n/Z_{n-1})],$$

where $\lambda_{n-1} \in \mathcal{F}_{n-1}$ is the solution of

$$\mathbb{E}_{n-1}^{\mathbb{P}}[\ell(I(\lambda_{n-1}Z_n/Z_{n-1}))] = \alpha_n, \quad (3.11)$$

and for $k < n$,

$$\widehat{V}_{k-1} = \mathbb{E}_{k-1}^{\mathbb{Q}}[\widehat{V}_k] + \mathbb{E}_{k-1}^{\mathbb{Q}}[(S_k - \widehat{V}_k + I(\lambda_{k-1} Z_k/Z_{k-1}))^+] \mathbf{1}_{\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] > \alpha_k\}},$$

where $\lambda_{k-1} \in \mathcal{F}_{k-1}$ is the solution of

$$\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(I(\lambda_{k-1} Z_k/Z_{k-1}) \vee (\widehat{V}_k - S_k))] = \alpha_k. \tag{3.12}$$

Proof In this proof, to save space, we denote the value function V^{NP} simply by V .

Step 1: Existence of λ . For any $k \in \{1, \dots, n\}$, let Λ_{k-1} be the set of strictly negative \mathcal{F}_{k-1} -measurable random variables Y such that

$$\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(I(Y Z_k/Z_{k-1}) \vee (\widehat{V}_k - S_k))] \leq \alpha_k.$$

The condition $\alpha_k > \lim_{x \rightarrow +\infty} \ell(x)$ and (3.10) imply that the family Λ_{k-1} is not empty by using the dominated convergence theorem. Let λ_{k-1} be the essential infimum of this family. Note that Λ_{k-1} is stable by taking the infimum of finitely many random variables. Therefore λ_{k-1} can be written as the limit of a decreasing sequence in Λ_{k-1} . The continuity of the functions I and ℓ , together with the monotone convergence theorem, show that λ_{k-1} lies in the family Λ_{k-1} . It remains to show the equality (3.12). If it does not hold, then for sufficiently small $\varepsilon > 0$, the set A_ε of $\omega \in \Omega$ such that

$$\mathbb{E}_{k-1}^{\mathbb{P}}\left[\ell\left(I((\lambda_{k-1} - \varepsilon)Z_k/Z_{k-1}) \vee (\widehat{V}_k - S_k)\right)\right](\omega) \leq \alpha_k$$

has strictly positive measure. This implies that $\lambda_{k-1} - \varepsilon \mathbf{1}_{A_\varepsilon}$ also lies in Λ_{k-1} , which leads to a contradiction.

Step 2: Representation for V_{n-1} . By Theorem 3.4,

$$V_{n-1} = \mathbb{E}_{n-1}^{\mathbb{Q}}[S_n] + \operatorname{ess\,inf}_{m \in \mathcal{F}_n} \{ \mathbb{E}_{n-1}^{\mathbb{Q}}[m] : \mathbb{E}_{n-1}^{\mathbb{P}}[\ell(m)] \leq \alpha_n \}$$

Taking $m = I(\lambda_{n-1} Z_n/Z_{n-1})$, we see that $V_{n-1} \leq \widehat{V}_{n-1}$. On the other hand, for every strictly negative random variable $\lambda \in \mathcal{F}_{n-1}$,

$$\ell(m) \geq \ell^*(\lambda Z_n/Z_{n-1}) + \lambda m Z_n/Z_{n-1},$$

where $\ell^*(u)$ is the Legendre transformation of ℓ , i.e., $\ell^*(u) := \inf_v (\ell(v) - uv)$. Note that one has $(\ell^*)' = I$. So

$$\begin{aligned} & V_{n-1} - \mathbb{E}_{n-1}^{\mathbb{Q}}[S_n | \mathcal{F}_{n-1}] \\ & \geq \operatorname{ess\,inf}_{m \in \mathcal{F}_n} \{ \mathbb{E}_{n-1}^{\mathbb{Q}}[m] : \mathbb{E}_{n-1}^{\mathbb{P}}[\ell^*(\lambda Z_n/Z_{n-1}) + \lambda m Z_n/Z_{n-1}] \leq \alpha_n \} \\ & = \operatorname{ess\,inf}_{m \in \mathcal{F}_n} \{ \mathbb{E}_{n-1}^{\mathbb{Q}}[m] : \mathbb{E}_{n-1}^{\mathbb{P}}[\ell^*(\lambda Z_n/Z_{n-1})] + \lambda \mathbb{E}_{n-1}^{\mathbb{Q}}[m] \leq \alpha_n \} \\ & = \frac{\alpha_n - \mathbb{E}_{n-1}^{\mathbb{P}}[\ell^*(\lambda Z_n/Z_{n-1})]}{\lambda} \\ & = \frac{\alpha_n - \mathbb{E}_{n-1}^{\mathbb{P}}[\ell(I(\lambda Z_n/Z_{n-1}))]}{\lambda} + \mathbb{E}_{n-1}^{\mathbb{Q}}[I(\lambda Z_n/Z_{n-1})], \end{aligned}$$

where the last equality comes from the relation $\ell^*(y) = \ell(I(y)) - yI(y)$. Taking λ to be the solution of (3.11), we find that $V_{n-1} \geq \widehat{V}_{n-1}$, which proves the desired representation of V_{n-1} .

Step 3: General case. We now proceed to the proof of the general case by induction on k . Assume that we have already established the equality $V_k = \widehat{V}_k$. By Theorem 3.4 and the induction hypothesis,

$$V_{k-1} = \operatorname{ess\,inf}_{m \in \mathcal{F}_k} \left\{ \mathbb{E}_{k-1}^{\mathbb{Q}}[m] : m \geq \widehat{V}_k, \mathbb{E}_{k-1}^{\mathbb{P}}[\ell(m - S_k)] \leq \alpha_k \right\}.$$

We choose

$$\tilde{m} := \widehat{V}_k + (S_k - \widehat{V}_k + I(\lambda_{k-1} Z_k / Z_{k-1}))^+ \mathbf{1}_{\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] > \alpha_k\}},$$

which is bounded from below by \widehat{V}_k and satisfies $\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\tilde{m} - S_k)] \leq \alpha_k$. Indeed, this inequality clearly holds on the set $\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] \leq \alpha_k\}$. On the other hand, on the complement $\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] > \alpha_k\}$, one has

$$\tilde{m} - S_k = \widehat{V}_k - S_k + (S_k - \widehat{V}_k + I(\lambda_{k-1} Z_k / Z_{k-1}))^+,$$

so

$$\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\tilde{m} - S_k)] = \mathbb{E}_{k-1}^{\mathbb{P}}[\ell(I(\lambda_{k-1} Z_k / Z_{k-1}) \vee (\widehat{V}_k - S_k))]$$

which implies $\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\tilde{m} - S_k)] \leq \alpha_k$ on this set by the definition of λ_{k-1} . Hence

$$V_{k-1} \leq \mathbb{E}_{k-1}^{\mathbb{Q}}[\tilde{m}] = \widehat{V}_{k-1}.$$

Let us now turn to the opposite inequality $\widehat{V}_{k-1} \leq V_{k-1}$. It clearly holds on the event $\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] \leq \alpha_k\}$. It remains to establish the inequality on the set $\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] > \alpha_k\}$. We have by a change of variables that

$$\begin{aligned} & V_{k-1} - \mathbb{E}_{k-1}^{\mathbb{Q}}[\widehat{V}_k] \\ &= \operatorname{ess\,inf}_{m \in \mathcal{F}_k, m \geq 0} \left\{ \mathbb{E}_{k-1}^{\mathbb{Q}}[m] : \mathbb{E}_{k-1}^{\mathbb{P}}[\ell(m + \widehat{V}_k - S_k)] \leq \alpha_k \right\}, \\ &\geq \operatorname{ess\,inf}_{m \in \mathcal{F}_k, m \geq 0} \left\{ \mathbb{E}_{k-1}^{\mathbb{Q}}[m] : \mathbb{E}_{k-1}^{\mathbb{P}} \left[\ell^* \left(\lambda \frac{Z_k}{Z_{k-1}} \right) + \lambda(m + \widehat{V}_k - S_k) \frac{Z_k}{Z_{k-1}} \right] \leq \alpha_k \right\} \\ &= \operatorname{ess\,inf}_{m \in \mathcal{F}_k, m \geq 0} \left\{ \mathbb{E}_{k-1}^{\mathbb{Q}}[m] : \mathbb{E}_{k-1}^{\mathbb{P}} \left[\ell^* \left(\lambda \frac{Z_k}{Z_{k-1}} \right) \right] + \lambda \mathbb{E}_{k-1}^{\mathbb{Q}}[m + \widehat{V}_k - S_k] \leq \alpha_k \right\} \\ &= \operatorname{ess\,inf}_{m \in \mathcal{F}_k, m \geq 0} \left\{ \mathbb{E}_{k-1}^{\mathbb{Q}}[m] : \mathbb{E}_{k-1}^{\mathbb{Q}}[m] \geq \frac{\alpha_k - \mathbb{E}_{k-1}^{\mathbb{P}}[\ell(I(\lambda \frac{Z_k}{Z_{k-1}}))]}{\lambda} \right. \\ &\quad \left. + \mathbb{E}_{k-1}^{\mathbb{Q}} \left[S_k - \widehat{V}_k + I \left(\lambda \frac{Z_k}{Z_{k-1}} \right) \right] \right\} \end{aligned}$$

