Modelling sovereign risks: from a hybrid model to the generalized density approach*

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Abstract

Motivated by the European sovereign debt crisis, we propose a hybrid sovereign default model which combines an accessible part taking into account the evolution of the sovereign solvency and the impact of critical political events, and a totally inaccessible part for the idiosyncratic credit risk. We obtain closed-form formulas for the probability that the default occurs at critical political dates in a Markovian setting. Moreover, we introduce a generalized density framework for the hybrid default time and deduce the compensator process of default. Finally, we apply the hybrid model and the generalized density to the valuation of sovereign bonds and explain the significant jumps in long-term government bond yields during the sovereign crisis.

Keywords : sovereign risk, sovereign solvency, decomposition of stopping times, generalized density of default, long-term government bond.

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1 Introduction

The European sovereign debt crisis started at the end of 2009, when the long-term interest rates of euro area countries began to diverge significantly. This made it difficult for several member states in the euro area (e.g., Greece, Ireland, Portugal, Cyprus) to refinance their public debts without aid of third parties. The crisis has also led to a growing amount of attention to sovereign risks from governments and financial markets.

Sovereign risk is the possibility that the government of a country defaults on its debt or other obligations. It belongs to the family of credit risks, and is a fundamental component of risks in government bond yield curves. The modelling of sovereign risks is a challenging subject and can differ from that of corporate credit risk. Firstly, a sovereign default is usually influenced by macroeconomic factors such as GDP, public debt, government revenue and expenditure, etc. Secondly, political events and decisions have important effects on the sovereign default, especially for a European Union country.

In studies such as Alogoskoufis (2012), the relevant macroeconomic variables can be summarized by a single one, known as the sovereign solvency, which can easily be measured and monitored. Regarding political decisions, we are notably interested in the impact of critical political events. In practice, we observe that important political events are accompanied by significant increases in the probability of sovereign default. When a sovereign is unable to repay its public debt, it solicits international financial aid as a last resort; if it is not able to receive the financial support, the sovereign can end up in default. We have chosen as examples three critical dates, all of which were related to the financial aid packages for Greece: on May 2, 2010 (T_1), the euro area member states and International Monetary Fund (IMF) agreed on a 110-billion-euro financial aid package for Greece; on July 21, 2011 (T_2), the government heads of the euro area agreed to support a new financial aid program of 109-billion-euro for Greece; on March 8, 2012 (T_3), the European Central Bank (ECB) governing council acknowledged the activation of the buy-back scheme for Greece and decided that debt instruments issued or fully guaranteed by Greece would again be accepted as collateral in European credit operations.

We notice that the critical dates T_1, T_2 and T_3 are predetermined political events which are in general arranged in advance. Moreover, they are publicly known to investors and can be found on the official website of the ECB. These political events have an impact on long-term Greek government bond yields. Figure 1 shows the historical 10-year bond yields of Greek government from 2003 to 2013 (source from Bloomberg), where we observe significant fluctuations during the sovereign crisis. In particular, as illustrated by Figure 2, where the three pictures are respectively extracted from Figure 1 around the three dates T_1, T_2 and T_3, the bond yields reach high levels before and have sudden negative jumps at or slightly after these critical dates.

The macroeconomic models of debt crisis explain such phenomena by the multiplicity of equilibria in debt markets in presence of credit risk. The prevailing equilibrium depends on the expectations of investors about the probability of default (c.f. Calvo (1988)). Before a critical political event, investors expect a sovereign default with high probability, so that the spreads on government bonds become very large, and the debt market is in a large-spread equilibrium. Shortly after the critical political event, the expectations of investors about the sovereign default suddenly decrease and a narrow-spread equilibrium prevails. Therefore, the probability of sovereign default at a critical political event is nonzero, characterized by a jump

Figure 1: Historical 10-year Greek bond yield from 2003 to 2013

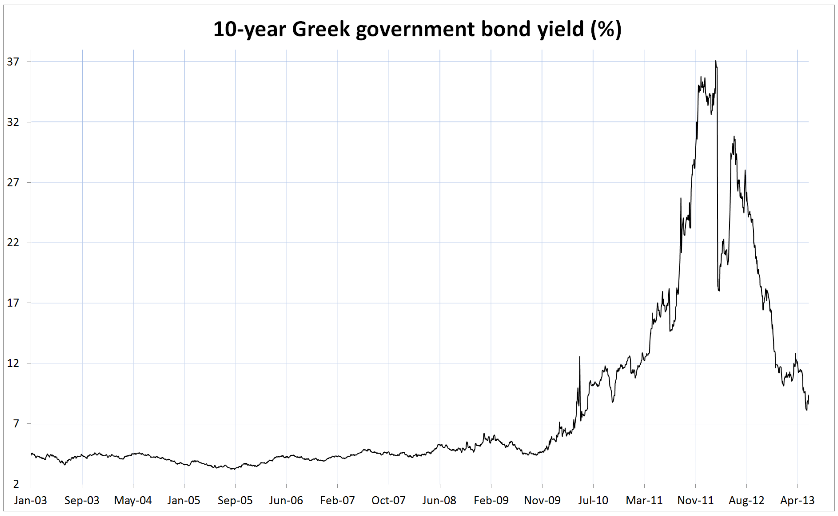
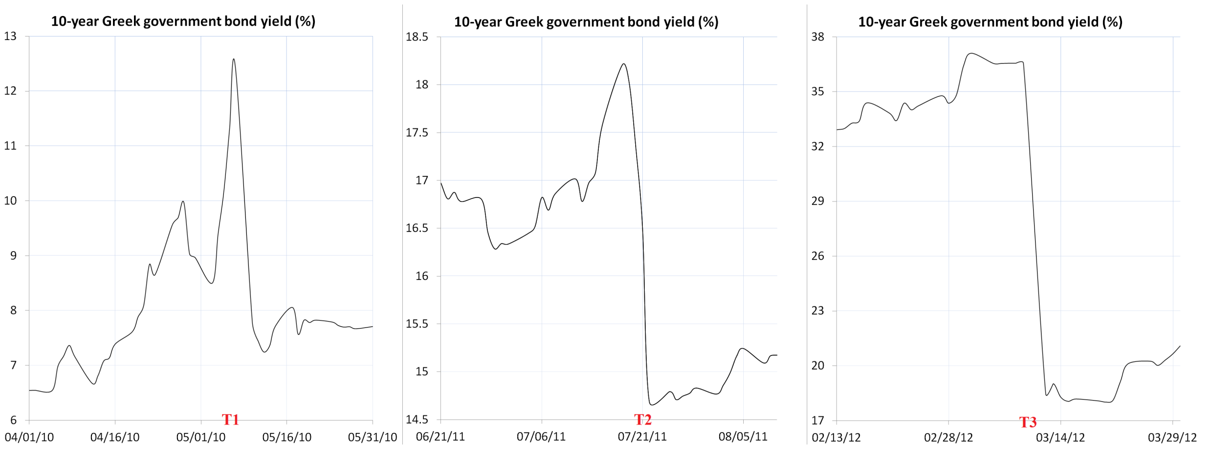


Figure 2: Greek bond yields around critical dates T_1, T_2 and T_3 (extracted from Figure 1)



in government bond yields.

From the mathematical point of view, the nonzero probability of default at a predetermined date means that the default time has a predictable component. In credit risk modelling, two main approaches exist: the structural approach and the reduced-form one. In a standard structural model, the default time is often a predictable stopping time which is defined as the first hitting time of a given default barrier by the asset value process of a firm; while in a reduced-form model, it is usually a totally inaccessible stopping time such as the first jump time of a point process with stochastic intensity. Both approaches have been widely used to model corporate credit risks (see the books of Bielecki and Rutkowski (2002) and Duffie and Singleton (2012) for a detailed description).

In this paper, we propose a hybrid default model based on both approaches of classic credit risk models, which takes into account the sovereign solvency and the impact of critical political events. We are inspired by the jump to default CEV (constant elasticity of variance) models in Carr and Linetsky (2006), Linetsky (2006) and Campi et al. (2009), which have been originally proposed for assessing corporate credit risks. In Carr and Linetsky (2006); Linetsky (2006), the equity value is a CEV diffusion punctuated by a possible jump to zero which corresponds to a default. The default time τ is decomposed into a predictable part, which is the first hitting time of zero by the equity value process, and a totally inaccessible part given by a Cox process. In Campi et al. (2009), the equity value is a CEV process, and the default time is the minimum of the first jump time of a Poisson process and the first absorption time of the equity value process by zero. So the default time can be either predictable according to the CEV process, or totally inaccessible according to the Poisson jump. In the literature, there also exist other hybrid models such as the generalized Cox process model by Bélanger et al. (2004) and the credit migration hybrid model by Chen and Filipović (2005). Recently, Gehmlich and Schmidt (2014) consider models where the Azéma supermartingale of τ contains jumps (so that the default intensity does not exist) and develop the associated term structures. The decomposition of random times also appears in the theory of enlargement of filtrations such as in the papers of Aksamit et al. (2016) and Coculescu (2010).

The hybrid sovereign default model that we propose in this paper also combines the structural and the reduced-form approaches. On one hand, the accessible part of the sovereign default time, which depends on the solvency process and other exogenous macroeconomic factors, describes the impact of political critical events. On the other hand, the totally inaccessible part, which represents the idiosyncratic credit risk, is given by the standard Cox process model. In this model, the probability law of the sovereign default time can have atoms and the political critical events have a significant impact on the default probability. The main contributions of the paper are threefold. First, we obtain closed-form formulas for the probability that the sovereign default occurs at critical dates, which permits the analysis of its political impact. More precisely, when solvency is modelled by a geometric Brownian motion, the default probability is given in terms of the solution of a Sturm-Liouville equation. In the case of a CEV model, we discuss the sign of the elasticity parameter and obtain the default probability by extending CEV ordinary differential equations in Davydov and Linetsky (2001); Linetsky (2004) and using Bessel functions. Second, we present a general framework which extends the default density approach introduced by El Karoui et al. (2010) to study the hybrid default model. In fact, due to the mixed characteristic of random times in the hybrid model, neither the classic intensity hypothesis in the credit risk modelling nor the density hypothesis in El Karoui et al. (2010) are satisfied. We introduce a more general setting with a weaker hypothesis, called the generalized density hypothesis, to describe random times which contain both predictable and totally inaccessible components. Moreover, this setting provides useful theoretical tools to deduce the compensator process of the sovereign default, which is a discontinuous process whose intensity does not exist. Third, we apply the hybrid model and the generalized density to the valuation of sovereign defaultable claims. The bond yields deduced from our model can have large jumps at the critical dates, which explains the significant fluctuations of sovereign bond yields during sovereign debt crises.

