8 Valuation and VaR Computation for CDOs Using Stein's Method

Nicole El Karoui, Ying Jiao, David Kurtz

8.1 Introduction

8.1.1 A Primer on CDO

Collateralized debt obligations (CDOs) are an innovation in the structured finance market that allow investors to invest in a diversified portfolio of assets at different risk attachment points to the portfolio. The basic concept behind a CDO is the redistribution of risk: some securities backed by a pool of assets in a CDO will be higher rated than the average rating of the portfolio and some will be lower rated.

Generally, CDOs take two forms, cash flow or synthetic. For a cash flow vehicle, investor capital is used directly to purchase the portfolio collateral and the cash generated by the portfolio is used to pay the investors in the CDO. Synthetic CDOs are usually transactions that involve an exchange of cash flow through a credit default swap or a total rate of return swap. The CDO basically sells credit protection on a reference portfolio and receives all cash generated on the portfolio. In these types of transaction, the full capital structure is exchanged and there is no correlation risk for the CDO issuer.

In this study, we are primarily interested in valuing (synthetic) single tranche CDO. It is very important to note that these products are exposed to correlation risk. In practice the CDO issuer sells protection on a portion of the capital structure on a reference portfolio of names. In exchange, he receives a running spread, usually paid quarterly, which value depends on the risk of the individual issuers in the reference portfolio and on a correlation hypothesis between those names. For liquid reference portfolios (indices) like Trac-X and iBoxx there exists now a liquid market for these single tranche CDOs and as a consequence for the correlation.

We now describe mathematically the payoff of a single tranche CDO on a reference portfolio of size n and maturity T. Let τ_i denote the default time of the i^{th} name in the underlying portfolio and N_i be its notional value. The total notional is $N = \sum_{i=1}^{n} N_i$. We use ω_i to represent the weight of the i^{th} name in the portfolio i.e. $\omega_i = N_i/N$. Let R_i be the recovery rate of name i. The cumulative loss process is given by $L_t = \sum_{i=1}^{n} N_i(1-R_i)\mathbf{1}_{\{\tau_i \le t\}}$ and the percentage loss process is

$$l_t = L_t/N = \sum_{i=1}^{n} \omega_i(1-R_i)\mathbf{1}_{\{\tau_i \le t\}}.$$

Usually the capital structure is decomposed in the following way: let us write the interval $(0,1]$ as the unions of the non-overlapping interval $(\alpha_{j-1}, \alpha_j]$ where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1$. The points α_{j-1} and α_j are called, respectively, the attachment and detachment points of the j^{th} tranche. At time t, the loss of the j^{th} tranche is given as a call spread i.e. $l_t^{(j)} = (l_t - \alpha_{j-1})^+ - (l_t - \alpha_j)^+$.

The cash flows of a single tranche CDO are as follows: The protection seller, on one hand, receives at times $\{t_1, \cdots, t_M = T\}$ the coupon $\kappa_j c_{t_u}^j$ $(u = 1, \cdots, M)$ where κ_j is called the spread of the tranche and $c_{t_u}^j = 1 - l_{t_u}^{(j)}/(\alpha_j - \alpha_{j-1})$ is the outstanding notional of the tranche at time t_u. The protection buyer, on the other hand, receives at each default time t that occurs before the maturity the amount $\Delta l_t^{(j)} l_t^{(j)} - l_{t-}^{(j)}$.

From the point of view of pricing, for the j^{th} tranche of the CDO, our objective is to find the value of the spread κ_j. From now on, we shall consider a continuously compounded CDO of maturity T. The value of the default leg and the premium leg are given respectively by the following formulas:

$$\text{Default Leg} = -(\alpha_j - \alpha_{j-1}) \int_0^T B(0,t) q(\alpha_{j-1}, \alpha_j, \mathrm{d}t),$$
$$\text{Premium Leg} = \kappa_j \times (\alpha_j - \alpha_{j-1}) \int_0^T B(0,t) q(\alpha_{j-1}, \alpha_j, t)\mathrm{d}t$$