for any strictly negative $\lambda \in \mathcal{F}_{k-1}$. On the set $\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] > \alpha_k\}$, the last constraint above can be replaced by

$$\mathbb{E}_{k-1}^{\mathbb{Q}}[m] \geq \frac{\alpha_k - \mathbb{E}_{k-1}^{\mathbb{P}}[\ell(I(\lambda \frac{Z_k}{Z_{k-1}}) \vee (\widehat{V}_k - S_k))]}{\lambda} + \mathbb{E}_{k-1}^{\mathbb{Q}}\left[\left(S_k - \widehat{V}_k + I\left(\lambda \frac{Z_k}{Z_{k-1}}\right)\right)^+\right].$$

Now we choose λ to be the solution of (3.12) to obtain the constraint

$$\mathbb{E}_{k-1}^{\mathbb{Q}}[m] \geq \mathbb{E}_{k-1}^{\mathbb{Q}}\left[\left(S_k - \widehat{V}_k + I\left(\lambda_{k-1} \frac{Z_k}{Z_{k-1}}\right)\right)^+\right].$$

Therefore on the set $\{\mathbb{E}_{k-1}^{\mathbb{P}}[\ell(\widehat{V}_k - S_k)] > \alpha_k\}$, we have

$$V_{k-1} - \mathbb{E}_{k-1}^{\mathbb{Q}}[\widehat{V}_k] \geq \mathbb{E}_{k-1}^{\mathbb{Q}}\left[\left(S_k - \widehat{V}_k + I\left(\lambda_{k-1} \frac{Z_k}{Z_{k-1}}\right)\right)^+\right],$$

which implies the inequality $\widehat{V}_{k-1} \leq V_{k-1}$. The theorem is thus proved. □

Remark 3.6 In the risk-neutral case where $\mathbb{P} = \mathbb{Q}$, we can relax Assumption 2.2 and obtain a similar result under Assumption 2.1 only. The following result can be proved by an approximation argument. Let Assumption (C) be satisfied with $\mathbb{Q} = \mathbb{P}$, let the loss function ℓ satisfy Assumption 2.1 and assume that $\alpha_k > \ell_\infty$ for all $k \in \{1, \dots, n\}$. Then the value function satisfies

$$\begin{aligned} V_{n-1}^{NP} &= \mathbb{E}_{n-1}[S_n] + \ell^{-1}(\alpha_n), \\ V_{k-1}^{NP} &= \mathbb{E}_{k-1}[V_k^{NP}] + \mathbb{E}_{k-1}[(S_k - V_k^{NP} + \lambda_k)^+] \mathbf{1}_{\{\mathbb{E}_{k-1}[\ell(V_k^{NP} - S_k)] > \alpha_k\}}, \quad k < n, \end{aligned}$$

where $\lambda_k \in \mathcal{F}_{k-1}$ is the solution of

$$\mathbb{E}_{k-1}[\ell(\lambda_k \vee (V_k^{NP} - S_k))] = \alpha_k.$$

In practice, one is often interested in the loss function $\ell(x) = x^-$ (expected shortfall). We close this section with an explicit solution of the superhedging problem for this loss function, in the risk-neutral setting. While in the European case only one constraint may be active, the form of the solution in the next-period case clearly shows that several constraints may be active at the same time.

Proposition 3.7 *Let Assumption (C) be satisfied with $\mathbb{Q} = \mathbb{P}$ and assume $\ell(x) = x^-$, with $\alpha_k > 0$ for all k . Then the value function for the next-period constraint satisfies*

$$\begin{aligned} V_{n-1}^{NP} &= \mathbb{E}_{n-1}[S_n] - \alpha_n, \\ V_{k-1}^{NP} &= \max \{ \mathbb{E}_{k-1}[V_k^{NP}], \mathbb{E}_{k-1}[V_k^{NP} \vee S_k - \alpha_k] \}. \end{aligned}$$

If in addition the benchmark process $(S_k)_{k=0}^n$ is nondecreasing, then

$$V_0^{EU} = \max_{k \in \{1, \dots, n\}} \{ \mathbb{E}[S_k] - \alpha_k \} \tag{3.13}$$

Proof The formula for V_{n-1}^{NP} is clear. When $\mathbb{E}_{k-1}[(V_k^{NP} - S_k)^-] \leq \alpha_k$, then clearly $\mathbb{E}_{k-1}[V_k^{NP}] \geq \mathbb{E}_{k-1}[V_k^{NP} \vee S_k - \alpha_k]$, and the formulas hold true. Assume that instead $\mathbb{E}_{k-1}[(V_k^{NP} - S_k)^-] > \alpha_k$. First we observe that $\lambda_k < 0$ a.s.; otherwise on the set where $\lambda_k \geq 0$, we get $\ell(\lambda_k \vee (V_k^{NP} - S_k)) = 0$ and the conditional expectation cannot be equal to α_k . Therefore,

$$\begin{aligned} & \mathbb{E}_{k-1}[(S_k + \lambda_k) \vee V_k^{NP}] \\ &= \mathbb{E}_{k-1}[S_k] + \mathbb{E}_{k-1}[\lambda_k \vee (V_k^{NP} - S_k)] \\ &= E_{k-1}[S_k] + \mathbb{E}_{k-1}[(\lambda_k \vee (V_k^{NP} - S_k))^+] - \mathbb{E}_{k-1}[(\lambda_k \vee (V_k^{NP} - S_k))^-] \\ &= E_{k-1}[S_k] + \mathbb{E}_{k-1}[(V_k^{NP} - S_k)^+] - \mathbb{E}_{k-1}[\ell(\lambda_k \vee (V_k^{NP} - S_k))] \\ &= \mathbb{E}_{k-1}[V_k^{NP} \vee S_k] - \alpha_k. \end{aligned}$$

To obtain (3.13), note that for any $k \in \{1, \dots, n\}$, we have

$$\begin{aligned} V_0^{EU} &= \inf_{(M_k)_{k=0}^n \in \mathcal{M}_{EU}} \{M_0 : \mathbb{E}[(S_k - M_k)^+] \leq \alpha_k, k = 1, \dots, n\} \\ &\geq \inf_{m \in \mathcal{F}_k} \{\mathbb{E}[m] : \mathbb{E}[(S_k - m)^+] \leq \alpha_k\} = \mathbb{E}[S_k] - \alpha_k \end{aligned}$$

and therefore

$$V_0^{EU} \geq \max_{k \in \{1, \dots, n\}} \{\mathbb{E}[S_k] - \alpha_k\}.$$

On the other hand, Jensen’s inequality yields $\mathbb{E}[(S_k - M_k)^+] \leq \mathbb{E}[(S_k - M_n)^+]$, so

$$\begin{aligned} V_0^{EU} &\leq \inf_{(M_k)_{k=0}^n \in \mathcal{M}_{EU}} \{M_0 : \mathbb{E}[(S_k - M_n)^+] \leq \alpha_k, k = 1, \dots, n\} \\ &= \inf_{m \in \mathcal{F}_n} \{\mathbb{E}[m] : \mathbb{E}[(S_k - m)^+] \leq \alpha_k, k = 1, \dots, n\}. \end{aligned}$$

Since $(S_t)_{t=0}^n$ is nondecreasing, (3.13) follows by Lemma A.2 in Appendix A. □

4 Examples and comparison of different risk constraints

In this section, we present explicit solutions and compare the different constraints in two settings: the simplified two-period risk-neutral setting, which allows us to illustrate the different ways of imposing the shortfall constraint on the superhedging price of a random benchmark, and the practically more relevant setting where the benchmark is deterministic and the market returns over different periods are independent.