The paper is organized as follows. In Section 2, we present the sovereign default model and describe the different components including the sovereign solvency, the political decision impact and the idiosyncratic credit risk. Section 3 concentrates on the computation of conditional default and survival probabilities, and in particular the probability of default at critical dates. In Section 4, we study the sovereign default model in a generalized density framework. We discuss theoretical properties of the random time such as the immersion property and the compensator process. In particular, we show that in this hybrid model, the default intensity does not exist in general. In Section 5, we apply the sovereign default model to the pricing of long-term sovereign bonds and we illustrate the jumps in bond yields in a numerical example.

2 Hybrid model of sovereign default time

In this section, we present a hybrid model of sovereign default which takes into account macroeconomic situations of the country, the impact of political events and the idiosyncratic default intensity.

2.1 Sovereign solvency: a structural model

We start by introducing the notion of solvency. The sovereign default is tightly related to macroeconomic factors of the country. Notably, the sovereign solvency is an essential indicator as it includes several relevant macroeconomic variables which are related to sovereign default. Here, we borrow the definition used in Alogoskoufis (2012).

Definition 2.1 *The sovereign solvency S at time t is defined by*

$$\ln S_t = \pi_t - \frac{d_{t-}(r_t - g_t)}{1 + g_t}, \tag{2.1}$$

where d_{t-} denotes the public debt to GDP ratio of the previous observation date, π_t is the primary surplus to GDP ratio, r_t is the real interest rate on government bonds, and g_t is the GDP growth rate. In particular, we say that the government is fiscally sustainable if $S_t \geq 1$, and is insolvent if $S_t < 1$.

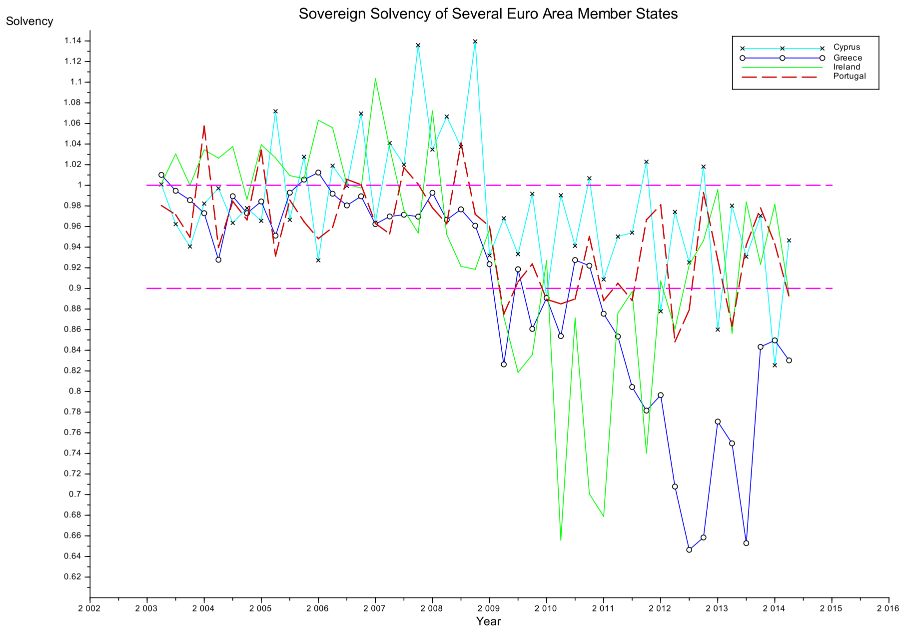
Slightly different from its initial definition, the solvency in Definition 2.1 is given in exponential form such that S_t takes only positive values. By (2.1), four factors determine the solvency. The predetermined historical debt is known from the government's balance sheet of the preceding year. The real interest rate on government bonds, which is the cost of debt refinancing, can be deduced from bond yield curves. The GDP growth rate is observable directly from economic cycles. The primary surplus, which is the measurement of government deficit, can be computed from government revenue and expenditure, as well as the fiscal dynamics. In practice, these data are available for discrete-time observations.

We illustrate in Figure 3 the solvency values computed by using (2.1) for the following four member states of the euro area: Cyprus, Greece, Ireland, and Portugal[1]. We notice that all these countries had been insolvent during the crisis, that is, their solvency values were lower than 1 from 2009, with Greece and Ireland in the worst situation. This observation is rather

[1]The data on interest rates come from the official website of the ECB (sdw.ecb.europa.eu) and on the other factors from the official website of the European Commission (ec.europa.eu/eurostat).

coherent with the reality as Greece and Ireland were the first countries which solicited aid from third parties. Furthermore, the crisis started from the end of 2009 when solvency values of several countries fell below 0.9, so we can consider 0.9 as an approximate threshold for the debt crisis. Indeed, fears about a debt crisis begin to propagate through financial markets when the solvency of a country hits a certain threshold.

Figure 3: Solvency of four countries of the euro area from 2003 to 2014.



On a long-term time scale, we model the sovereign solvency by a continuous-time process. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $W = (W_t, t \geq 0)$ be an $\mathbb{F}$-adapted Brownian motion. For a given country, we assume that the solvency is governed by the process $S = (S_t, t \geq 0)$ which satisfies the following diffusion:

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t), \quad S_0 = x, \tag{2.2}$$

where μ and σ are $\mathbb{F}$-predictable processes such that $\int_0^T |\mu_t| dt + \int_0^T |\sigma_t|^2 dt < \infty$ for any $T > 0$. Let L be a real positive constant such that $L < S_0$, which represents a threshold for the debt crisis. More precisely, if S falls below L, the sovereign is considered to be insolvent. We then define a random time τ_0 as the first hitting time of the barrier L by the solvency process S, i.e.,

$$\tau_0 := \inf\{t \geq 0 : S_t \leq L\}, \tag{2.3}$$

with convention $\inf \emptyset = \infty$. Note that τ_0 is a predictable $\mathbb{F}$-stopping time.

2.2 Critical political event

In practice, when a sovereign becomes fiscally vulnerable, a political meeting will be organized where a multitude of political decisions need to be made concerning the relevant sovereign country. In this paper, we propose a simplified model to describe the political impact, while real situations tend to be more complicated and less transparent. For the concerned sovereign, the meeting date is a critical date and often comes shortly after the solvency barrier hitting time τ_0. In our model, we assume that the critical date coincides with τ_0. We also assume that the outcome of political decisions depends on some exogenous factor, such as economic or financial shocks: if a shock has occurred before the critical date, then the sovereign can possibly end up in default at τ_0; otherwise, it receives a financial aid package which prevents an immediate default at τ_0. The financial aid package is perceived as an assistance from third parties with the aim of improving the solvency of the country in crisis, including a bailout loan, a quantitative easing policy, etc. From an economic point of view, when solvency falls below the threshold, an exogenous shock can make things worse so that the aid from third parties can be too costly (for example, austerity policies can harm the economy) and the political process favors a sovereign default.

We model the exogenous shock by the jump of a Poisson process $N = (N_t, t \geq 0)$ with intensity $\lambda^N > 0$, which is independent of the filtration $\mathbb{F}$. Suppose that the result of political decisions depends on the value of N at the critical date τ_0. More precisely, we define

$$\zeta = \begin{cases} \tau_0, & \text{on } \{N_{\tau_0} \geq 1\}, \\ \infty, & \text{on } \{N_{\tau_0} = 0\}. \end{cases} \tag{2.4}$$

The random time ζ takes into account both the sovereign solvency and the political decisions. Obviously, ζ is not an $\mathbb{F}$-stopping time. However, ζ is an honest time (e.g. Barlow (1978)). Indeed, for any $t \geq 0$, $1_{\{\zeta \leq t\}}\zeta = 1_{\{\zeta \leq t\}}\tau_0 = 1_{\{\zeta \leq t\}}(t \wedge \tau_0)$, where $t \wedge \tau_0$ is $\mathcal{F}_t$-measurable.

2.3 Idiosyncratic credit risk: a Cox process model

Besides macroeconomic and political factors, we also consider the default risk related to the idiosyncratic financial situation of the sovereign and adopt the widely-used Cox process model.

Let the idiosyncratic default intensity $\lambda = (\lambda_t, t \geq 0)$ be a positive $\mathbb{F}$-adapted process. In Carr and Linetsky (2006); Linetsky (2006), the default intensity depends on the pre-default equity price process. In our case, λ depends on the solvency S. For example, let $\lambda_t = \lambda(t, S_t)$ where $\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic function. In general, the intensity λ is assumed to be decreasing with respect to the solvency S, which implies that in a healthy situation of solvency, the idiosyncratic default risk is relatively low.

We introduce the default hazard process $\Lambda = (\Lambda_t, t \geq 0)$ as $\Lambda_t = \int_0^t \lambda_s ds$. Let η be an $\mathcal{A}$-measurable random variable which is exponentially distributed with unit parameter and independent of both the filtration $\mathbb{F}$ and the Poisson process N. Define ξ to be the time of default due to idiosyncratic financial risks, given by a Cox process model, i.e.,

$$\xi := \inf \{t \geq 0 : \Lambda_t \geq \eta\}. \tag{2.5}$$

The random time ξ is a totally inaccessible stopping time with respect to the progressive enlargement of the filtration $\mathbb{F}$ by ξ, namely the filtration $\mathbb{F}^\xi = (\mathcal{F}_t^\xi)_{t\geq 0}$, where $\mathcal{F}_t^\xi = \cap_{s>t}(\sigma(\{\xi \leq u\} : u \leq s) \vee \mathcal{F}_s)$.

2.4 Sovereign default time: a hybrid model

We now model the sovereign default by combining the two components: the economic and political influences described by ζ and the idiosyncratic default risk described by ξ. We define the sovereign default time as

$$\tau := \zeta \wedge \xi. \tag{2.6}$$

In this model, the sovereign default can result from either macroeconomic and political events, or its own idiosyncratic risks. The default time τ has a hybrid nature of both structural and reduced-form approaches.

We make some comparisons with the jump to default extended CEV model.

1. Note that the default time τ in our model is not bounded by its predictable component τ_0. In fact, on the set $\{\tau_0 < \xi\} \cap \{N_{\tau_0} = 0\}$, $\tau = \xi > \tau_0$. If the solvency process S follows a CEV model, then the default time τ defined in (2.6) is similar to the jump to default extended CEV model. We will discuss in detail the CEV case in Section 3.2.2 and refer the readers to Davydov and Linetsky (2001); Delbaen and Shirakawa (2002) for backgrounds about CEV processes.

2. If the Poisson intensity of the exogenous shock $\lambda^N \to 0$, then the default never occurs at τ_0, and our model converges to a Cox process model. If $\lambda^N \to \infty$, we have $\tau = \tau_0 \wedge \xi$. In this case, we recover the jump to default extended CEV model.

2.5 Extension to re-adjusted solvency thresholds

If the solvency of the sovereign does not improve, the authorities can become less confident and relax the solvency barrier requirement. In this case, other critical political events can be gradually anticipated. This motivates us to extend the hybrid model to include multiple critical dates where solvency thresholds can be re-adjusted.