where $B(0,t)$ is the value at time 0 of a zero coupon maturing at time t assuming deterministic interest rates and $q(\alpha_{j-1}, \alpha_j, t) := \mathsf{E}(c_t^j)$ is the tranche survival probability at time t computed under a given risk-neutral probability. Thanks to the integration by part formula, the fair spread κ_j is then computed as

$$\kappa_j = \frac{1 - B(0,T)q(\alpha_{j-1},\alpha_j,T) + \displaystyle\int_0^T q(\alpha_{j-1},\alpha_j,t)B(0,\mathrm{d}t)}{\displaystyle\int_0^T B(0,t)q(\alpha_{j-1},\alpha_j,t)\mathrm{d}t}. \tag{8.1}$$

To obtain the value of the preceding integrals, the key term to compute is the functions q, which can be expressed as a linear combination of call prices of the form

$$C(t,k) = \mathsf{E}\{(l_t - k)_+\}. \tag{8.2}$$

8.1.2 Factor Models

The main element in computing CDO value is the distribution of the percentage loss l. As mentioned earlier, this distribution depends in a critical manner on the spread (or market implied default probabilities) of the individual names and their correlation as quoted for instance in the liquid tranche market. As a consequence, we need a way to model the correlation between default times of individual names. In practice and in order to obtain tractable results, the market adopts a simplified approach - the factor models.

The main characteristic of the factor models, e.g. see Andersen and Sidenius and Basu (2003), is the conditional independence between the default times $\tau_1, \cdots, \tau_n$. In this framework, the market is supposed to contain some latent factors which impact all concerning firms at the same time. Conditionally on these factors, denoted by U (and we may assume U is uniformly distributed on $(0,1)$ without loss of generality), the default events $E_i = \{\tau_i \le t\}$ are supposed to be independent. To define the correlation structure using the factor framework, it is sufficient to define the conditional default probabilities. In a nutshell, this tantamounts to choose a function F such that $0 \le F \le 1$ and

$$\int_0^1 F(p,u)\mathrm{d}u = p, \quad 0 \le p \le 1.$$

If $p_i = \mathrm{P}(E_i)$, the function $F(p_i,u)$ is to be interpreted as $\mathrm{P}(E_i|U=u)$.

The standard Gaussian copula case with correlation ρ corresponds to the function F defined by

$$F(p,u) = \Phi\left\{\frac{\Phi^{-1}(p) - \sqrt{\rho}\Phi^{-1}(u)}{\sqrt{1-\rho}}\right\}$$

where $\Phi(x)$ is the distribution function of the standard normal distribution $N(0,1)$. Other copula functions, which corresponds to different types of correlation structure, can be used in a similar way. The main drawback of the Gaussian correlation approach is the fact that one cannot find a unique model parameter ρ able to price all the observed market tranches on a given basket. This phenomenon is referred to as *correlation skew* by the market practitioners. One way to take into account this phenomenon is to consider that the correlation ρ is itself dependent on the factor value. See Burtschell, Gregory and Laurent (2007) for a discussion of this topic.

In the factor framework, the conditional cumulative loss l can be written as a sum of independent random variables given U. It is then possible to calculate (8.2) by analytical or numerical methods:

- Firstly, calculate the conditional call value using exact or approximated numerical algorithms,
- secondly, integrate the result over the factor U.

In the sequel, we will explore new methodologies to compute approximations of the conditional call value in an accurate and very quick manner.

8.1.3 Numerical Algorithms

The challenge for the practitioners is to compute quickly prices for their (usually large) books of CDOs in a robust way.

Several methods are proposed to speed up the numerical calculations, such as the recursive method: Hull and White (2004), Brasch (2004), saddle-point method: Martin, Thompson and Browne (2001), Antonov, Mechkov and Misirpashaev (2005) and the Gaussian approximation method: Vasicek (1991). In this paper, we propose a new numerical method which is based on the Stein's method and the zero-bias transformation.