4.1 The risk-neutral setting with random benchmark

In this section, we assume that $n = 2$, the market is complete, and we consider the risk-neutral case where $\mathbb{P} = \mathbb{Q}$. We consider the European-style constraint, the next-period constraint and the general shortfall risk constraint with the “lookback-style”

loss function

$$L(M_1 - S_1, \dots, M_n - S_n) = \max_{k=1, \dots, n} \{\ell(M_k - S_k) - \alpha_k\},$$

where $\ell : \mathbb{R} \rightarrow \mathbb{R}$ is a one-dimensional loss function satisfying Assumption 2.1.

The following results are easy consequences of Theorem 3.4 and Jensen’s inequality:

– *European-style constraint:*

$$V_1^{EU}(N_1) = \text{ess inf}_{M_2 \in \mathcal{F}_2} \{\mathbb{E}_1[M_2] : \mathbb{E}_1[\ell(M_2 - S_2)] \leq N_1\} = \mathbb{E}_1[S_2] + \ell^{-1}(N_1),$$

$$V_0^{EU} = \inf_{m \in \mathcal{F}_1} \{\mathbb{E}[m] : \mathbb{E}[\ell(m - S_1)] \leq \alpha_1, \mathbb{E}[\ell(m - \mathbb{E}_1[S_2])] \leq \alpha_2\}.$$

– *Next-period constraint:*

$$V_1^{NP} = \text{ess inf}_{M_2 \in \mathcal{F}_2} \{\mathbb{E}_1[M_2] : \mathbb{E}_1[\ell(M_2 - S_2)] \leq \alpha_2\} = \mathbb{E}_1[S_2] + \ell^{-1}(\alpha_2),$$

$$V_0^{NP} = \inf_{m \in \mathcal{F}_1} \{\mathbb{E}[m] : \mathbb{E}[\ell(m - S_1)] \leq \alpha_1, \ell(m - \mathbb{E}_1[S_2]) \leq \alpha_2\}.$$

– *Shortfall risk constraint:*

$$V_1^{SR}(N_1, M_1) = \text{ess inf}_{M_2 \in \mathcal{F}_2} \{\mathbb{E}_1[M_2] : \mathbb{E}_1[\max\{\ell(M_1 - S_1) - \alpha_1, \ell(M_2 - S_2) - \alpha_2\}] \leq N_1\}$$

$$= \text{ess inf}\{m \in \mathcal{F}_1 : \max\{\ell(M_1 - S_1) - \alpha_1, \ell(m - \mathbb{E}_1[S_2]) - \alpha_2\} \leq N_1\},$$

$$V_0^{SR} = \inf_{m \in \mathcal{F}_1} \{\mathbb{E}[m] : \mathbb{E}[\max\{\ell(m - S_1) - \alpha_1, \ell(m - \mathbb{E}_1[S_2 | \mathcal{F}_1]) - \alpha_2\}] \leq 0\}.$$

For the loss function $\ell(x) = (-x)^+$ which is often used in practice, explicit results are obtained, thanks to Lemma A.1 in Appendix A (for the next-period constraint, the result follows by taking $\beta = 0$ and $Y = \mathbb{E}_1[S_2 | \mathcal{F}_1] - \alpha_2$), as follows: we get

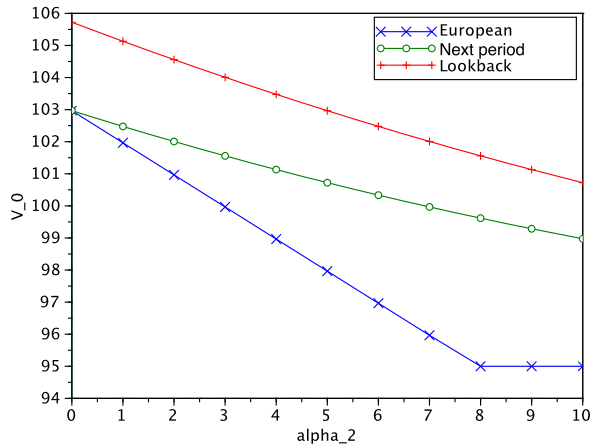
$$V_0^{EU} = \max\{\mathbb{E}[S_1] - \alpha_1, \mathbb{E}[S_2] - \alpha_2, \mathbb{E}[(S_1 \vee \mathbb{E}_1[S_2])] - \alpha_1 - \alpha_2\},$$

$$V_0^{NP} = \max\{\mathbb{E}[S_2] - \alpha_2, \mathbb{E}[S_1 \vee (\mathbb{E}_1[S_2] - \alpha_2)] - \alpha_1\},$$

$$V_0^{SR} = \mathbb{E}[\max\{S_1 - \alpha_1, (\mathbb{E}_1[S_2] - \alpha_2)\}].$$

Numerical illustration We compare the cost for hedging two liabilities under the three probabilistic constraints in Fig. 1. The loss function is $\ell(x) = (-x)^+$. The model is given by $S_1 = S_0 e^{\sigma Y_1 - \frac{\sigma^2}{2}}$ and $S_2 = S_0 e^{\sigma Y_2 - \frac{\sigma^2}{2}}$, where $S_0 = 100$, $\sigma = 0.2$

Fig. 1 Cost of hedging two objectives with probabilistic constraints of three different styles. The cost of superreplicating both objectives in an almost sure way is equal to 107.966 in this case



and Y_1 and Y_2 are standard normal random variables with correlation $\rho = 50\%$. We fix the loss tolerance at the first date to be $\alpha_1 = 5$ and plot the hedging value V_0 for three different constraint styles as a function of the loss tolerance α_2 at the second date.

Not surprisingly, all three curves are decreasing with respect to the constraint level α_2 . Moreover, among the three constraints we consider, the European constraint corresponds to the lowest hedging cost and the “lookback-style” shortfall risk constraint to the highest one. Finally, the cost of almost sure hedging is higher than the three probabilistic constraints.

4.2 Market with independent returns and deterministic benchmark

In this section, we provide explicit formulas for the superhedging price in the case of next-period and European-style constraints, under some simplifying but nevertheless realistic assumptions on the market. More precisely, we assume that

- the benchmark values $(S_k)_{k=1}^n$ are deterministic

and use the following two alternative assumptions to describe the market structure:

- (C') The financial market satisfies the assumption (C); the unique martingale measure \mathbb{Q} has density process

$$Z_k = \mathbb{E}_k^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right],$$

and the variable $U_k := Z_k/Z_{k-1}$ is independent of \mathcal{F}_{k-1} for $k = 1, \dots, n$.

- (NC') The financial market satisfies the assumption (NC) and for $k = 0, \dots, n - 1$, we have $X_k^{(i)} > 0$ a.s. for $i = 1, \dots, d$, and the vector of returns $(R_{k+1}^{(i)})_{i=1}^d$ defined by $R_{k+1}^{(i)} = (X_{k+1}^{(i)} - X_k^{(i)})/X_k^{(i)}$ is independent of \mathcal{F}_k .

4.2.1 Next-period constraint, incomplete market

Proposition 4.1 *Let Assumption (NC') be satisfied. Then the value function $(V_k^{NP})_{k=0}^{n-1}$ is deterministic and given recursively by*

$$V_k^{NP} = \inf \{x \in \mathbb{R} : \exists \eta \in \mathbb{R}^d \text{ with } \mathbb{E}^{\mathbb{P}}[\ell(x + \eta \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1} \\ \text{and } x + \eta \cdot R_{k+1} \geq V_{k+1}^{NP}\}.$$

Proof We prove the result by induction, starting with $k = n - 1$. Let $(x, \eta) \in \mathbb{R}^{d+1}$ be such that $\mathbb{E}^{\mathbb{P}}[\ell(x + \eta \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1}$. Then by the independence assumption, it holds that $\mathbb{E}_k^{\mathbb{P}}[\ell(x + \eta \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1}$ and so $V_k^{NP} \leq x$. On the other hand, let $\hat{x} \in \mathcal{F}_k$ and $\hat{\eta} \in \mathcal{F}_k$ be such that $\mathbb{E}_k^{\mathbb{P}}[\ell(\hat{x} + \hat{\eta} \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1}$. Then taking the conditional expectation with respect to the σ -field generated by R_{k+1} and using Jensen's inequality, we have that

$$\mathbb{E}^{\mathbb{P}}[\ell(\mathbb{E}[\hat{x}] + \mathbb{E}[\hat{\eta}] \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1},$$

and therefore also

$$\mathbb{E}_k^{\mathbb{P}}[\ell(\mathbb{E}[\hat{x}] + \mathbb{E}[\hat{\eta}] \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1},$$

which means that the set of random variables $\xi \in \mathcal{F}_k$ such that there exists $\eta \in \mathcal{F}_k$ with $\mathbb{E}_k^{\mathbb{P}}[\ell(\xi + \eta \cdot R_{k+1} - S_{k+1})] \leq \alpha_{k+1}$ is stable with respect to taking the expectation. Therefore, using the monotone convergence theorem, we conclude that the essential lower bound of this set is a constant.