Let $n \in \mathbb{N}$, and $L_1, L_2, \ldots, L_n \in \mathbb{R}_+$ such that $S_0 > L_1 > L_2 > \ldots > L_n$, representing different levels of solvency requirement. We define a sequence of solvency barrier hitting times $\{\tau_i\}_{i=1}^n$ as

$$\tau_i = \inf\{t \geq 0 : S_t \leq L_i\}, \quad i \in \{1, \ldots, n\}. \tag{2.7}$$

The sequence $\{\tau_i\}_{i=1}^n$ is increasing because the solvency thresholds are decreasing. When the solvency falls below a certain level L_i, we assume that a critical political event is organized immediately at τ_i. If an exogenous shock has already arrived, then the sovereign can probably default at τ_i and no more critical political events will be planned; if no exogenous shock has arrived before τ_i, the sovereign will obtain financial aid to avoid an immediate default, which raises the possibility of another critical political event when the solvency falls below L_{i+1}, and so on and so forth, until all requirements on solvency are exhausted.

The exogenous factor is modeled by an inhomogeneous Poisson process N with intensity function $\lambda^N(t)$, and we define the random time ζ^* as

$$\zeta^* = \tau_i, \quad \text{ on } \{N_{\tau_{i-1}} = 0\} \cap \{N_{\tau_i} \geq 1\}, \quad i \in \{1, \ldots, n+1\} \tag{2.8}$$

with convention $\tau_0 = 0$ and $\tau_{n+1} = \infty$. Note that for $t \geq 0$, one has

$$1_{\{\zeta^* > t\}} = \sum_{i=1}^{n+1} 1_{\{\tau_i > t\}}(1_{\{N_{\tau_{i-1}}=0\}} - 1_{\{N_{\tau_i}=0\}}) = \sum_{i=1}^{n+1} 1_{\{\tau_{i-1} \leq t < \tau_i\}} 1_{\{N_{\tau_{i-1}}=0\}}. \tag{2.9}$$

We have $\mathbb{P}(\cup_{i=1}^n \{\omega : \tau_i(\omega) = \zeta^*(\omega) < \infty\}) = \mathbb{P}(\zeta^* < \infty)$, then ζ^* is an accessible stopping time (c.f. Protter (2005)[Chapter III.2]).

Similar to the case of a single critical date, the sovereign can be hit either by successive solvency downgrades or by idiosyncratic credit risks. The sovereign default time is defined as

$$\tau = \zeta^* \wedge \xi, \tag{2.10}$$

where ξ is still given by (2.5). In this case, the default time τ is decomposed into an accessible part which has n predictable components and a totally inaccessible part.

3 Probability of default on multiple critical dates

In this section, we focus on the computation of probabilities that the sovereign default occurs at political critical dates. As we show, such default probabilities are nonzero in the hybrid model, which means that the probability law of default has atoms.

3.1 Conditional default and survival probability

We consider the sovereign default given by the hybrid model (2.10). For any $i \in \{1, \ldots, n\}$, let the $\mathcal{F}_t$-conditional probability that the sovereign default τ coincides with τ_i be denoted by $p_t^i := \mathbb{P}(\tau = \tau_i | \mathcal{F}_t)$, $t \geq 0$.

Proposition 3.1 *The process $(p_t^i, t \geq 0)$ is a stopped $\mathbb{F}$-martingale at τ_i and is given by*

$$p_t^i = \mathbb{E}\Big[\big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\big)e^{-\Lambda_{\tau_i}} | \mathcal{F}_t\Big]. \tag{3.1}$$

Proof: The event $\{\tau = \tau_i\}$ equals $\{\tau_i \leq \xi, N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1\}$. As τ_i is an $\mathbb{F}$-stopping time, the Poisson process N and the random variable η are mutually independent and in addition independent of $\mathbb{F}$, one has

$$\begin{aligned}
\mathbb{P}(\tau = \tau_i | \mathcal{F}_\infty) &= \mathbb{P}(\tau_i \leq \xi | \mathcal{F}_\infty)\mathbb{P}(N_{\tau_{i-1}} = 0, N_{\tau_i} \geq 1 | \mathcal{F}_\infty) \\
&= \mathbb{P}(\Lambda_{\tau_i} \leq \eta | \mathcal{F}_\infty)\big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\big)
\end{aligned} \tag{3.2}$$

which implies (3.1) and the fact that p_t^i is stopped at τ_i. □

The conditional probability process p_t^i is stopped at τ_i, which means that the impact of a political decision is neutralized after the time τ_i. In particular, we have

$$\mathbb{P}(\tau = \tau_i) = p_0^i = \mathbb{E}\big[\big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\big)e^{-\Lambda_{\tau_i}}\big]. \tag{3.3}$$

We now compute the conditional survival probability of the sovereign and study the immersion property.

Proposition 3.2 *For all $u, t \in \mathbb{R}_+$, the $\mathbb{F}$-conditional survival probability is given by*

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{E}\big[\exp\big(-\sum_{i=1}^{n} 1_{\{\tau_i \leq u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \Lambda_u\big)\big|\mathcal{F}_t\big]. \tag{3.4}$$

Proof: For all $u, t \in \mathbb{R}_+$, by (2.9), one has

$$\begin{aligned}
\mathbb{P}(\tau > u|\mathcal{F}_t) &= \mathbb{P}(\zeta^* > u, \xi > u|\mathcal{F}_t) \\
&= \mathbb{E}\big[\big(\sum_{i=1}^{n+1} 1_{\{\tau_{i-1} \leq u < \tau_i\}} 1_{\{N_{\tau_{i-1}} = 0\}}\big) 1_{\{\xi > u\}}\Big|\mathcal{F}_t\big].
\end{aligned}$$

If $u \leq t$, then

$$\begin{aligned}
\mathbb{P}(\tau > u|\mathcal{F}_t) &= \sum_{i=1}^{n+1} 1_{\{\tau_{i-1} \leq u < \tau_i\}} \mathbb{E}\big[1_{\{N_{\tau_{i-1}} = 0\}} 1_{\{\xi > u\}}\big|\mathcal{F}_t\big] \\
&= \sum_{i=1}^{n+1} 1_{\{\tau_{i-1} \leq u < \tau_i\}} \mathbb{E}\big[1_{\{N_{\tau_{i-1}} = 0\}} 1_{\{\eta > \Lambda_u\}}\big|\mathcal{F}_t\big] \\
&= \big(\sum_{i=1}^{n+1} 1_{\{\tau_{i-1} \leq u < \tau_i\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds}\big) e^{-\Lambda_u} \\
&= \exp\big(-\sum_{i=1}^{n+1} 1_{\{\tau_{i-1} \leq u < \tau_i\}} \int_0^{\tau_{i-1}} \lambda^N(s)ds\big) e^{-\Lambda_u} \\
&= \exp\big(-\sum_{i=1}^{n} 1_{\{\tau_i \leq u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds - \Lambda_u\big).
\end{aligned}$$

If $u > t$, we calculate $\mathbb{P}(\tau > u|\mathcal{F}_t)$ as the $\mathcal{F}_t$-conditional expectation of $\mathbb{P}(\tau > u|\mathcal{F}_u)$, which implies (3.4). □

Let the global information structure be given as usual by the progressive enlargement of the filtration $\mathbb{F}$ by the sovereign default time τ, that is,

$$\mathcal{G}_t = \bigcap_{s > t} \big(\sigma(\{\tau \leq u\} : u \leq s) \vee \mathcal{F}_s\big), \quad t \geq 0.$$

Then the couple $(\mathbb{F}, \mathbb{G})$ satisfies the immersion property, that is, any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale. Indeed, by Proposition 3.2, when $u \leq t$, the $\mathbb{F}$-conditional probability does not depend on t, i.e.,

$$\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{P}(\tau > u|\mathcal{F}_u), \quad u \leq t.$$

This last equality is equivalent to the immersion property (see Elliott et al. (2000)). The following result is a direct consequence of Elliott et al. (2000)[Lemma 3.1] and Proposition 3.2.

Corollary 3.3 *For all $t, T \in \mathbb{R}_+$ such that $t \le T$, the $\mathbb{G}$-conditional survival probability is given by*

$$\mathbb{P}(\tau > T|\mathcal{G}_t) = 1_{\{\tau>t\}}\mathbb{E}\big[\exp\big(-\sum_{i=1}^{n} 1_{\{t<\tau_i\le T\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds - \int_t^T \lambda_s\,ds\big)\big|\mathcal{F}_t\big]. \qquad (3.5)$$

3.2 Default probability in a Markovian setting

The general form of sovereign default probabilities at critical dates p_t^i is given by Proposition 3.1. We now consider several specific settings where the solvency process is a geometric Brownian motion or a CEV process.

We first make simplifying assumptions. We suppose that the equation (2.2) is homogeneous and that the solvency process is given by

$$dS_t = S_t\big(\mu(S_t)dt + \sigma(S_t)dW_t\big), \quad S_0 = x$$

where $\mu(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ and $\sigma(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy regular enough conditions to guarantee the existence and uniqueness of a strong solution $(S_t^x, t \ge 0)$ (see e.g. Revuz and Yor (1999)[Theorem 3.5] for details). Let $\mathcal{L}$ denote the generator of S, i.e., for any function $f \in C^2 : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathcal{L}f(z) = z\mu(z)f'(z) + \frac{z^2}{2}\sigma^2(z)f''(z).$$

Furthermore, we specify the idiosyncratic default intensity process $\lambda = (\lambda_t, t \ge 0)$ as a decreasing function of the solvency, i.e., $\lambda_t = \lambda(S_t)$ with $\lambda(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ being decreasing. When $S \to \infty$, the sovereign has almost no chance to default, hence λ remains bounded. When $S \to 0$, the solvency is in an unfavorable situation and may trigger a default, so that the idiosyncratic default intensity can explode in this case. Suppose in addition that the intensity of the exogenous shock is constant, that is, in the inhomogeneous Poisson process, the intensity function is $\lambda^N(t) = \lambda^N > 0$ for any $t \ge 0$.