Stein's method is an efficient tool to estimate the approximation errors in the limit theorem problems. We shall combine the Stein's method and the zero bias transformation to propose first-order approximation formulas in both Gauss and Poisson cases. The error estimations of the corrected approximations are obtained. We shall compare our method with other methods numerically. Thanks to the simple closed-form formulas, we reduce largely the computational burden for standard single tranche deals.

In financial problems, the binomial-normal approximation has been studied in different contexts. In particular, Vasicek (1991) has introduced the normal approximation to a homogeneous portfolio of loans. In general, this approximation is of order $\mathcal{O}(1/\sqrt{n})$. The Poisson approximation, less discussed in the financial context, is known to be robust for small probabilities in the approximation of binomial laws. (One usually asserts that the normal approximation remains robust when $np \geq 10$. If np is small, the binomial law approaches a Poisson law.) In our case, the size of the portfolio is fixed for a standard synthetic CDO tranche and $n \approx 125$. In addition, the default probabilities are usually small. Hence we may encounter both cases and it is mandatory to study the convergence speed since n is finite.

The rest of this study is organized as follows: We present in Section 8.2 the theoretical results; Section 8.3 is devoted to numerical tests; finally Section 8.4 explores real life applications, namely, efficient pricing of single tranche CDO and application of this new methodology to VaR computation.

8.2 First Order Gauss-Poisson Approximations

8.2.1 Stein's Method - the Normal Case

Stein's method is an efficient tool to study the approximation problems. In his pioneer paper, Stein (1972) first proposed this method to study the normal approximation in the central limit theorem. The method has been extended to the Poisson case later by Chen (1975).

Generally speaking, the zero bias transformation is characterized by some functional relationship implied by the reference distributions, normal or Poisson, such that the "distance" between one distribution and the reference distribution can be measured by the "distance" between the distribution and its zero biased distribution.

In the framework of Stein's method, the zero bias transformation in the normal case is introduced by Goldstein and Reinert (1997), which provides practical and concise notation for the estimations. In the normal case, the zero biasing is motivated by the following observation of Stein: a random variable Z has the centered normal distribution $N(0, \sigma^2)$ if and only if $\mathsf{E}\{Zf(Z)\} = \sigma^2\,\mathsf{E}\{f'(Z)\}$ for all regular enough functions f. In a more general context, Goldstein and Reinert propose to associate with any random variable X of mean zero and variance $\sigma^2 > 0$ its zero bias transformation random variable X^* if the following relationship (8.3) holds for any function f of C^1-type,

$$\mathsf{E}\{Xf(X)\} = \sigma^2\,\mathsf{E}\{f'(X^*)\}. \tag{8.3}$$

The distribution of X^* is unique with density function given by $p_{X^*}(x) = \sigma^{-2}\,\mathsf{E}(X\mathbf{1}_{\{X>x\}})$.

The centered normal distribution is invariant by the zero bias transformation. In fact, X^* and X have the same distribution if and only if X is a centered Gaussian variable.

We are interested in the error of the normal approximation $\mathsf{E}\{h(X)\} - \mathsf{E}\{h(Z)\}$ where h is some given function and Z is a centered normal r.v. with the same variance σ^2 of X. By *Stein's equation*:

$$xf(x) - \sigma^2 f'(x) = h(x) - \Phi_\sigma(h) \tag{8.4}$$

where $\Phi_\sigma(h) = \mathsf{E}\{h(Z)\}$. We have that

$$\begin{aligned}\mathsf{E}\{h(X)\} - \Phi_\sigma(h) &= \mathsf{E}\{Xf_h(X) - \sigma^2 f_h'(X)\} = \sigma^2\,\mathsf{E}\{f_h'(X^*) - f_h'(X)\} \\ &\le \sigma^2 \|f_h''\|_{\sup}\,\mathsf{E}(|X^* - X|)\end{aligned} \tag{8.5}$$

where f_h is the solution of (8.4). Here the property of the function f_h and the difference between X and X^* are important for the estimations.