Suppose that the result is proved for $k + 1 \leq n - 1$, that is, V_{k+1}^{NP} is deterministic. Using the same argument as above, we can show that V_k^{NP} is deterministic as well. □

For specific loss functions, explicit formulas may be obtained. As a first example, we consider the constraints imposed in terms of the conditional entropic risk measure

$$\frac{1}{p} \log \mathbb{E}_{k-1}^{\mathbb{P}}[e^{-p(M_k - S_k)}] \leq \rho_k, \tag{4.1}$$

which is equivalent to taking $\ell(x) = e^{-px}$ and $\alpha_k = e^{p\rho_k}$, and our second example uses the power loss function $\ell(x) = (x^-)^p$ with $p > 1$. Moreover, we impose the following full support assumption on the law of the returns:

(FS) For all $k = 1, \dots, n$ and all $\eta \in \mathbb{R}^d$ with $\eta \neq 0$, if $\eta^{(i)} \geq 0$ for all i , then the support of the random vector $\eta \cdot R_{k+1}$ is equal to $(-\sum_{i=1}^d \eta_i, +\infty)$; if $\eta^{(i)} \leq 0$ for all i , then the support of the random vector $\eta \cdot R_{k+1}$ is equal to $(-\infty, -\sum_{i=1}^d \eta_i)$; otherwise the support of $\eta \cdot R_{k+1}$ is equal to $(-\infty, +\infty)$.

Proposition 4.2 1) *Let Assumption (NC') be satisfied, assume that the constraints are imposed in terms of the conditional entropic risk measure, that the vector of returns*

R_{k+1} satisfies the assumption (FS), and that for every $k \in \{0, 1, \dots, n - 1\}$, the convex function

$$\eta \mapsto \log \mathbb{E}^{\mathbb{P}}[e^{-p\eta \cdot R_{k+1}}]$$

attains its unique minimum at a finite point. Then

$$V_{n-1}^{NP} = S_n - \rho_n + \frac{1}{p} \min_{\eta \in \mathbb{R}^d} \log \mathbb{E}^{\mathbb{P}}[e^{-p\eta \cdot R_n}] \tag{4.2}$$

and for $k < n - 1$,

$$V_k^{NP} = \min_{\eta \in \mathbb{R}^d, \eta \geq 0} \max \left\{ V_{k+1}^{NP} + \sum_{i=1}^d \eta_i, S_{k+1} - \rho_{k+1} + \frac{1}{p} \log \mathbb{E}^{\mathbb{P}}[e^{-p\eta \cdot R_{k+1}}] \right\}. \tag{4.3}$$

The optimal strategy consists of maintaining a self-financing portfolio which invests into the risky asset the amount η_k^* at date $k - 1$, for $k = 1, \dots, n$, where η_n^* is the minimizer in (4.2) and η_k^* for $k < n$ is the minimizer in (4.3).

2) Let Assumption (NC') be satisfied, assume that the loss function is given by $\ell(x) = (x^-)^p$ with $p > 1$, that the vector of returns R_{k+1} satisfies the assumption (FS), and that for every $k \in \{0, 1, \dots, n - 1\}$, the convex function

$$\eta \mapsto \mathbb{E}^{\mathbb{P}}[((1 - \eta \cdot R_{k+1})^+)^p]$$

attains its unique minimum at a finite point. Then

$$V_{n-1}^{NP} = S_n - \left(\frac{\alpha_n}{\min_{\eta \in \mathbb{R}^d} \mathbb{E}^{\mathbb{P}}[((1 - \eta \cdot R_n)^+)^p]} \right)^{\frac{1}{p}} \tag{4.4}$$

and for $k < n - 1$,

$$V_k^{NP} = S_{k+1} - \max_{\eta \in \mathbb{R}^d, \eta \geq 0} \min \left\{ \frac{S_{k+1} - V_{k+1}^{NP}}{1 + \sum_{i=1}^d \eta_i}, \left(\frac{\alpha_{k+1}}{\mathbb{E}^{\mathbb{P}}[((1 - \eta \cdot R_{k+1})^+)^p]} \right)^{\frac{1}{p}} \right\} \tag{4.5}$$

whenever $S_{k+1} > V_{k+1}^{NP}$, and $V_k^{NP} = V_{k+1}^{NP}$ otherwise. The optimal strategy consists of maintaining a self-financing portfolio which invests into the risky asset the amount $(S_k - V_{k-1}^{NP})\eta_k^*$ at date $k - 1$ if $S_k > V_k^{NP}$ (or $k = n$) and zero otherwise, where η_n^* is the minimizer in (4.4) and η_k^* for $k < n$ is the minimizer in (4.5).

Proof We use Proposition 4.1. The computation of V_{n-1}^{NP} is straightforward in both cases. For the computation of V_k^{NP} , observe that under the full support assumption, when V_{k+1}^{NP} is deterministic, the condition

$$x + \eta \cdot R_{k+1} \geq V_{k+1}^{NP} \quad \text{a.s.}$$

is equivalent to

$$\eta^i \geq 0, \quad i = 1, \dots, d \quad \text{and} \quad x - \sum_{i=1}^d \eta^{(i)} \geq V_{k+1}^{NP}. \quad \square$$

4.2.2 Next-period constraint, complete market

Suppose that assumption (C') holds true and the loss function satisfies Assumption 2.2. We begin by solving the hedging problem under the next-period constraint by using Proposition 3.5. Since U_k is independent of \mathcal{F}_{k-1} , it follows that the value functions $(V_k^{NP})_{k=0}^{n-1}$ and the Lagrange multipliers $(\lambda_k)_{k=0}^{n-1}$ are deterministic, and can be found by solving the sequence of algebraic equations

$$V_{n-1}^{NP} = S_n + \mathbb{E}^{\mathbb{P}}[U_n I(\lambda_{n-1} U_n)],$$

where $\lambda_{n-1} \in \mathbb{R}$ satisfies $\mathbb{E}^{\mathbb{P}}[\ell(I(\lambda_{n-1} U_n))] = \alpha_n$ and for $1 \leq k \leq n - 1$,

$$V_{k-1}^{NP} = V_k^{NP} + \mathbb{E}^{\mathbb{P}}[U_k (S_k - V_k^{NP} + I(\lambda_{k-1} U_k))^+] \mathbf{1}_{\{\ell(V_k^{NP} - S_k) > \alpha_k\}},$$

where $\lambda_{k-1} \in \mathbb{R}$ satisfies

$$\mathbb{E}^{\mathbb{P}}[\ell(I(\lambda_{k-1} U_k) \vee (V_k^{NP} - S_k))] = \alpha_k.$$

The optimal strategy at date $n - 1$ consists of replicating the claim $S_n + I(\lambda_{n-1} U_n)$, and at date $k - 1 < n - 1$, the strategy consists of replicating the claim

$$V_k^{NP} + (S_k - V_k^{NP} + I(\lambda_{k-1} U_k))^+$$

if $\ell(V_k^{NP} - S_k) > \alpha_k$, and investing everything into the risk-free asset otherwise.

To obtain more explicit results, we once again consider the entropic risk measure constraints (4.1) and the power loss function. For the power loss function, to avoid cumbersome developments, we consider the case $p = 2$, which means that the penalty is given by the second moment of the loss.