We consider the Laplace transform for the $\mathbb{F}$-stopping time

$$\rho_x = \inf\{t \ge 0 : S_t^x \le L\} \quad \text{with} \quad S_0^x = x.$$

For any $k \ge 0$, let

$$Q(x; k, L) := \mathbb{E}\big[\exp\big(-k\rho_x - \int_0^{\rho_x}\lambda(S_s^x)\,ds\big)\big] \qquad (3.6)$$

We can prove that (3.6) is the solution to the following Dirichlet problem

$$\begin{aligned} \mathcal{L}u(z) - \big(\lambda(z) + k\big)u(z) = 0 \quad \text{on } \{z > L\} \\ u(L) = 1. \end{aligned} \qquad (3.7)$$

Indeed, as ρ_x is a predictable stopping time, there exists an increasing sequence of stopping times $(\rho_m)_{m\ge1}$ such that $\rho_m < \rho_x$ and $\lim_{m\to\infty}\rho_m = \rho_x$, $\mathbb{P}$-a.s.. Let $\beta_t^x = \exp(-\int_0^t(k + \lambda(S_s^x))ds)$ for any $t \ge 0$. Then on the set $\{t < \rho_x\}$, one has

$$d(\beta_t^x u(S_t^x)) = \beta_t^x u'(S_t^x)\sigma(S_t^x)S_t^x dW_t$$

where u is a solution to the Dirichlet problem (3.7). We then have

$$\mathbb{E}[\beta^x_{\rho_m} u(S^x_{\rho_m})] - u(x) = \mathbb{E}\big[\int_0^{\rho_m} \beta^x_s u'(S^x_s)\sigma(S^x_s)S^x_s dW_s\big],$$

where the right-hand side vanishes thanks to the boundedness of β and the smoothness of u. Thus, when m tends to infinity, $u(x) = \mathbb{E}[\beta^x_{\rho_x} u(L)] = \mathbb{E}[\exp(-\int_0^{\rho_x}(\lambda(S^x_s)+k)ds)]$. We refer to Karatzas and Shreve (2002)[Chapter 5, Proposition 7.2] and Touzi (2013)[Theorem 2.8] for a general representation of this kind of Dirichlet problem.

Proposition 3.4 *For any $i \in \{1,\dots,n\}$, the $\mathbb{F}$-martingale $(p^i_t, t\geq 0)$ is computed as*

$$p^i_t = e^{-\int_0^{t\wedge\tau_i}\lambda(S_u)du} Q(S_{t\wedge\tau_{i-1}};\lambda^N, L_{i-1}) \\ \cdot\big[e^{-\lambda^N(t\wedge\tau_{i-1})}Q(S_{(t\wedge\tau_i)\vee\tau_{i-1}};0,L_i) - e^{-\lambda^N(t\wedge\tau_i)}Q(S_{(t\wedge\tau_i)\vee\tau_{i-1}};\lambda^N,L_i)\big], \quad t\geq 0,$$

with $L_0 = S_0 = x$.

Proof: By Proposition 3.1 and the section assumptions, we have for any $i \in \{1,\dots,n\}$ and $t \geq 0$ that

$$p^i_t = \mathbb{E}\big[(e^{-\lambda^N\tau_{i-1}} - e^{-\lambda^N\tau_i})e^{-\int_0^{\tau_i}\lambda(S_u)du}\big|\mathcal{F}_t\big].$$

As p^i is a martingale stopped at τ_i, it suffices to compute p^i_t for $t\leq\tau_i$. On the set $\{\tau_{i-1}\leq t<\tau_i\}$, by the Markovian property of the process S, we obtain

$$\begin{aligned} p^i_t &= e^{-\lambda^N\tau_{i-1}}\mathbb{E}\big[e^{-\int_0^{\tau_i}\lambda(S_u)du}\big|\mathcal{F}_t\big] - \mathbb{E}\big[e^{-\lambda^N\tau_i-\int_0^{\tau_i}\lambda(S_u)du}\big|\mathcal{F}_t\big] \\ &= e^{-\int_0^t\lambda(S_u)du}\big[e^{-\lambda^N\tau_{i-1}}Q(S_t;0,L_i) - e^{-\lambda^N t}Q(S_t;\lambda^N,L_i)\big]. \end{aligned}$$

In particular,

$$p^i_{\tau_{i-1}} = e^{-\lambda^N\tau_{i-1}-\int_0^{\tau_{i-1}}\lambda(S_u)du}\left[Q(L_{i-1};0,L_i) - Q(L_{i-1};\lambda^N,L_i)\right],$$

which yields that on the set $\{t<\tau_{i-1}\}$,

$$p^i_t = e^{-\lambda^N t-\int_0^t\lambda(S_u)du}Q(S_t;\lambda^N,L_{i-1})\left[Q(L_{i-1};0,L_i) - Q(L_{i-1};\lambda^N,L_i)\right].$$

Finally, we note that $Q(S_{\tau_{i-1}};k,L_{i-1}) = Q(L_{i-1};k,L_{i-1}) = 1$ for any k, which implies the proposition. □

The above proposition shows that it is essential to calculate the quantity $Q(x;k,L)$ to obtain explicit forms of atom probabilities. We present below two cases.

3.2.1 Case of geometric Brownian motion

Let the solvency process S be a geometric Brownian motion which is the solution to the SDE $dS_t = S_t(\mu dt + \sigma dW_t)$, $t\geq 0$, where W is a standard Brownian motion, $\mu,\sigma\in\mathbb{R}$ with $\sigma>0$ and $S_0 = x$. Similar as in Carr and Linetsky (2006), we suppose that the idiosyncratic default intensity λ is a decreasing function of the solvency S as

$$\lambda_t = \lambda(S_t) = \frac{a}{S_t^{2\beta}} + b \tag{3.8}$$

where $a > 0$, $b, \beta \geq 0$ represent respectively the scale parameter governing the sensitivity of λ to S, the constant lower bound and the elasticity parameter. Then we have by (3.7) that $u(x) = Q(x; k, L)$ is the solution to the following Sturm-Liouville equation (see Everitt (2005)):

$$\begin{aligned} \frac{1}{2}\sigma^2 x^2 u''(x) + \mu x u'(x) - (a x^{-2\beta} + b + k) u(x) = 0 \quad \text{on } (L, +\infty); \\ u(L) = 1. \end{aligned} \tag{3.9}$$

The computation is similar to that in Linetsky (2006). The difference between the two cases is that in Linetsky (2006), S represents a traded stock price under risk-neutral probability, so that the default intensity is added in the drift of S to compensate for the jump to default.

If $\beta = 0$, let $\hat{k} = a + b + k$. We can compute $\mathbb{E}[e^{-\hat{k}\rho_x}]$ directly by noticing that $(\exp(-\sqrt{2\hat{k}}W_t - \hat{k}t), t \geq 0)$ is a martingale. Then the optional sampling theorem yields (see Borodin and Salminen (2002)[Part II, Chapter 9, 2.0.1])

$$Q(x; k, L) = \left(\frac{L}{x}\right)^{\sqrt{\nu^2 + 2\hat{k}/\sigma^2} + \nu}$$

where $\nu = \mu/\sigma^2 - 1/2$.

If $\beta > 0$, we let $w(z) = C u(z^{-\frac{1}{\beta}}) z^{-\frac{\nu}{\beta}}$, where $C = w(L^{-\beta}) L^{-\nu}$. Then, w satisfies the following Bessel equation in a modified form (c.f. Everitt (2005)[Chapter 17]):

$$(z w'(z))' - \frac{1}{\beta^2}\left(\nu^2 + 2(k+b)/\sigma^2\right) z^{-1} w'(z) = \frac{2az}{\beta^2\sigma^2} w(z). \tag{3.10}$$

Let $\psi = \frac{1}{\beta}\sqrt{\nu^2 + 2(k+b)/\sigma^2}$, then the above equation admits two basic solutions $I_\psi(z\sqrt{2a}/\sigma\beta)$ and $K_\psi(z\sqrt{2a}/\sigma\beta)$, where I and K are modified Bessel functions with the following properties (c.f. Borodin and Salminen (2002)[Appendix 2.4]):

$$(z^{-\psi} I_\psi(z))' = z^{-\psi} I_{\psi+1}(z), \quad (z^{-\psi} K_\psi(z))' = -z^{-\psi} K_{\psi+1}(z),$$

which implies that $z^{-\psi} I_\psi(z\sqrt{2a}/\sigma\beta)$ is increasing and $z^{-\psi} K_\psi(z\sqrt{2a}/\sigma\beta)$ is decreasing. Moreover, $u(z^{-\frac{1}{\beta}}) = (z^{-\psi} w(z)) z^{\psi - \frac{\nu}{\beta}}$. We have, by Kent (1978)[Theorem 3.1] (see also Borodin and Salminen (2002)[Part II, Chapter 9, 2.8.3]), that

$$Q(x; k, L) = \frac{x^{-\nu} w(x^{-\beta})}{L^{-\nu} w(L^{-\beta})} = \left(\frac{L}{x}\right)^{\nu} \frac{I_\psi(\sqrt{2a}/\sigma\beta x^\beta)}{I_\psi(\sqrt{2a}/\sigma\beta L^\beta)}$$

where I is modified Bessel function of the first kind, defined as

$$I_\psi(x) := \sum_{i=0}^{\infty} \frac{(x/2)^{\psi+2i}}{i!\Gamma(\psi + i + 1)}$$

with Γ being the gamma function.

3.2.2 Case of the CEV process

We now consider the CEV case where the volatility is a monotonic function of the solvency. Let the solvency process be a CEV process driven by the following diffusion:

$$dS_t = \mu S_t \, dt + \delta S_t^{\beta+1} dW_t, \qquad S_0 = x, \tag{3.11}$$

where $\beta \in \mathbb{R}$ and $\delta > 0$ are respectively the elasticity parameter and the scale parameter of the volatility. In particular, the process S is a geometric Brownian motion if $\beta = 0$. We distinguish two cases according to the sign of β. For $\beta < 0$, the volatility $\sigma(S) = \delta S^\beta$ is a decreasing function of S. From a financial point of view, lower solvency indicates higher deficit and government borrowings, leading to a lower growth rate, as well as smaller future expenditures to improve the budget. All these add uncertainty to the solvency. For $\beta > 0$, the volatility is an increasing function of S. When the solvency increases, besides higher growth rate, it may imply higher surplus and fiscal revenues that the government is pressured to disburse for social welfare, leading to increased uncertainty about solvency. In other words, the volatility can be either increasing or decreasing with respect to the solvency. We note in addition that when $\beta > 0$, the process S is a strictly local martingale (c.f. Emanuel and MacBeth (1982)) , which describes situations where bubbles may exist in financial markets.

The specification of the idiosyncratic default intensity $\lambda(S)$ also depends on the sign of the parameter β. More precisely, when $\beta > 0$ (respectively $\beta < 0$), $\lambda(S)$ is an affine function of $\frac{1}{\sigma^2(S)}$ (respectively $\sigma^2(S)$), i.e.,

$$\lambda(S) = \frac{a}{S^{2|\beta|}} + b, \quad a > 0, \quad b \geq 0, \quad \beta \in \mathbb{R}. \tag{3.12}$$

Then, $u(x) = Q(x; k, L)$ is the decreasing solution of the following equation:

$$\begin{aligned} \frac{1}{2}\delta^2 x^{2+2\beta} u'' + \mu x u' - (a x^{-2|\beta|} + b + k) u = 0, \quad & \text{on } (L, +\infty); \\ u(L) = 1. & \end{aligned} \tag{3.13}$$

The fundamental solutions to this last equation are different according to the sign of β.