The Stein's equation can be solved explicitly. If $h(t)\exp(-\frac{t^2}{2\sigma^2})$ is integrable on $\mathbb{R}$, then one solution of (8.4) is given by

$$f_h(x) = \frac{1}{\sigma^2\phi_\sigma(x)}\int_x^\infty \{h(t) - \Phi_\sigma(h)\}\phi_\sigma(t)dt \tag{8.6}$$

where $\phi_\sigma(x)$ is the density function of $N(0,\sigma^2)$. The function f_h is one order more differentiable than h. Stein has established that $\|f_h''\|_{\sup} \le 2\|h'\|_{\sup}/\sigma^2$ if h is absolutely continuous.

For the term $X - X^*$, the estimations are easy when X and X^* are independent by using a symmetrical term $X^s = X - \widetilde{X}$ where $\widetilde{X}$ is an independent duplicate of X:

$$\mathsf{E}(|X^* - X|) = \frac{1}{4\sigma^2}\,\mathsf{E}\left(|X^s|^3\right), \quad \mathsf{E}(|X^* - X|^k) = \frac{1}{2(k+1)\sigma^2}\,\mathsf{E}\left(|X^s|^{k+2}\right). \tag{8.7}$$

When it concerns dependent random variables, a typical example is the sum of independent random variables. We present here a construction of zero biased variable introduced in Goldstein and Reinert (1997) using a random index to well choose the weight of each summand variable.

Proposition 8.1 *Let X_i $(i = 1, \ldots, n)$ be independent zero-mean r.v. of finite variance $\sigma_i^2 > 0$ and X_i^* having the X_i-zero normal biased distribution.*

We assume that $(\bar{X}, \bar{X}^*) = (X_1, \ldots, X_n, X_1^*, \ldots, X_n^*)$ *are independent r.v. Let* $W = X_1 + \cdots + X_n$ *and denote its variance by* σ_W^2*. Let* $W^{(i)} = W - X_i$ *and* I *be an random index which is independent of* $(\bar{X}, \bar{X}^*)$ *such that* $\mathrm{P}(I = i) = \sigma_i^2/\sigma_W^2$*. Then* $W^* = W^{(I)} + X_I^*$ *has the* W*-zero biased distribution.*

Although W and W^* are dependent, the above construction based on a random index choice enables us to obtain the estimation of $W - W^*$, which is of the same order of $X - X^*$ in the independent case:

$$\mathsf{E}\left(|W^* - W|^k\right) = \frac{1}{2(k+1)\sigma_W^2} \sum_{i=1}^n \mathsf{E}\left(|X_i^s|^{k+2}\right), \quad k \geq 1. \tag{8.8}$$

8.2.2 First-Order Gaussian Approximation

In the classical binomial-normal approximation, as discussed in Vasicek (1991), the expectation of functions of conditional losses can be calculated using a Gaussian expectation. More precisely, the expectation $\mathsf{E}\{h(W)\}$ where W is the sum of conditional independent individual loss variables can be approximated by $\Phi_{\sigma_W}(h)$ where

$$\Phi_{\sigma_W}(h) = \frac{1}{\sqrt{2\pi}\sigma_W} \int_{-\infty}^{\infty} h(u) \exp\Big(-\frac{u^2}{2\sigma_W^2}\Big) \mathrm{d}u$$

and σ_W is the standard deviation of W. The error of this zero-order approximation is of order $\mathcal{O}(1/\sqrt{n})$ by the well-known Berry-Esseen inequality using the Wasserstein distance, e.g. Petrov (1975), Chen and Shao (2005), except in the symmetric case.

We shall improve the approximation quality by finding a correction term such that the corrected error is of order $\mathcal{O}(1/n)$ even in the asymmetric case. Some regularity condition is required on the considered function. Notably, the call function, not possessing second order derivative, is difficult to analyze. In the following theorem, we give the explicit form of the corrector term alongside the order of the approximation.