Proposition 4.3 1) *Let Assumption (C') be satisfied and assume that the constraints are imposed in terms of the conditional entropic risk measure. Then*

$$V_{n-1}^{NP} = S_n - D_n - \rho_n$$

and for $1 \leq k \leq n - 1$,

$$V_{k-1}^{NP} = \begin{cases} V_k^{NP}, & \text{if } V_k^{NP} \geq S_k - \rho_k, \\ V_k^{NP} + \mathbb{E}^{\mathbb{Q}}[(S_k - V_k^{NP} - \frac{1}{p} \log \tilde{\lambda}_{k-1} - \frac{1}{p} \log U_k)^+], & \text{otherwise,} \end{cases}$$

where $\tilde{\lambda}_{k-1}$ is the solution to

$$\mathbb{E}^{\mathbb{P}}[(\tilde{\lambda}_{k-1} U_k) \wedge e^{-p(V_k^{NP} - S_k)}] = \alpha_k$$

and we denote

$$D_k = \frac{1}{p} \mathbb{E}^{\mathbb{P}} [U_k \log U_k]$$

for $k = 1, \dots, n$.

2) Let Assumption (C') be satisfied and assume that the loss function is given by $\ell(x) = (x^-)^2$, so that the constraint is

$$\mathbb{E}^{\mathbb{P}} [(M_k - S_k)^2 \mathbf{1}_{\{M_k \leq S_k\}} | \mathcal{F}_{k-1}]^{\frac{1}{2}} \leq \sigma_k = \sqrt{\alpha_k}.$$

Then

$$V_{n-1}^{NP} = S_n - \sigma_n \sqrt{\mathbb{E}^{\mathbb{P}} [U_n^2]}$$

and for $k < n$,

$$V_{k-1}^{NP} = \begin{cases} V_k^{NP} + \mathbb{E}^{\mathbb{Q}} [(S_k - V_k^{NP} + \tilde{\lambda}_{k-1} U_k)^+], & \text{if } S_k - V_k^{NP} > \sigma_k, \\ V_k^{NP}, & \text{otherwise,} \end{cases}$$

where $\tilde{\lambda}_{k-1} < 0$ is the solution to $\mathbb{E}^{\mathbb{P}} [\tilde{\lambda}_{k-1}^2 U_k^2 \wedge (V_k^{NP} - S_k)^2] = \sigma_k^2$.

4.2.3 European-style constraint

For the European-style constraints, the computations are generally more difficult, owing to the high dimension of the problem, but the entropic penalty simplifies matters once again since for this penalty, the optimal strategy of investing into the risky assets does not depend on the constraints. Therefore in this situation, all constraints but one may be removed without changing the value function. We insist that this feature is very specific to the exponential loss function.

Proposition 4.4 1) Let Assumption (NC') be satisfied, let $\ell(x) = e^{-px}$ and for $k = 1, \dots, n$, let $\alpha_k = e^{p\rho_k}$ and assume that the vector of returns R_k satisfies the assumption (FS) and that the convex function

$$\eta \mapsto \log \mathbb{E}^{\mathbb{P}} [e^{-p\eta \cdot R_k}]$$

attains its unique minimum at a finite point. Then

$$V_0^{EU}(\alpha_1, \dots, \alpha_n) = \max_{j=1, \dots, n} \left\{ S_j - \rho_j - \sum_{i=1}^j D_i \right\},$$

where

$$D_k = \frac{1}{p} \log \max_{\eta \in \mathbb{R}^d} \mathbb{E}^{\mathbb{P}} [e^{-p\eta \cdot R_k}].$$

The optimal strategy consists of maintaining a self-financing portfolio which invests the amount $\eta_k^* = \arg \max_{\eta \in \mathbb{R}^d} \mathbb{E}^{\mathbb{P}} [e^{-p\eta \cdot R_k}]$ into the risky asset between dates $k - 1$ and k .

2) Let Assumption (C') be satisfied and $\ell(x) = e^{-px}$ and $\alpha_k = e^{p\rho_k}$, $k = 1, \dots, n$. Then

$$V_0^{EU}(\alpha_1, \dots, \alpha_n) = \max_{j=1, \dots, n} \left\{ S_j - \rho_j - \sum_{i=1}^j D_i \right\},$$

where

$$D_k = \frac{1}{p} \mathbb{E}^{\mathbb{P}} [U_k \log U_k].$$

The optimal strategy consists of replicating the claim

$$-\frac{1}{p} \sum_{i=1}^n \log U_i$$

and investing the remaining amount into the risk-free asset.

Proof We carry out the proof simultaneously for both parts of the proposition. By removing all constraints except the k th one, we see that

$$\begin{aligned} V_0^{EU}(\alpha_1, \dots, \alpha_n) &\geq \inf_{V \in \mathcal{A}} \{V_0 : \mathbb{E}^{\mathbb{P}}[\ell(V_k - S_k)] \leq \alpha_k\} \\ &= S_k - \rho_k + \frac{1}{p} \log \max_{V \in \mathcal{A}: V_0=0} \mathbb{E}[e^{-pV_k}]. \end{aligned}$$

Under the assumption (NC'), using the independence of returns,

$$\max_{V \in \mathcal{A}: V_0=0} \mathbb{E}[e^{-pV_k}] = \prod_{i=0}^{k-1} \max_{\eta \in \mathbb{R}^d} \mathbb{E}[e^{-p\eta \cdot R_k}].$$

On the other hand, under the assumption (C'),

$$\max_{V \in \mathcal{A}: V_0=0} \mathbb{E}[e^{-pV_k}] = \max_{X \in \mathcal{F}_k: \mathbb{E}^{\mathbb{Q}}[X] \leq 0} \mathbb{E}[e^{-pX}] = e^{-\mathbb{E}[Z_k \log Z_k]} = e^{-\sum_{i=1}^k \mathbb{E}[U_i \log U_i]}.$$

Therefore, under both assumptions,

$$V_0^{EU}(\alpha_1, \dots, \alpha_n) \geq \max_{j=1, \dots, n} \left\{ S_j - \rho_j - \sum_{i=1}^j D_i \right\}.$$

To show the reverse inequality, we need to prove that the candidate optimal strategy satisfies the constraints.

In the first part of the proposition, the value of the portfolio corresponding to the candidate optimal strategy at date t_k is given by

$$\tilde{V}_k = \max_{j=1, \dots, n} \left\{ S_j - \rho_j - \sum_{i=1}^j D_i \right\} + \sum_{i=1}^k \eta_i^* R_i \geq S_k - \rho_k + \sum_{i=1}^k \eta_i^* R_i - \sum_{i=1}^k D_i$$

and therefore

$$\frac{1}{p} \log \mathbb{E}^{\mathbb{P}}[\ell(\tilde{V}_k - S_k)] \leq \rho_k + \frac{1}{p} \log \prod_{i=1}^k \mathbb{E}^{\mathbb{P}}[e^{-p\eta_i^* R_i + pD_i}] = \rho_k.$$

In the second part of the proposition, the value of the portfolio corresponding to the candidate optimal strategy at date t_k is given by

$$\begin{aligned} \tilde{V}_k &= \max_{j=1, \dots, n} \left\{ S_j - \rho_j - \sum_{i=1}^j D_i \right\} - \frac{1}{p} \mathbb{E}_k^{\mathbb{Q}} \left[\sum_{i=1}^n \log U_i \right] + \frac{1}{p} \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^n \log U_i \right] \\ &\geq S_k - \rho_k - \frac{1}{p} \sum_{i=1}^k \log U_i \end{aligned}$$

and therefore

$$\frac{1}{p} \log \mathbb{E}^{\mathbb{P}}[\ell(\tilde{V}_k - S_k)] \leq \frac{1}{p} \log \mathbb{E}^{\mathbb{P}} \left[e^{p\rho_k} \prod_{i=1}^k U_i \right] = \rho_k. \quad \square$$

5 Application to the ALM problem of a nuclear power plant operator

In this section, we present an application of our results to the actual ALM problem of a nuclear power plant operator in a realistic but simplified setting. In several countries, energy companies operating nuclear power plants are required by law to hold decommissioning funds, to cover the future costs of decontaminating and dismantling the plants as well as for treatment and long-term storage of radioactive waste. As an example, we refer the reader to [6, Chap. 3] for a detailed description of the current practices of evaluating the future charges related to nuclear power plant decommissioning and waste management in France.

Definition of the benchmark process Quoting from [6, p. 88], “dismantling costs are calculated in the form of estimates which are drawn up and regularly revised, based on all the technical, financial and contractual data available at the time at which they are produced.” Therefore, the future dismantling charges, even though they are periodically revised to account for inflation and changes in industrial processes, are assumed to be known by the nuclear operator company from the point of view of asset–liability management. For this reason, in this simplified example, we suppose that the benchmark process S is deterministic.