In the following, we consider the two cases separately according to the sign of β. In the literature, the case $\beta < 0$ has been studied for the valuation of path-dependent options in Linetsky (2004) and the jump to default extended CEV model in Mendoza-Arriaga et al. (2010); Mendoza-Arriaga and Linetsky (2011). The case $\beta > 0$ is unusual. We shall use another equation, called CEV ordinary differential equation (ODE), which has been studied in Davydov and Linetsky (2001), where the coefficient of u is a negative constant. We make use of the solutions to the CEV ODE to solve the equation (3.13).

Case $\beta > 0$: We let $v(x) = u(x) e^{\kappa x^{-2\beta}}$, where $\kappa = \frac{1}{2\beta\delta^2}(\sqrt{\mu^2 + 2a\delta^2} - \mu) > 0$. Then, v satisfies the following CEV ODE:

$$\frac{1}{2}\delta^2 x^{2+2\beta} v'' + \sqrt{\mu^2 + 2a\delta^2} x v' - \left(\kappa\beta(2\beta+1)\delta^2 + b + k\right) v = 0. \tag{3.14}$$

The above equation admits a general solution (c.f. Davydov and Linetsky (2001)[Proposition 5]) of the form $v(x) = A v_+(x) + B v_-(x)$ where $A, B \in \mathbb{R}$, and v_+ and v_- are two fundamental solutions that are respectively increasing and decreasing. As $\beta > 0$, the infinity is an entrance boundary. Then, v_+ and v_- satisfy the following boundary conditions

$$\lim_{x\to\infty} v_+(x) = +\infty, \quad \lim_{x\to\infty} v_-(x) > 0.$$

Moreover, u is decreasing and $0 < \lim_{x\to\infty} u(x) < 1$. Then we have $0 < \lim_{x\to\infty} v(x) < +\infty$. Therefore $A = 0$ and $v = Bv_-$. By the condition $u(L) = 1$, we obtain the coefficient $B = e^{\kappa L^{-2\beta}}/v_-(L)$. Note that the solution v_- can be written explicitly in the form

$$v_-(x) = x^{\beta+\frac{1}{2}} e^{\frac{\sqrt{\mu^2+2a\delta^2}}{2\beta\delta^2}x^{-2\beta}} M_{n,m}\Big(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2}x^{-2\beta}\Big),$$

where $n = \frac{1}{2}+\frac{1}{4\beta}-\frac{\kappa\beta(2\beta+1)\delta^2+b+k}{2\beta\sqrt{\mu^2+2a\sigma^2}} = \frac{\mu(2\beta+1)+2b+2k}{4\beta\sqrt{\mu^2+2a\delta^2}}$, $m = \frac{1}{4\beta}$ and $M_{n,m}(z) := z^{m+1/2}e^{-z/2}F_1(m-n+1/2, 2m+1, z)$ is Whittaker function of the first kind with

$$F_1(a,b,z) := 1 + \sum_{j=1}^{\infty} \frac{a(a+1)\dots(a+j-1)z^j}{b(b+1)\dots(b+j-1)j!}$$

being Kummer confluent hypergeometric function of the first kind. This fundamental solution implies that

$$Q(x;k,L) = v(x)e^{-\kappa x^{-2\beta}} = \frac{x^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2}x^{-2\beta}} M_{n,m}\big(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2}x^{-2\beta}\big)}{L^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2}L^{-2\beta}} M_{n,m}\big(\frac{\sqrt{\mu^2+2a\delta^2}}{\beta\delta^2}L^{-2\beta}\big)},$$

which is valid for any $\mu \in \mathbb{R}$.

Case $\beta < 0$: We let $y(z) = u(z^{\frac{1}{\gamma}})z^{\frac{1}{2}-\frac{1}{2\gamma}}$, where

$$\gamma = \begin{cases} \sqrt{1+8a/\delta^2}, & \mu > 0, \\ -\sqrt{1+8a/\delta^2}, & \mu \le 0. \end{cases}$$

If $\gamma < -1$, $y(z)$ is an increasing function on $(0, L^\gamma)$ and $\lim_{z\to 0+} y(z) = 0$; if $\gamma > 1$, $y(z)$ is a decreasing function on $(L^\gamma, +\infty)$ and $\lim_{z\to+\infty} y(z) = 0$. Moreover, the function y satisfies a CEV ODE as follows:

$$\frac{1}{2}\delta^2\gamma^2 z^{2+2\hat{\beta}}y'' + \mu\gamma z y' - \Big(b+k+\frac{\mu\gamma-\mu}{2}\Big)y = 0, \tag{3.15}$$

where $\hat{\beta} = \frac{\beta}{\gamma}$, the sign of which depends on the sign of μ, and we note that $b+k+\frac{\mu\gamma-\mu}{2} > 0$ for any $\mu \in \mathbb{R}$.

Let y_- and y_+ be the two fundamental solutions of the equation (3.15) on $(0,+\infty)$ with y_- decreasing and y_+ increasing. If $\hat{\beta} < 0$ (namely $\mu > 0$ and $\gamma > 1$), then the infinity $+\infty$ is a natural boundary, and one has

$$\lim_{z\to+\infty} y_-(z) = 0 \text{ and } \lim_{z\to+\infty} y_+(z) = +\infty$$

and hence y is proportional to y_-. If $\hat{\beta} > 0$ (namely $\mu \leq 0$ and $\gamma < -1$), then 0 is a natural boundary and one has

$$\lim_{z\to 0+} y_-(z) = +\infty \text{ and } \lim_{z\to 0+} y_+(z) = 0$$

and hence y is proportional to y_+. Therefore, by Davydov and Linetsky (2001)[Proposition 5], when $\mu \neq 0$ there exists a constant $C > 0$ such that

$$y(z) = C z^{\frac{\beta}{\gamma}+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2} z^{-2\beta/\gamma}} W_{n',m'}\Big(-\frac{|\mu|}{\beta\delta^2} z^{-2\beta/\gamma}\Big),$$

where $n' = \mathrm{sgn}(\mu\beta)(\frac{1}{2}+\frac{\gamma}{4\beta}) - \frac{2b+2k+\mu\gamma-\mu}{4|\mu\beta|} = \frac{2b+2k-\mu(2\beta+1)}{4|\mu|\beta}$, $m' = -\frac{|\gamma|}{4\beta} = -\frac{\sqrt{1+8a/\delta^2}}{4\beta}$ and the function $W_{n,m}(x) := x^{m+1/2}e^{-x/2}F_2(m-n+1/2, 2m+1, x)$ is Whittaker function of the second kind with

$$F_2(a,b,x) := \frac{\Gamma(1-b)}{\Gamma(1+a-b)} F_1(a,b,x) + \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} F_1(1+a-b, 2-b, x)$$

being Kummer confluent hypergeometric function of the second kind. One can compute the constant C by using the relation $y(L^\gamma) = L^{\gamma/2-1/2}$. If $\mu = 0$ and thus $\gamma < -1$ and $\hat{\beta} > 0$, then

$$y(z) = C y_+(z) = C\sqrt{z} K_{2m'}\Big(-\frac{z^{-\beta/\gamma}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\Big),$$

where $K_\psi(x)$ is modified Bessel function of the second kind, defined as

$$K_\psi(x) := \frac{\pi}{2\sin(\psi\pi)}\left(I_{-\psi}(x) - I_\psi(x)\right).$$

Therefore we obtain

$$Q(x;k,L) = y(x^\gamma)x^{\frac{1}{2}-\frac{\gamma}{2}} = \begin{cases} \dfrac{x^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2}x^{-2\beta}} W_{n',m'}\Big(-\frac{|\mu|}{\beta\delta^2}x^{-2\beta}\Big)}{L^{\beta+\frac{1}{2}} e^{\frac{\mu}{2\beta\delta^2}L^{-2\beta}} W_{n',m'}\Big(-\frac{|\mu|}{\beta\delta^2}L^{-2\beta}\Big)}, & \mu \neq 0, \\ \dfrac{\sqrt{x}K_{2m'}\Big(-\frac{x^{-\beta}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\Big)}{\sqrt{L}K_{2m'}\Big(-\frac{L^{-\beta}}{\delta\beta}\sqrt{2b+2k+\mu\gamma-\mu}\Big)}, & \mu = 0. \end{cases}$$

4 Generalized density framework

In this section, we determine the default compensator process for the hybrid sovereign default model that we developed in the previous section. For this purpose, we present a generalized density framework which provides a suitable theoretical setting for hybrid random times and valuation of defaultable claims. In the literature, the default density approach has been proposed in El Karoui et al. (2010) to study the impact of default events where the key hypothesis is the existence of the conditional density of default time τ with respect to the reference filtration $\mathbb{F}$. Under the density hypothesis, the default time is totally inaccessible. The sovereign default time we consider in (2.10) contains both accessible and totally inaccessible parts. Hence the density hypothesis does not hold and we need a more general setting. The generalized density framework permits a larger family of hybrid random times. In particular, we will show that the default compensator process in the hybrid model is in general discontinuous and the default intensity does not necessarily exist.

4.1 Generalized density hypothesis

We first introduce the following assumption, called the generalized density hypothesis, which implies that, when avoiding a family of $\mathbb{F}$-stopping times, the random time τ admits a conditional density with respect to $\mathbb{F}$.

Assumption 4.1 *We assume that there exist a finite family of $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$ satisfying $\mathbb{P}(\tau_i = \tau_j) = 0$ for all $i \neq j$, $(i, j = 1, \cdots, n)$, together with a family of $\mathbb{F} \otimes \mathcal{B}(\mathbb{R}_+)$-adapted processes $\alpha(\cdot)$ such that*

$$\mathbb{E}\big[h(\tau)\prod_{i=1}^n 1_{\{\tau\neq\tau_i\}}\,\big|\,\mathcal{F}_t\big] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)du \qquad \mathbb{P}\text{-}a.s.$$

for any bounded Borel function h. We call $\alpha(\cdot)$ the generalized $\mathbb{F}$-density of τ.

In the model (2.10), $(\tau_i)_{i=1}^n$ correspond to the successive political critical dates which are predictable $\mathbb{F}$-stopping times. In the general case, they can be $\mathbb{F}$-stopping times with both accessible and inaccessible parts. Without loss of generality, $(\tau_i)_{i=1}^n$ can be a family of strictly increasing $\mathbb{F}$-stopping times, which corresponds in (2.10), to the sequence of hitting times of re-adjusted solvency thresholds. The generalized density approach provides a suitable setting for hybrid default models. In the credit risk literature, the hybrid models such as the generalized Cox process model in Bélanger et al. (2004), the jump to default extended CEV models in Campi et al. (2009); Carr and Linetsky (2006) as well as the credit migration model in Chen and Filipović (2005) satisfy the generalized density hypothesis.