PROPOSITION 8.1 *Let* $X_1, \ldots, X_n$ *be independent random variables of mean zero such that* $\mathsf{E}(X_i^4)$ $(i = 1, \ldots, n)$ *exists. Let* $W = X_1 + \cdots + X_n$ *and* $\sigma_W^2 = \mathsf{Var}(W)$*. For any function* h *such that* h'' *is bounded, the normal approximation* $\Phi_{\sigma_W}(h)$ *of* $\mathsf{E}\{h(W)\}$ *has the corrector:*

$$C_h = \frac{\mu_{(3)}}{2\sigma_W^4} \Phi_{\sigma_W} \left\{ \Big(\frac{x^2}{3\sigma_W^2} - 1\Big) x h(x) \right\} \tag{8.9}$$

where $\mu_{(3)} = \sum_{i=1}^n \mathsf{E}(X_i^3)$. *The corrected approximation error is bounded by*

$$\begin{aligned}
&\Big| \mathsf{E}\{h(W)\} - \Phi_{\sigma_W}(h) - C_h \Big| \\
&\le \|f_h^{(3)}\|_{\sup} \Big\{ \frac{1}{12} \sum_{i=1}^n \mathsf{E}\left(|X_i^s|^4\right) + \frac{1}{4\sigma_W^2} \Big| \sum_{i=1}^n \mathsf{E}(X_i^3) \Big| \sum_{i=1}^n \mathsf{E}\left(|X_i^s|^3\right) \\
&+ \frac{1}{\sigma_W} \sqrt{\sum_{i=1}^n \sigma_i^6} \Big\}.
\end{aligned}$$

Proof:
By taking first order Taylor expansion, we obtain

$$\begin{aligned}
&\mathsf{E}\{h(W)\} - \Phi_{\sigma_W}(h) = \sigma_W^2 \,\mathsf{E}\{f_h'(W^*) - f_h'(W)\} \\
&= \sigma_W^2 \,\mathsf{E}\{f_h''(W)(W^* - W)\} + \sigma_W^2 \,\mathsf{E}\left[f_h^{(3)}\{\xi W + (1-\xi)W^*\}\xi(W^* - W)^2 \right]
\end{aligned} \tag{8.10}$$

where ξ is a random variable on $[0,1]$ independent of all X_i and X_i^*. First, we notice that the remaining term is bounded by

$$\mathsf{E}\left[\left| f_h^{(3)}\{\xi W + (1-\xi)W^*\} \right| \xi (W^* - W)^2 \right] \le \frac{\|f_h^{(3)}\|_{\sup}}{2} \mathsf{E}\{(W^* - W)^2\}.$$

Then we have

$$\sigma_W^2 \Big| \mathsf{E}\left[f_h^{(3)}\{\xi W + (1-\xi)W^*\}\xi(W^* - W)^2 \right] \Big| \le \frac{\|f_h^{(3)}\|_{\sup}}{12} \sum_{i=1}^n \mathsf{E}\left(|X_i^s|^4\right). \tag{8.11}$$

Secondly, we consider the first term in the right-hand side of (8.10). Since X_I^* is independent of W, we have

$$\begin{aligned}
\mathsf{E}\{f_h''(W)(W^* - W)\} &= \mathsf{E}\{f_h''(W)(X_I^* - X_I)\} \\
&= \mathsf{E}(X_I^*)\,\mathsf{E}\{f_h''(W)\} - \mathsf{E}\{f_h''(W)X_I\}.
\end{aligned} \tag{8.12}$$

For the second term $\mathsf{E}\{f_h''(W)X_I\}$ of (8.12), since

$$\mathsf{E}\{f_h''(W)X_I\} = \mathsf{E}\left\{ f_h''(W)\,\mathsf{E}(X_I | \bar{X}, \bar{X}^*) \right\},$$

we have using the conditional expectation that

$$\Big| \mathsf{E}\{f_h''(W)X_I\} \Big| \le \frac{1}{\sigma_W^2} \sqrt{\mathsf{Var}\{f_h''(W)\}} \sqrt{\sum_{i=1}^n \sigma_i^6}. \tag{8.13}$$

Notice that

$$\mathsf{Var}\{f_h''(W)\} = \mathsf{Var}\{f_h''(W) - f_h''(0)\} \leq \mathsf{E}[\{f_h''(W) - f_h''(0)\}^2] \leq \|f_h^{(3)}\|_{\sup}^2 \sigma_W^2.$$