According to the law, the dedicated assets held by the nuclear company must be sufficient to cover the discounted value of future liabilities, where the discount rate “may not exceed the confidently expected yield rate on collateral assets managed with a degree of security and liquidity sufficient to meet their purpose” [6, p. 174]. It is fixed by the regulator using a complex formula resulting from negotiations with the nuclear companies and was equal to 5.24% on December 31, 2010 [6, p. 170], well in

excess of what can be expected of a risk-free portfolio of Eurozone instruments.³ We denote this discount rate by γ . Since we want to work with self-financing portfolios, the nondiscounted benchmark process S contains the future cash flows discounted with rate γ as well as the past cash flows accrued with the risk-free rate r (assumed constant in this simplified setting); so

$$S_t = \sum_{i \geq 1 : t_i \leq t} P_i e^{r(t-t_i)} + \sum_{i \geq 1 : t_i > t} P_i e^{-\gamma(t_i-t)}. \tag{5.1}$$

This definition of the benchmark process is different from the one in (2.1) because (a) in this section we make the interest rate appear explicitly and (b) we assume, following the industry practice, that the portfolio must not only remain positive at all future dates, but also remain above the sum of future cash flows discounted at the rate γ . Given that the discount rate for future charges γ exceeds the risk-free rate, it is impossible to construct a portfolio whose value at time $t = 0$ equals S_0 and which dominates the benchmark portfolio at all future payment dates with probability 1. In other words, using a discount rate $\gamma > r$ leads us to consider the case when the company holds risky assets and therefore faces a risk that the dedicated portfolio will fall below the benchmark level at some or all future payment dates. For the purpose of understanding and managing this risk, it is therefore essential to answer the following questions:

- Given a value of the discount rate γ , what is the minimum risk level of a portfolio whose value is at most S_0 at time $t = 0$, and which allows hedging the future liabilities with this risk level?
- For a given level of risk, what is the maximum value of the discount rate γ such that there exists a portfolio with initial value of at most S_0 , and which allows hedging future liabilities without exceeding this risk level?
- For a given value of discount rate γ and a given risk tolerance, what is the minimum required risk premium of the risky asset which allows hedging the future liabilities without exceeding the risk tolerance, given that the initial portfolio value is at most S_0 ?

Our results presented in Sect. 3 provide a precise mathematical framework in which these questions can be answered. In this section, we give an illustration with a realistic and simple model. We assume that the financial market consists of a risk-free asset evolving with constant interest rate r and a risky asset which follows the Black–Scholes model

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t.$$

We consider two alternative settings: the complete market setting, where the agent can rebalance the portfolio dynamically between the benchmark dates, and the in-

³At this point, one may argue, see e.g. [15], that discounting fixed future liabilities at an expected rate of return of the assets runs counter to the logic of financial economics. Nevertheless, regulations that mandate such discounting practices are in place in many countries, and it is important to understand their impact in practice, and in particular to quantify the actual risk of not being able to meet the future liabilities and to identify the strategies which minimize this risk.

Table 1 Estimation of future decommissioning charges (nondiscounted values). The data were obtained from the graphs on pp. 87 and 93 of [6]

Year	2015	2020	2025	2030	2035	2040	2045
Cash flow, M€	200	950	5550	7950	2700	1500	500

complete market setting, where the agent can only rebalance the portfolio at benchmark dates and uses buy-and-hold strategy between these dates (this corresponds to the situation when the portfolio contains very illiquid assets which cannot be easily traded, such as real estate).

We assume that the risk constraint is imposed on the mean square loss, that is, the loss function is given by

$$\ell(x) = (x^-)^2.$$

We use the next-period constraint formulation (2.2), which now (with nonzero interest rate) means that we need to find the minimal value of M_0 such that there is a process $(M_k)_{k=0}^n$ with $\text{ess sup}_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{k-1}^{\mathbb{Q}} [e^{-r(t_k - t_{k-1})} M_k] \leq M_{k-1}$ for all k and

$$\mathbb{E}_{k-1}^{\mathbb{P}} [(M_k - S_k)^2 \mathbf{1}_{\{M_k \leq S_k\}}]^{\frac{1}{2}} \leq \sigma_k, \quad k = 1, \dots, n,$$

where $\sigma_k = \sqrt{\alpha_k}$ is the maximum allowed loss standard deviation at date t_k , expressed in monetary units. It is clear that this formulation easily reduces to that of Sect. 4.2 by discounting the benchmark values and σ_k , and we can apply the results of this section without further discussion. Precise computations in the Black–Scholes model are provided in Appendix B.

Numerical illustrations Our numerical example is based on the sequence of cash flows given in Table 1. For these cash flows, we first computed the benchmark values (5.1) and then evaluated the value function V_0^{NP} which corresponds to the minimum price for hedging the benchmark with a given risk tolerance. This computation has been carried out for different sets of model parameters, in both complete and incomplete market settings. The risk tolerance values were chosen as $\sigma_k = \sigma_0 e^{r t_k}$, with σ_0 fixed as explained below.

Tables 2 (complete market) and 3 (incomplete market) show the value V_0^{NP} as a function of the rate of return μ of the risky asset and of the risk tolerance parameter σ_0 , expressed as a percentage of the total net present value of the cashflows. The other parameters are $r = 3\%$, $\sigma = 20\%$ and $\gamma = 5\%$. For this value of γ , the initial value of the benchmark is $S_0 = 8682$ M€. The first lines of the tables correspond to no risk tolerance: the value function is simply equal to the net present value of the future cash flows. Note that this value is much greater than S_0 because the computation of S_0 uses a discount rate γ which is larger than the risk-free interest rate. The other parts of the tables are more interesting: here we see that the agent can benefit from the probabilistic nature of the risk constraint and achieve an additional risk reduction (which is minimal for $\mu = 5\%$, because this value corresponds to the smallest Sharpe ratio of the market). However, for this additional reduction to become significant,

Table 2 Value function V_0^{NP} (the price of hedging the benchmark subject to risk constraints) for different values of the rate of return μ and the risk tolerance σ_0 , expressed as a percentage of the total net present value of the cash flows, in the complete market setting (dynamic rebalancing). The values are given in M€

Risk tolerance σ_0	$\mu = 0\%$	$\mu = 5\%$	$\mu = 10\%$	$\mu = 15\%$
0%	11858	11858	11858	11858
5%	11231	11250	11156	10927
10%	10604	10650	10498	10106
15%	9976	10053	9844	9299
20%	9349	9456	9191	8496

Table 3 Value function V_0^{NP} (the price of hedging the benchmark subject to risk constraints) for different values of the rate of return μ and the risk tolerance σ_0 , expressed as a percentage of the total net present value of the cash flows in the incomplete market setting (buy-and-hold strategy between benchmark dates). The values are given in M€

Risk tolerance σ_0	$\mu = 0\%$	$\mu = 5\%$	$\mu = 10\%$	$\mu = 15\%$
0%	11858	11858	11858	11858
5%	11247	11253	11215	11131
10%	10654	10654	10586	10465
15%	10061	10059	9974	9799
20%	9468	9465	9370	9161

the rate of return of the risky asset should be quite high. In particular, even in the complete market setting, for a risk tolerance level of 20%, a hedging portfolio for the benchmark exists (that is to say, $V_0^{NP} \leq S_0$) only if the rate of return is at least 15%.

Tables 4 (complete market) and 5 (incomplete market) study the effect of changing the parameters γ and μ . The risk tolerance is fixed to 10% of the total cash flow, and the other parameters are chosen as above. Since both the benchmark S_0 and the value function V_0^{NP} depend on γ , we show the difference between the two values $S_0 - V_0^{NP}$. This means that the hedging portfolio exists whenever the value in the corresponding cell is positive. Here once again we see that if one uses a discount rate γ which is much higher than the risk-free rate r to compute the initial benchmark value, an unrealistically high market return is needed to ensure that the portfolio remains above the benchmark at all dates with a reasonable confidence level.