There exists a martingale version of the generalized density such that $\alpha(\theta)$ is a càdlàg $\mathbb{F}$-martingale for any $\theta \in \mathbb{R}_+$ (c.f. Jiao and Li (2015)[Proposition 2.3]). For each $i \in \{1, \cdots, n\}$, denote by $p^i = (p^i_t, t \geq 0)$ a càdlàg version of the $\mathbb{F}$-martingale where $p^i_t = \mathbb{P}(\tau = \tau_i|\mathcal{F}_t)$. Assumption 4.1 implies that, for any $t \geq 0$,

$$\int_{\mathbb{R}_+} \alpha_t(u)du + \sum_{i=1}^n p^i_t = 1, \qquad \mathbb{P}\text{-a.s..}$$

Moreover, for any bounded Borel function h, one has

$$\mathbb{E}[h(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} h(u)\alpha_t(u)du + \sum_{i=1}^n \mathbb{E}[h(\tau_i)p^i_{\tau_i\vee t}|\mathcal{F}_t]. \tag{4.1}$$

In particular, the Azéma supermatingale of the random time τ is given by

$$G_t := \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t(u)du + \sum_{i=1}^n 1_{\{\tau_i>t\}}p^i_t, \quad t \geq 0. \tag{4.2}$$

4.2 Sovereign default model revisited

In this subsection, we revisit the sovereign default model (2.10), which is a special case in the generalized density framework.

Proposition 4.2 *The random time τ defined in* (2.10) *satisfies Assumption 4.1, and the generalized $\mathbb{F}$-density $\alpha(\cdot)$ is given for all $u, t \in \mathbb{R}_+$, on the set $\cap_{i=1}^n \{\tau_i \neq u\}$, by*

$$\alpha_t(u) = \mathbb{E}\big[\lambda_u \exp\big(-\int_0^u \lambda_s\, ds - \sum_{i=1}^n 1_{\{\tau_i<u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds\big)\big|\mathcal{F}_t\big] \tag{4.3}$$

PROOF: For any $w \leq t$, denote $J_t(w) := \mathbb{P}(\tau \leq w|\mathcal{F}_t) = \int_0^w \alpha_t(u)du + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} p_t^i$ where we recall that $p_t^i = \mathbb{P}(\tau = \tau_i|\mathcal{F}_t)$, $i \in \{1, \cdots, n\}$, are given by Proposition 3.1 as

$$p_t^i = \big(e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} - e^{-\int_0^{\tau_i} \lambda^N(s)ds}\big)e^{-\int_0^{\tau_i} \lambda_s\, ds}, \quad \text{on } \{\tau_i \leq t\}, \quad i \in \{1, \dots, n\}.$$

Indeed, for any $w \leq t$,

$$\begin{aligned}
J_t(w) &= \int_0^w \lambda_u e^{-\int_0^u \lambda_s\, ds - \sum_{i=1}^n 1_{\{\tau_i<u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} du + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} p_t^i \\
&= -\int_0^w e^{-\sum_{i=1}^n 1_{\{\tau_i<u\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} d(e^{-\int_0^u \lambda_s\, ds}) + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} p_t^i \\
&= -\int_0^w \big(\sum_{i=1}^{n+1} 1_{\{\tau_{i-1}<u\leq\tau_i\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds}\big) d(e^{-\int_0^u \lambda_s\, ds}) + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} p_t^i \\
&= -\sum_{i=1}^{n+1} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} \int_0^w 1_{\{\tau_{i-1}<u\leq\tau_i\}} d(e^{-\int_0^u \lambda_s\, ds}) + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} p_t^i \\
&= -\sum_{i=1}^{n+1} 1_{\{\tau_{i-1}\leq w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds} \big(e^{-\int_0^{w\wedge\tau_i} \lambda_s\, ds} - e^{-\int_0^{\tau_{i-1}} \lambda_s\, ds}\big) + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} p_t^i
\end{aligned}$$

By rewriting explicitly p_t^i, one has

$$\begin{aligned}
J_t(w) &= 1 - \sum_{i=1}^{n+1} 1_{\{\tau_{i-1}\leq w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds - \int_0^{w\wedge\tau_i} \lambda_s\, ds} + \sum_{i=1}^n 1_{\{\tau_i \leq w\}} e^{-\int_0^{\tau_{i-1}} \lambda^N(s)ds - \int_0^{\tau_i} \lambda_s\, ds} \\
&= 1 - \sum_{i=1}^{n+1} 1_{\{\tau_{i-1}\leq w<\tau_i\}} e^{-\int_0^w \lambda_s\, ds - \int_0^{\tau_{i-1}} \lambda^N(s)ds} \\
&= 1 - e^{-\int_0^w \lambda_s\, ds - \sum_{i=1}^n 1_{\{\tau_i\leq w\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds} = 1 - \mathbb{P}(\tau > w|\mathcal{F}_t) = \mathbb{P}(\tau \leq w|\mathcal{F}_t).
\end{aligned}$$

When $u > t$, $\alpha_t(u) = \mathbb{E}[\alpha_u(u)|\mathcal{F}_t]$ by the martingale property, which finishes the proof. $\square$

Remark 4.3 We note that in the sovereign default model (2.10), the following equalities are satisfied: $\alpha_t(u) = \alpha_u(u)$ for $0 \leq u \leq t$ on $\bigcap_{i=1}^n \{\tau_i \neq u\}$ (see the proposition above) and $p_t^i = p_{\tau_i\wedge t}^i$ for any $i \in \{1, \cdots, n\}$ (see Proposition 3.1). In the generalized density framework, these two equalities imply $\mathbb{P}(\tau > u|\mathcal{F}_t) = \mathbb{P}(\tau > u|\mathcal{F}_u)$ for $u \leq t$ (c.f. Jiao and Li (2015)[Proposition 5.1]) and hence the immersion property holds (see also Proposition 3.2).

4.3 Compensator process

The intensity and compensator processes are important quantities in the reduced-form modelling approach of default. For random times which have both accessible and totally inaccessible parts, the intensity does not exist in general. By adopting the generalized density framework, we deduce the compensator process for a hybrid default time, which is discontinuous. Moreover, we show that the compensator is not necessarily equal to the hazard process of default and we compare these two quantities.

Recall that an increasing càdlàg $\mathbb{F}$-predictable process $\Lambda^{\mathbb{F}}$ is called $\mathbb{F}$-compensator process of a random time τ if the process $(1_{\{\tau\le t\}} - \Lambda^{\mathbb{F}}_{t\wedge\tau}, t\ge 0)$ is a $\mathbb{G}$-martingale. The process $\Lambda^{\mathbb{G}} = (\Lambda^{\mathbb{F}}_{t\wedge\tau}, t\ge 0)$ is called $\mathbb{G}$-compensator of τ. The general method for computing the compensator is given in Jeulin and Yor (1978)[Proposition 2] and Elliott et al. (2000) by using the Doob-Meyer decomposition of the Azéma supermartingale G.

When the immersion property holds, G is the unique solution of the following stochastic differential equation:

$$dG_t = -G_{t^-}d\Lambda^{\mathbb{F}}_t, \quad G_0 = 0.$$

Then one has $G = \mathcal{E}(-\Lambda^{\mathbb{F}})$ where $\mathcal{E}$ denotes the Doléan-Dade exponential. Under the generalized density hypothesis 4.1, in terms of $\alpha(\cdot)$ and $(p^i)_{i=1}^n$, the $\mathbb{F}$-compensator process $\Lambda^{\mathbb{F}}$ of τ is then given as

$$\Lambda^{\mathbb{F}}_t = \int_0^t \frac{\alpha_s(s)ds}{G_{s^-}} + \sum_{i=1}^n 1_{\{\tau_i\le t\}}\frac{p^i_{\tau_i}}{G_{\tau_i^-}}, \quad t\in\mathbb{R}_+. \tag{4.4}$$

In the sovereign default model (2.10), the Azéma supermartingale (4.2) has an explicit form given by Proposition 3.2:

$$G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \exp\big(-\int_0^t \lambda_s\,ds - \sum_{i=1}^n 1_{\{\tau_i\le t\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds\big), \quad t\in\mathbb{R}_+, \tag{4.5}$$

which is a decreasing process due to the immersion property. By consequence, we obtain the compensator of the sovereign default time as

$$\Lambda^{\mathbb{F}}_t = \int_0^t \lambda_s\,ds + \sum_{i=1}^n 1_{\{\tau_i\le t\}}\big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds}\big), \quad t\in\mathbb{R}_+ \tag{4.6}$$

which contains an absolutely continuous part depending on the idiosyncratic default intensity and a discontinuous part depending on the critical dates and the intensity of the inhomogeneous Poisson shock. We underline that the intensity of sovereign default does not exist because of the discontinuity of the compensator process at the $\mathbb{F}$-stopping times $(\tau_i)_{i=1}^n$. Recently, Gehmlich and Schmidt (2014) propose a class of models where the Azéma supermartingale contains an additional stochastic integral containing jumps at predictable stopping times.

We can also compute the hazard process Γ (see Bielecki and Rutkowski (2002)[Chapter 5]) of the sovereign default τ as

$$\Gamma_t = -\ln G_t = \int_0^t \lambda_s\,ds + \sum_{i=1}^n 1_{\{\tau_i\le t\}}\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)ds, \quad t\in\mathbb{R}_+. \tag{4.7}$$

Thus, the processes $\Lambda^{\mathbb{F}}$ and Γ have the following relationship

$$\Lambda_t^{\mathbb{F}} = \Gamma_t^c + \sum_{0<s\le t} \left(1 - e^{-\Delta\Gamma_s}\right), \quad t \in \mathbb{R}_+, \tag{4.8}$$

where Γ^c is the continuous part of Γ and $\Delta\Gamma_t = \Gamma_t - \Gamma_{t-}$. We observe that the absolutely continuous parts of $\Lambda^{\mathbb{F}}$ and Γ are identical and depend on the idiosyncratic default intensity λ. Their jump parts are different and both depend on the solvency (through the political critical dates) and the exogenous shock.

Remark 4.4 It is known (c.f. Bielecki and Rutkowski (2002)[Proposition 6.2.2]) that if the compensator process is continuous, then the hazard process is also continuous and coincides with the compensator. In the generalized density setting, we provide a natural counterexample where the hazard process is not equal to the compensator. We note from (4.8) that in the sovereign default model (2.10), the compensator process $\Lambda^{\mathbb{F}}$ is smaller than the hazard process Γ.

5 Applications to sovereign defaultable claims

In this section, we apply the sovereign default model to financial assets which are subject to sovereign risks such as the government bonds. We are particularly interested in long-term government bond yields during the sovereign crisis and we illustrate the jump behavior of bond yields around political critical dates.

5.1 Sovereign bond and credit spread

We consider a defaultable sovereign zero-coupon bond of maturity T. The recovery payment at default, if the sovereign default τ occurs prior to the maturity, is represented by an $\mathbb{F}$-predictable process $R = (R_t, t \ge 0)$ valued in $[0, 1)$. In a financial market with credit risk, when the immersion property holds, the risk-neutral probability in $\mathbb{F}$ is also a risk-neutral probability in $\mathbb{G}$ (c.f. Coculescu et al. (2012)). Let $\mathbb{Q}$ be a risk-neutral probability and assume that all dynamics of the sovereign default model are given under $\mathbb{Q}$. The generalized density hypothesis remains valid under an equivalent probability change. We denote by $r = (r_t, t \ge 0)$ the default-free interest rate process and by $D(t, T)$ the value at $t < T$ of the zero-coupon bond.