Therefore

$$\Big| \mathsf{E}\{f_h''(W)X_I\}\Big| \leq \frac{\|f_h^{(3)}\|_{\sup}}{\sigma_W} \sqrt{\sum_{i=1}^n \sigma_i^6}\,.$$

For the first term $\mathsf{E}(X_I^*)\,\mathsf{E}\{f_h''(W)\}$ of (8.12), we write it as the sum of two parts

$$\mathsf{E}(X_I^*)\,\mathsf{E}\{f_h''(W)\} = \mathsf{E}(X_I^*)\Phi_{\sigma_W}(f_h'') + \mathsf{E}(X_I^*)\,\mathsf{E}\{f_h''(W) - \Phi_{\sigma_W}(f_h'')\}.$$

The first part is the candidate for the corrector. We apply the zero order estimation to the second part and get

$$\Big|\mathsf{E}(X_I^*)\Big[\,\mathsf{E}\{f_h''(W)\} - \Phi_{\sigma_W}(f_h'')\Big]\Big| \leq \frac{\big\|f_h^{(3)}\big\|_{\sup}}{4\sigma_W^4}\Big|\sum_{i=1}^n \mathsf{E}(X_i^3)\Big|\sum_{i=1}^n \mathsf{E}\left(|X_i^s|^3\right). \tag{8.14}$$

Then, it suffices to write

$$\begin{aligned}
&\mathsf{E}\{h(W)\} - \Phi_{\sigma_W}(h)\\
&= \sigma_W^2\Big[\,\mathsf{E}(X_I^*)\Phi_{\sigma_W}(f_h'') + \mathsf{E}(X_I^*)\big[\,\mathsf{E}\{f_h''(W)\} - \Phi_{\sigma_W}(f_h'')\big] - \mathsf{E}\{f_h''(W)X_I\}\Big]\\
&+ \sigma_W^2\,\mathsf{E}\left[f_h^{(3)}\{\xi W + (1-\xi)W^*\}\xi(W^* - W)^2\right].
\end{aligned} \tag{8.15}$$

Combining (8.11), (8.13) and (8.14), we let $C_h = \sigma_W^2\,\mathsf{E}(X_I^*)\Phi_{\sigma_W}(f_h'')$ and we deduce the error bound. Finally, we use the invariant property of the normal distribution under zero bias transformation and the Stein's equation to obtain (8.9). □

The corrector is written as the product of two terms: the first one depends on the moments of X_i up to the third order and the second one is a normal expectation of some polynomial function multiplying h. Both terms are simple to calculate, even in the inhomogeneous case.

To adapt to the definition of the zero biasing random variable, and also to obtain a simple representation of the corrector, the variables X_i's are set to be of expectation zero in Theorem 8.1. This condition requires a normalization step when applying the theorem to the conditional loss. A useful example concerns the centered Bernoulli random variables which take two real values and are of expectation zero.

Note that the moments of X_i play an important role here. In the symmetric case, we have $\mu_{(3)} = 0$ and as a consequence $C_h = 0$ for any function h. Therefore, C_h can be viewed as an asymmetric corrector in the sense that, after correction, the approximation realizes the same error order as in the symmetric case.

To precise the order of the corrector, let us consider the normalization of an homogeneous case where X_i's are i.i.d. random variables whose moments may depend on n. Notice that

$$\Phi_{\sigma_W}\left\{\Big(\frac{x^2}{3\sigma_W^2}-1\Big)xh(x)\right\} = \sigma_W\Phi_1\left\{\Big(\frac{x^2}{3}-1\Big)xh(\sigma_W x)\right\}.$$

To ensure that the above expectation term is of constant order, we often suppose that the variance of W is finite and does not depend on n. In this case, we have $\mu_{(3)} \sim \mathcal{O}(1/\sqrt{n})$ and the corrector C_h is also of order $\mathcal{O}(1/\sqrt{n})$. Consider now the percentage default indicator variable $\mathbf{1}_{\{\tau_i\le t\}}/n$, whose conditional variance given the common factor equals to $p(1-p)/n^2$ where p is the conditional default probability of i^{th} credit, identical for all in the homogeneous case. Hence, we shall fix p to be zero order and let $X_i = (\mathbf{1}_{\{\tau_i\le t\}} - p)/\sqrt{n}$. Then σ_W is of constant order as stated above. Finally, for the percentage conditional loss, the corrector is of order $\mathcal{O}(1/n)$ because of the remaining coefficient $1/\sqrt{n}$.