Finally, Table 6 shows the evolution of the value function and of the optimal investment strategy at the beginning of each period, for risk tolerance $\sigma_0 = 15\%$ and market return rate $\mu = 15\%$. We see that in this case, the strategy does not invest into the risky asset during the first three periods in the complete market setting and during the first four periods in the incomplete market setting. To understand why this happens, recall the formula for the value function of the next-period problem. At each date $k - 1$, the value V_{k-1}^{NP} must be sufficient to superhedge the value V_k^{NP} of the following date in an almost sure way, and to superhedge the benchmark S_k in a probabilistic sense. At the first dates, the constraint of superhedging the benchmark is not binding, so the strategy simply superhedges the value at the following date,

Table 4 Difference between the benchmark S_0 and the value function V_0^{NP} (price of hedging the benchmark subject to risk constraints) for different values of the rate of return μ and the discount rate γ , with $\sigma_0 = 10\%$ in the complete market setting (dynamic rebalancing). The values are given in M€

	$\mu = 0\%$	$\mu = 5\%$	$\mu = 10\%$	$\mu = 15\%$
$\gamma = 3\%$	1254	1185	1209	1344
$\gamma = 3.5\%$	353	296	406	716
$\gamma = 4\%$	-471	-524	-391	-41
$\gamma = 4.5\%$	-1227	-1277	-1133	-761
$\gamma = 5\%$	-1921	-1967	-1815	-1423
$\gamma = 5.5\%$	-2558	-2601	-2440	-2031
$\gamma = 6\%$	-3142	-3183	-3014	-2590

Table 5 Difference between the benchmark S_0 and the value function V_0^{NP} (price of hedging the benchmark subject to risk constraints) for different values of the rate of return μ and the discount rate γ , with $\sigma_0 = 10\%$ in the incomplete market setting (buy-and-hold between benchmark dates). The values are given in M€

	$\mu = 0\%$	$\mu = 5\%$	$\mu = 10\%$	$\mu = 15\%$
$\gamma = 3\%$	1185	1185	1185	1188
$\gamma = 3.5\%$	289	291	325	402
$\gamma = 4\%$	-530	-529	-480	-373
$\gamma = 4.5\%$	-1282	-1281	-1218	-1103
$\gamma = 5\%$	-1971	-1971	-1903	-1782
$\gamma = 5.5\%$	-2604	-2604	-2537	-2405
$\gamma = 6\%$	-3184	-3187	-3118	-2976

Table 6 Evolution of the benchmark S_k , the value function V_k^{NP} and the optimal strategy (amount to invest into the risky asset at the beginning of the period) for risk tolerance $\sigma_0 = 15\%$ and rate of return $\mu = 15\%$. The amounts are given in M€

Year	2013	2015	2020	2025	2030	2035	2040	2045
Benchmark	8682	9596	12297	15645	19242	22754	26616	30971
Value function, complete market	9299	9874	11472	13328	15272	16970	16822	
Amount to invest, complete market	0	0	0	768	2176	5727	29508	
Value function, incomplete market	9799	10405	12089	14046	16319	18505	19833	
Amount to invest, incomplete market	0	0	0	0	235	741	4944	

which does not require an investment into the risky asset since this value is deterministic. During the last dates, however, the benchmark constraint becomes binding, and it becomes optimal to invest into the risky asset due to the probabilistic nature of the benchmark constraints. Note also that in the incomplete market setting, the strategy is much more prudent due to the additional risk of holding illiquid stocks which cannot be easily traded.

Appendix A: Technical lemmas

Lemma A.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $X, Y \in \mathcal{F}$ and $\alpha, \beta \geq 0$. Then*

$$\begin{aligned} V_0 &= \inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[(X - m)^+] \leq \alpha, \mathbb{E}[(Y - m)^+] \leq \beta \} \\ &= \max \{ \mathbb{E}[X - \alpha], \mathbb{E}[Y - \beta], \mathbb{E}[X \vee Y - \alpha - \beta] \} \end{aligned}$$

and

$$\begin{aligned} V'_0 &= \inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[\max\{(X - m)^+ - \alpha, (Y - m)^+ - \beta\}] \leq 0 \} \\ &= \mathbb{E}[(X - \alpha) \vee (Y - \beta)]. \end{aligned}$$

Proof We begin with V_0 . By considering only one of the two constraints, we have clearly

$$V_0 \geq \max\{\mathbb{E}[X - \alpha], \mathbb{E}[Y - \beta]\}.$$

In addition,

$$(X - m)^+ + (Y - m)^+ \geq X \vee Y - m$$

and so

$$V_0 \geq \mathbb{E}[X \vee Y - \alpha - \beta].$$

To finish the proof, we need to find m^* which satisfies the constraint and realizes the equality.

Assume first that

$$\mathbb{E}[X \vee Y - \alpha - \beta] \geq \mathbb{E}[X - \alpha] \quad \text{and} \quad \mathbb{E}[X \vee Y - \alpha - \beta] \geq \mathbb{E}[Y - \beta].$$

This is equivalent to

$$\mathbb{E}[(X - Y)^+] \geq \alpha \quad \text{and} \quad \mathbb{E}[(Y - X)^+] \geq \beta.$$

In this case, we can take

$$m^* = X \vee Y - \alpha \frac{(X - Y)^+}{\mathbb{E}[(X - Y)^+]} - \beta \frac{(Y - X)^+}{\mathbb{E}[(Y - X)^+]}.$$

If

$$\mathbb{E}[X - \alpha] \geq \mathbb{E}[X \vee Y - \alpha - \beta] \quad \text{and} \quad \mathbb{E}[X - \alpha] \geq \mathbb{E}[Y - \beta],$$

we take

$$m^* = X \wedge Y - \alpha + \mathbb{E}[(X - Y)^+].$$

If finally

$$\mathbb{E}[Y - \beta] \geq \mathbb{E}[X \vee Y - \alpha - \beta] \quad \text{and} \quad \mathbb{E}[Y - \beta] \geq \mathbb{E}[X - \alpha],$$

then we can take

$$m^* = X \wedge Y - \beta + \mathbb{E}[(Y - X)^+].$$

Now we turn to V'_0 . Since $m = (X - \alpha) \vee (Y - \beta)$ satisfies the constraint, it holds that $V'_0 \leq \mathbb{E}[(X - \alpha) \vee (Y - \beta)]$. On the other hand,

$$\begin{aligned} V'_0 &= \inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[\max\{(X - m)^+ - \alpha, (Y - m)^+ - \beta\}] \leq 0 \} \\ &\geq \inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[\max\{X - m - \alpha, Y - m - \beta\}] \leq 0 \} \\ &= \inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[\max\{X - \alpha, Y - \beta\}] \leq m \} \\ &= \mathbb{E}[(X - \alpha) \vee (Y - \beta)]. \end{aligned} \quad \square$$

We now consider the case when n is arbitrary, but the objectives are ordered.

Lemma A.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Z_1, \dots, Z_n \in \mathcal{F}$ with $Z_1 \leq Z_2 \leq \dots \leq Z_n$ a.s. and $\alpha_1, \dots, \alpha_n \geq 0$. Then*

$$\inf_{M \in \mathcal{F}} \{ \mathbb{E}[M] : \mathbb{E}[(Z_k - M)^+] \leq \alpha_k, k = 1, \dots, n \} = \max_{k=1, \dots, n} \{ \mathbb{E}[Z_k] - \alpha_k \}.$$

Proof 1) Assume that there exists $k \in \{1, \dots, n\}$ such that

$$\mathbb{E}[Z_k] - \alpha_k \geq \mathbb{E}[Z_{k+1}] - \alpha_{k+1}. \tag{A.1}$$

Then the constraint $\mathbb{E}[(Z_k - m)^+] \leq \alpha_k$ implies $\mathbb{E}[(Z_{k+1} - m)^+] \leq \alpha_{k+1}$ since

$$\mathbb{E}[(Z_{k+1} - m)^+] \leq \mathbb{E}[(Z_k - m)^+] + \mathbb{E}[Z_{k+1} - Z_k] \leq \alpha_k + \alpha_{k+1} - \alpha_k = \alpha_{k+1}.$$

We can then remove the constraint $\mathbb{E}[(Z_{k+1} - m)^+] \leq \alpha_{k+1}$ without modifying the value function. Repeating the same argument for all other indices k satisfying (A.1), we can assume without loss of generality that

$$\mathbb{E}[Z_1] - \alpha_1 < \dots < \mathbb{E}[Z_n] - \alpha_n. \tag{A.2}$$

We then need to prove that

$$\inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[(Z_k - m)^+] \leq \alpha_k, k = 1, \dots, n \} = \mathbb{E}[Z_n] - \alpha_n.$$

2) Removing all the constraints except the last one, it is easy to see that

$$\inf_{m \in \mathcal{F}} \{ \mathbb{E}[m] : \mathbb{E}[(Z_k - m)^+] \leq \alpha_k, k = 1, \dots, n \} \geq \mathbb{E}[Z_n] - \alpha_n.$$

To finish the proof, it is then enough to find m which satisfies the constraints and is such that $\mathbb{E}[m] = \mathbb{E}[Z_n] - \alpha_n$.