Proposition 5.1 *The value of the defaultable zero-coupon bond is given by*

$$D(t,T) = D^0(t,T) + D^1(t,T), \tag{5.1}$$

where D^0 is the pre-default price related to the payment at maturity, computed as

$$D^0(t,T) = 1_{\{\tau>t\}}\mathbb{E}_{\mathbb{Q}}\Big[\exp\big(-\int_t^T (r_s+\lambda_s)ds - \sum_{i=1}^n 1_{\{t<\tau_i\le T\}}\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)ds\big)\big|\mathcal{F}_t\Big], \tag{5.2}$$

and D^1 is related to the recovery payment, given by

$$D^1(t,T) = \frac{1_{\{\tau>t\}}}{G_t}\,\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T e^{-\int_t^u r_s\,ds} R_u \alpha_u(u)du + \sum_{i=1}^n 1_{\{t<\tau_i\le T\}} e^{-\int_t^{\tau_i} r_s\,ds} R_{\tau_i} p^i_{\tau_i}\big|\mathcal{F}_t\Big], \tag{5.3}$$

where $G_t = \mathbb{Q}(\tau > t|\mathcal{F}_t)$.

Proof: The value of the bond is given by

$$D(t,T) = \mathbb{E}_{\mathbb{Q}}\big[e^{-\int_t^T r_s\,ds} 1_{\{\tau>T\}} | \mathcal{G}_t\big] + \mathbb{E}_{\mathbb{Q}}\big[e^{\int_t^\tau r_s\,ds} 1_{\{t<\tau\le T\}} R_\tau | \mathcal{G}_t\big] =: D^0(t,T) + D^1(t,T).$$

The first term $D^0(t,T)$ is obtained by

$$D^0(t,T) = 1_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big[\frac{G_T}{G_t} e^{-\int_t^T r_s\,ds} | \mathcal{F}_t\Big]$$

together with (4.5), and the second term results from Bielecki and Rutkowski (2002)[Proposition 5.1.1] as

$$\begin{aligned} D^1(t,T) &= \frac{1_{\{\tau>t\}}}{G_t} \mathbb{E}_{\mathbb{Q}}\big[\mathbb{E}_{\mathbb{Q}}\big[1_{\{t<\tau\le T\}} \exp(-\int_t^\tau r_s\,ds) R_\tau | \mathcal{F}_T\big] | \mathcal{F}_t\big] \\ &= \frac{1_{\{\tau>t\}}}{G_t} \mathbb{E}_{\mathbb{Q}}\big[\int_t^T e^{-\int_t^u r_s\,ds} R_u \alpha_T(u) du + \sum_{i=1}^n 1_{\{t<\tau_i\le T\}} e^{-\int_t^{\tau_i} r_s\,ds} R_{\tau_i} p_T^i | \mathcal{F}_t\big]. \end{aligned}$$

We complete the proof by using the equality (4.5) and the following properties: $\alpha_T(u) = \alpha_u(u)$ for $t \le u \le T$ on $\bigcap_{i=1}^n \{\tau_i \ne u\}$ and $p_T^i = p_{\tau_i}^i$ on $\{t < \tau_i \le T\}$ for any $i \in \{1, \cdots, n\}$ (see Remark 4.3). □

We are interested in the bond prices at the political critical dates $(\tau_i)_{i=1}^n$ and in particular the jump behavior. Let

$$\Delta D(t,T) := D(t,T) - D(t-,T), \quad t \le T$$

which is the sum of $\Delta D^0(t,T)$ and $\Delta D^1(t,T)$ that we compute next. In order to determine the jumps of the processes $D^0(t,T)$ and $D^1(t,T)$, we assume that the filtration $\mathbb{F}$ only supports continuous martingales. On one hand,

$$\begin{aligned} D^0(t,T) = 1_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\big[\exp\big(-\int_0^T (r_s+\lambda_s)\,ds - \sum_{i=1}^n 1_{\{\tau_i\le T\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)\,ds\big) \,\big|\, \mathcal{F}_t\big] \\ \cdot \exp\big(\int_0^t (r_s+\lambda_s)\,ds + \sum_{i=1}^n 1_{\{\tau_i\le t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)\,ds\big), \quad t \le T \end{aligned}$$

where the conditional expectation term on the right-hand side is a continuous process on $t \in [0,T]$ due to the above assumption. Hence

$$\Delta D^0(t,T) = D^0(t,T) \sum_{i=1}^n 1_{\{\tau_i=t\}} \big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s)\,ds}\big), \quad \text{on } \{\tau > t\}. \tag{5.4}$$

On the other hand, for the similar reason, we deduce from (5.3) the following formula

$$\begin{aligned} \Delta D^1(t,T) = \Delta(G_t^{-1}) \mathbb{E}_{\mathbb{Q}}\big[\int_t^T e^{-\int_t^u r_s ds} R_u \alpha_T(u) du + \sum_{i=1}^n 1_{\{t<\tau_i\le T\}} e^{-\int_t^{\tau_i} r_s\,ds} R_{\tau_i} p_{\tau_i}^i | \mathcal{F}_t\big] \\ - \frac{1}{G_{t-}} \sum_{i=1}^n 1_{\{\tau_i=t\}} R_{\tau_i} p_{\tau_i}^i, \quad \text{on } \{\tau > t\}. \end{aligned}$$

Moreover, by (4.5),

$$\Delta(G_t^{-1}) = \frac{1}{G_t}\sum_{i=1}^n 1_{\{\tau_i=t\}}\big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\big).$$

We then deduce

$$\Delta D^1(t,T) = D^1(t,T)\sum_{i=1}^n 1_{\{\tau_i=t\}}\big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\big) - \frac{1}{G_{t-}}\sum_{i=1}^n 1_{\{\tau_i=t\}}R_{\tau_i}p^i_{\tau_i} \quad \text{on } \{\tau > t\}, \tag{5.5}$$

which implies, combining with (5.4), that

$$\Delta D(t,T) = \sum_{i=1}^n 1_{\{\tau_i=t\}}D(\tau_i,T)\big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\big) - \sum_{i=1}^n 1_{\{\tau_i=t\}}\frac{1}{G_{\tau_i-}}R_{\tau_i}p^i_{\tau_i} \quad \text{on } \{\tau > t\}.$$

By using the relation

$$\frac{p^i_{\tau_i}}{G_{\tau_i-}} = 1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds},$$

we obtain finally

$$\Delta D(t,T) = \sum_{i=1}^n 1_{\{\tau_i=t\}}\big(D(\tau_i,T) - R_{\tau_i}\big)\big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\big), \tag{5.6}$$

and in particular,

$$\Delta D(\tau_i,T) = \big(D(\tau_i,T) - R_{\tau_i}\big)\big(1 - e^{-\int_{\tau_{i-1}}^{\tau_i}\lambda^N(s)\,ds}\big) \quad \text{on} \quad \{\tau_i \le T\}. \tag{5.7}$$

Let the pre-default yield to maturity of the defaultable bond on $\{t < \tau\}$ be

$$Y^d(t,T) = -\frac{\ln D(t,T)}{T-t}. \tag{5.8}$$

Similarly, the yield to maturity of the corresponding default-free zero-coupon bond is given as

$$Y(t,T) = -\frac{\ln B(t,T)}{T-t}.$$

where $B(t,T) = \mathbb{E}_{\mathbb{Q}}[e^{-\int_t^T r_s\,ds}|\mathcal{F}_t]$ denotes the price at $t \le T$ of the default-free zero-coupon bond of maturity T. Let the pre-default credit spread, $S(t,T)$, be defined as the difference between the two yields to maturity, i.e.,

$$S(t,T) := Y^d(t,T) - Y(t,T) = -\frac{1}{T-t}\ln\frac{D(t,T)}{B(t,T)}.$$

Then,

$$\Delta S(t,T) = S(t,T) - S(t-,T) = -\frac{\Delta \ln D(t,T)}{T-t} = -\frac{1}{T-t}\ln\left(1 + \frac{\Delta D(t,T)}{D(t-,T)}\right) \tag{5.9}$$

which implies by (5.7) that the jump of bond yields at a critical date τ_i is negative if and only if $\Delta D(\tau_i, T)$ is positive. More precisely, $\Delta S(\tau_i, T) < 0$ on $\{\tau_i < T \wedge \tau\}$ if and only if

$$D(\tau_i, T) > R_{\tau_i} \quad \text{a.s..} \tag{5.10}$$

In practice, the recovery rate is often assumed to be of expectation 0.48 according to Moody's service. As the bond price is in general higher than this value, the inequality (5.10) is often satisfied, which means that at critical dates, the credit spread is likely to have negative jumps. In particular, if $R \equiv 0$, one has $D^1 = 0$ and $\Delta \ln D^0(t, T) = \sum_{i=1}^n 1_{\{\tau_i = t\}} \int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds$. Then, we can compute the jump in the credit spread at τ_i as

$$\Delta S(\tau_i, T) = -1_{\{\tau_i < T\}} \frac{\Delta \ln D^0(\tau_i, T)}{T - \tau_i} = -1_{\{\tau_i < T\}} \frac{\int_{\tau_{i-1}}^{\tau_i} \lambda^N(s) ds}{T - \tau_i}. \tag{5.11}$$

We notice that whether the jump of sovereign bond yields at a critical date τ_i is negative depends on the intensity of the exogenous shock, the elapsed time between τ_{i-1} and τ_i (the solvency indirectly), and the value of recovery payment at τ_i. When the recovery payment is small enough, the condition in (5.10) is satisfied. Moreover, if no recovery payment is made, the size of jumps only depends on the solvency and the exogenous shock.

5.2 Numerical illustrations

We now present numerical examples to illustrate the previous results on sovereign default probabilities and defaultable bond yields.