3) If $\mathbb{E}[Z_n - Z_1] \geq \alpha_n$, we let

$$k^* = \max\{i : \mathbb{E}[Z_n - Z_i] \geq \alpha_n\}$$

and

$$m := wZ_{k^*} + (1 - w)Z_{k^*+1}, \quad w = \frac{\mathbb{E}[Z_{k^*+1}] - \mathbb{E}[Z_n] + \alpha_n}{\mathbb{E}[Z_{k^*+1}] - \mathbb{E}[Z_{k^*}]} \in [0, 1].$$

Otherwise, let $k^* = 0$ and

$$m := Z_1 - C, \quad C = \alpha_n - \mathbb{E}[Z_n - Z_1].$$

By construction, we have $\mathbb{E}[m] = \mathbb{E}[Z_n] - \alpha_n$ and since $m \leq Z_n$, it also holds that $\mathbb{E}[(Z_n - m)^+] = \alpha_n$. To check the other constraints, observe that if $k \leq k^*$, then $m \geq Z_k$ a.s. and so $\mathbb{E}[(Z_k - m)^+] = 0$. On the other hand, if $k > k^*$, then $m \leq Z_k$ a.s. and so by (A.2),

$$\mathbb{E}[(Z_k - m)^+] = \mathbb{E}[Z_k] - \mathbb{E}[m] = \mathbb{E}[Z_k] - \mathbb{E}[Z_n] + \alpha_n < \alpha_k. \quad \square$$

Appendix B: Explicit computations in the Black–Scholes model

In this appendix, we specialize the formulas of Propositions 4.2 and 4.3 to the Black–Scholes market with a risk-free asset evolving with zero interest rate and a risky asset which follows the Black–Scholes model

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t,$$

where the rate of return μ of the risky asset is assumed to be positive.

Complete market We assume that the agent can trade dynamically between the benchmark dates, which means that the market is complete, and the unique risk-neutral measure has density

$$Z_T = e^{-\frac{\mu}{\sigma}W_T - \frac{\mu^2}{2\sigma^2}T}.$$

In this case,

$$D_k = \frac{\mu^2}{2\rho\sigma^2}(t_k - t_{k-1}).$$

To compute the value function in the case of the entropic penalty, introduce the functions

$$\text{BSPut}(S, K, v) = \mathbb{E}\left[\left(K - Se^{vZ - \frac{v^2}{2}}\right)^+\right], \quad \text{and} \quad \text{LBPut}(S, K) = \mathbb{E}[(K - SZ)^+],$$

where $Z \sim N(0, 1)$ and LB stands for the Louis Bachelier model. Then we obtain for $1 \leq k \leq n - 1$ that $V_{k-1}^{NP} = V_k^{NP}$ if $V_k^{NP} \geq S_k - \rho_k$ and

$$V_{k-1}^{NP} = V_k^{NP} + \text{LBPuT} \left(\frac{|\mu|}{\sigma p} \sqrt{t_k - t_{k-1}}, S_k - V_k^{NP} - \frac{1}{p} \log \tilde{\lambda}_{k-1} - \frac{\mu^2(t_k - t_{k-1})}{2\sigma^2 p} \right)$$

otherwise, where $\tilde{\lambda}_{k-1}$ is the solution to

$$\text{BSPuT} \left(\tilde{\lambda}_{k-1}, e^{-p(V_k^{NP} - S_k)}, \frac{|\mu|}{\sigma} \sqrt{t_k - t_{k-1}} \right) = e^{-p(V_k^{NP} - S_k)} - e^{p\rho k}.$$

The optimal strategy at date $n - 1$ consists of replicating the claim

$$S_n - \rho_n - \frac{\log U_n}{p}.$$

This is achieved by investing a constant amount of $\mu/(\sigma^2 p)$ into the risky asset between t_{n-1} and t_n . The optimal strategy at date $k - 1 < n - 1$ consists of replicating the claim with payoff at date k given by

$$V_k^{NP} + \left(S_k - V_k^{NP} - \frac{\log \tilde{\lambda}_{k-1}}{p} - \frac{\mu(\mu - \sigma^2)}{2p\sigma^2}(t_k - t_{k-1}) + \frac{\mu}{\sigma^2 p} \log \frac{X_{t_k}}{X_{t_{k-1}}} \right)^+.$$

For the power loss function with $p = 2$, at the last date,

$$V_{n-1}^{NP} = S_n - \sigma_n \sqrt{\mathbb{E}^{\mathbb{P}}[U_n^2]},$$

and at previous dates,

$$V_{k-1}^{NP} = V_k^{NP} + \mathbb{E}^{\mathbb{P}}[U_k^2] \text{BSPuT} \left(-\tilde{\lambda}_{k-1}, \frac{S_k - V_k^{NP}}{\mathbb{E}^{\mathbb{P}}[U_k^2]}, \frac{|\mu|}{\sigma} \sqrt{t_k - t_{k-1}} \right)$$

whenever $S_k - V_k^{NP} > \sigma_k$, and $V_{k-1}^{NP} = V_k^{NP}$ otherwise, where $\tilde{\lambda}_{k-1} < 0$ is the solution of

$$\frac{(V_k^{NP} - S_k)^2 - \sigma_k^2}{\mathbb{E}^{\mathbb{P}}[U_k^2]} = \text{BSPuT} \left(\tilde{\lambda}_{k-1}^2, \frac{(V_k^{NP} - S_k)^2}{\mathbb{E}^{\mathbb{P}}[U_k^2]}, 2 \frac{|\mu|}{\sigma} \sqrt{t_k - t_{k-1}} \right).$$

The optimal strategy at date $n - 1$ is to replicate the claim

$$S_n - \frac{\sigma_n}{\mathbb{E}^{\mathbb{P}}[U_n^2]^{\frac{1}{2}}} U_n = S_n - \sigma_n \left(\frac{X_{t_n}}{X_{t_{n-1}}} \right)^{-\frac{\mu}{\sigma^2}} \exp(-\mu(t_n - t_{n-1})/2),$$

which can be achieved with a constant proportion portfolio strategy. On the other hand, at date $k - 1 < n - 1$, one should replicate the claim with the option-like payoff

$$V_k^{NP} + \left(S_k - V_k^{NP} + \tilde{\lambda}_{k-1} \left(\frac{X_{t_k}}{X_{t_{k-1}}} \right)^{-\frac{\mu}{\sigma^2}} \exp\left(\frac{\mu}{2\sigma^2}(\mu - \sigma^2)(t_k - t_{k-1})\right) \right)^+.$$

Incomplete market Now we assume that the agent cannot modify her position between the benchmark dates. This means that the market is incomplete, and the return of the risky asset between date t_{k-1} and t_k is given by

$$R_k = \frac{X_{t_k} - X_{t_{k-1}}}{X_{t_{k-1}}} = e^{(\mu - \frac{\sigma^2}{2})(t_k - t_{k-1}) + \sigma(W_{t_k} - W_{t_{k-1}})} - 1.$$

Let $h = t_k - t_{k-1}$. Then

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\left((1 - \eta R_k)^+ \right)^2 \right] &= 2(1 + \eta) \text{BSPut}(\eta e^{\mu h}, 1 + \eta, \sigma \sqrt{h}) \\ &\quad - \text{BSPut}(\eta^2 e^{2\mu h + \sigma^2 h}, (1 + \eta)^2, 2\sigma \sqrt{h}) \end{aligned}$$

for $\eta > 0$,

$$\mathbb{E}^{\mathbb{P}} \left[\left((1 - \eta R_k)^+ \right)^2 \right] = (1 + \eta)^2 - 2\eta(1 + \eta)e^{\mu h} + \eta^2 e^{2\mu h + \sigma^2 h}$$

for $\eta \in [-1, 0]$, and

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\left((1 - \eta R_k)^+ \right)^2 \right] &= 2(1 + \eta) \text{BSCall}(-\eta e^{\mu h}, -1 - \eta, \sigma \sqrt{h}) \\ &\quad + \text{BSCall}(\eta^2 e^{2\mu h + \sigma^2 h}, (1 + \eta)^2, 2\sigma \sqrt{h}) \end{aligned}$$

for $\eta < -1$.

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