In the first example, we consider three political critical dates and compute the default probability p_0^i at τ_i, $(i = 1, 2, 3)$, given by (3.3). We assume that the solvency process S is modelled by a geometric Brownian motion as in Section 3.2.1, and we use the solvency data of Greece from 2003 to 2013 to estimate the parameters and obtain $S_0 = 1.01$, $\mu = -0.01$ and $\sigma = 0.14$. Let the idiosyncratic default intensity process λ be specified by $\lambda(S) = \frac{a}{S^{2\beta}} + b$ as in (3.8) and the Poisson intensity be a constant λ^N. The solvency barrier is re-adjustable with three values $L_1 = 0.9$, $L_2 = 0.8$ and $L_3 = 0.7$. Figure 4 and Figure 5 plot the probabilities that the sovereign default occurs at τ_1 and respectively at τ_2 and τ_3 as functions of the Poisson intensity λ^N for different parameters a, b and β. We observe that the default probability at τ_1 is an increasing function of λ^N as it is more probable for the exogenous shock to occur when λ^N is larger, in which case the sovereign has higher likelihood of defaulting at the first critical date τ_1 due to an unfavorable political decision. However, when λ^N is large, the default probability at other critical dates τ_2 or τ_3 after τ_1 is reduced because the exogenous shock has more chance to occur before τ_1. As a result, the probabilities of default at τ_2 and τ_3 are increasing functions of λ^N for small λ^N and decreasing for large λ^N. To analyze the parameters of the idiosyncratic intensity process, we set the parameters $a = 0.05$, $b = 0.01$ and $\beta = 1$ and examine the impact of each parameter by also considering the values $a = 0.25$, $b = 0.05$ and $\beta = 4$ respectively. Other things being equal, the default probabilities at τ_1, τ_2 and τ_3 are smaller for bigger a and respectively bigger b because the sovereign is more likely to default due to the idiosyncratic credit risk when $\lambda(S)$ is bigger. The impact of the elasticity parameter β depends on the level of solvency, more precisely, $\lambda(S)$ is decreasing (respectively increasing) when $S \geq 1$ (respectively $S < 1$). Consequently, the default probability on τ_1 (respectively τ_2, τ_3) is smaller for smaller β (respectively bigger β).

Figure 4: Probability of sovereign default at τ_1

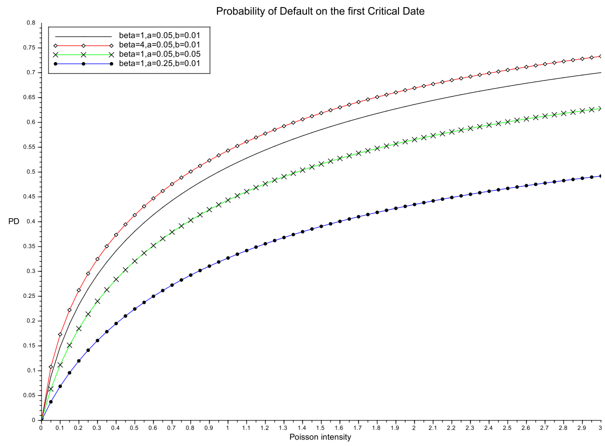
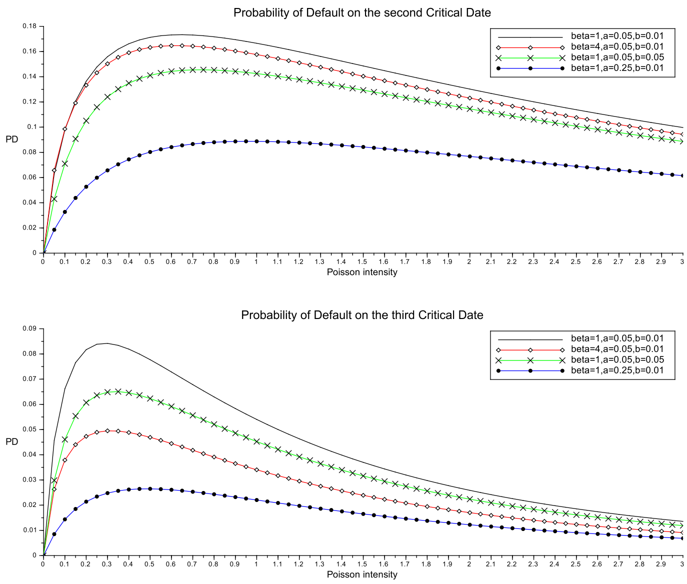


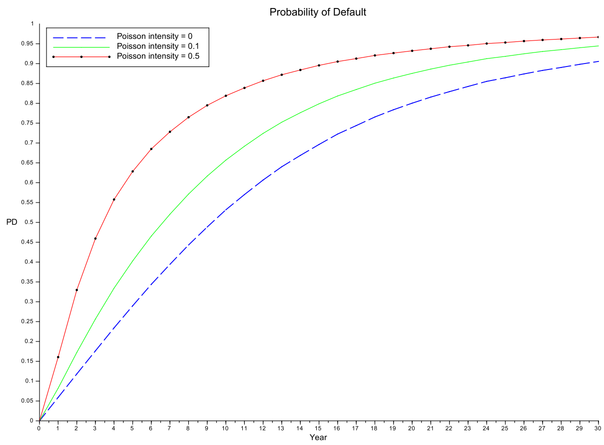
Figure 5: Probability of sovereign default at τ_2 and τ_3 respectively.



In the second example, we consider the sovereign default probability $\mathbb{P}(\tau \leq T)$, computed by Proposition 3.2. The solvency process S is given as a geometric Brownian motion with the same parameters as in the previous example. We fix the values $a = 0.05$, $b = 0.01$ and $\beta = 1$ for idiosyncratic default intensity. Figure 6 plots the default probability with T from 1 to 30 years for different values of the Poisson intensity: $\lambda^N = 0$ (the Cox process model), $\lambda^N = 0.05$ and

0.2 respectively. We note that unsurprisingly, an exogenous shock with larger intensity value increases the sovereign default probability.

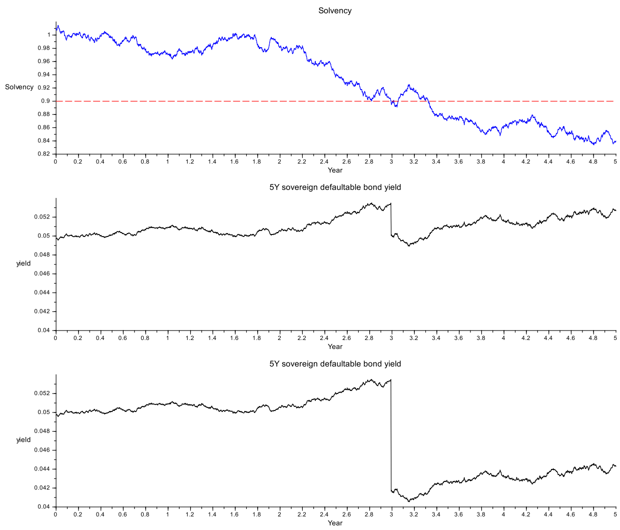
Figure 6: Sovereign default probability.



In the third example, we present the bond yields and in particular the jump at a critical date. The solvency is described by a CEV process as in (3.11) and we set the parameters to be $S_0 = 1.01$, $\mu = -0.01$, $\delta = 0.03$ and $\beta = 1$. The idiosyncratic default intensity process λ is specified by $\lambda(S) = \frac{a}{S^{2|\beta|}} + b$ as in (3.12) with coefficients $a = 0.005$ and $b = 0.01$. We assume that there is only one critical date with the solvency barrier $L = 0.9$ and that the risk-free interest rate is 0. Figure 7 plots bond yields of a defaultable zero-coupon bond of maturity $T = 5$ without recovery payment, as well as the corresponding simulated trajectory of the solvency process. We consider two different exogenous shock intensities: $\lambda^N = 0.05$ and $\lambda^N = 0.2$. We observe in this example that when the solvency hits the threshold 0.9, the bond yield has a negative jump, the size of which depends on the value of λ^N. More precisely, a larger value of exogenous shock intensity λ^N results in a larger jump in bond yields.

In the last example, we consider Greek government bond yields of maturity $T = 10$ years. The solvency of Greece is described by a CEV process. We estimate the parameters by using the solvency data in Figure 3 where δ and β are jointly calibrated (c.f. Chesney et al. (1993) and Yuen et al. (2001)) and obtain $S_0 = 1.01$, $\mu = -0.01$, $\delta = 0.03$ and $\beta = -4.92$. The coefficients of the idiosyncratic default intensity (as in (3.12)) are $a = 0.013$ and $b = 0.035$, estimated from the 3-month Greek bond yields. The solvency barrier is re-adjustable with three values $L_1 = 0.9$, $L_2 = 0.8$ and $L_3 = 0.7$. We suppose that the intensity of the inhomogeneous Poisson process for the exogenous shock is a piecewise constant function which changes its value at each critical date. By Figure 1, given the sizes of the three jumps, we let $\lambda^N(t) = 0.07$ for $t \in [0, \tau_1]$, $\lambda^N(t) = 0.16$ for $t \in (\tau_1, \tau_2]$ and $\lambda^N(t) = 3.15$ for $t \in (\tau_2, \tau_3]$, which are computed using (5.11). Figure 8 plots the bond yields of a 10-year Greek government zero-coupon bond, together with a sample path of the solvency of Greece which corresponds to the period of 2003-2013. We observe that the solvency of Greece tends to fall gradually through time and hits the

Figure 7: Jump at a critical date in sovereign defaultable bond yields (with the corresponding solvency sample path): $\lambda^N = 0.05$ and 0.2 respectively.



three thresholds successively. The bond yields have three negative jumps at the barrier hitting times: in particular, there is a large negative jump when the solvency falls below 0.7 because the exogenous shock intensity is at a high level, while the first two values of exogenous intensity are relatively small. Figure 8 provides an illustration for long-term Greek bond yields by using the hybrid sovereign default model where the three jumps correspond respectively to the critical dates in Figure 2.

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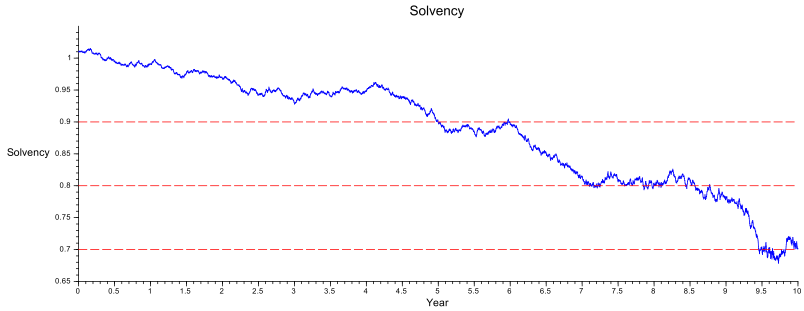
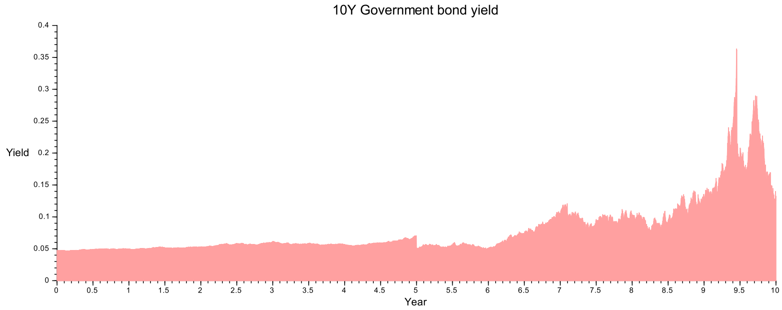
Figure 8: Simulated 10-year Greek government bond yields with re-adjustable Poisson intensities and the corresponding solvency sample path.





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