The X_i's are not required to have the same distribution: we can handle easily different recovery rates (as long as they are independent r.v.) by computing the moments of the product variables $(1-R_i)\mathbf{1}_{\{\tau_i\le t\}}$. The corrector depends only on the moments of R_i up to the third order. Note however that the dispersion of the recovery rates, also of the nominal values can have an impact on the order of the corrector.

We now concentrate on the call function $h(x) = (x-k)_+$. The Gauss approximation corrector is given in this case by

$$C_h = \frac{\mu_{(3)}}{6\sigma_W^2}k\phi_{\sigma_W}(k) \tag{8.16}$$

where $\phi_\sigma(x)$ is the density function of the distribution $N(0,\sigma^2)$. When the strike $k=0$, the corrector $C_h = 0$. On the other hand, the function $k\exp\big(-\frac{k^2}{2\sigma_W^2}\big)$ reaches its maximum and minimum values when $k=\sigma_W$ and $k=-\sigma_W$, and then tends to zero quickly.

The numerical computation of this corrector is extremely simple since there is no need to take expectation. Observe however that the call function is a Lipschitz function with $h'(x) = \mathbf{1}_{\{x>k\}}$ and h'' exists only in the distribution sense. Therefore, we can not apply directly Theorem 8.1 and the error estimation deserves a more subtle analysis. The main tool we used to establish the error estimation for the call function is a concentration inequality in Chen and Shao (2001). For detailed proof, interested reader may refer to El Karoui and Jiao (2007).

We shall point out that the regularity of the function h is essential in the above result. For more regular functions, we can establish correction terms of corresponding order. However, for the call function, the second order correction can not bring further improvement to the approximation results in general.

8.2.3 Stein's Method - the Poisson Case

The Poisson case is parallel to the Gaussian one. Recall that Chen (1975) has observed that a non-negative integer-valued random variable Λ of expectation λ follows the Poisson distribution if and only if $\mathsf{E}\{\Lambda g(\Lambda)\} = \lambda\,\mathsf{E}\{g(\Lambda+1)\}$ for any bounded function g. Similar as in the normal case, let us consider a random variable Y taking non-negative integer values and $\mathsf{E}(Y) = \lambda < \infty$. A r.v. Y^* is said to have the *Y-Poisson zero biased distribution* if for any function g such that $\mathsf{E}\{Yg(Y)\}$ exists, we have

$$\mathsf{E}\{Yg(Y)\} = \lambda\,\mathsf{E}\{g(Y^*+1)\}. \tag{8.17}$$

Stein's Poisson equation is also introduced in Chen (1975):

$$yg(y) - \lambda g(y+1) = h(y) - \mathcal{P}_\lambda(h) \tag{8.18}$$

where $\mathcal{P}_\lambda(h) = \mathsf{E}\{h(\Lambda)\}$ with $\Lambda \sim P(\lambda)$. Hence, for any non-negative integer-valued r.v. V with expectation λ_V, we obtain the error of the Poisson approximation

$$\mathsf{E}\{h(V)\} - \mathcal{P}_\lambda(h) = \mathsf{E}\left\{Vg_h(V) - \lambda_V g_h(V+1)\right\} = \lambda_V\,\mathsf{E}\left\{g_h(V^*+1) - g_h(V+1)\right\} \tag{8.19}$$

where g_h is the solution of (8.18) and is given by

$$g_h(k) = \frac{(k-1)!}{\lambda^k}\sum_{i=k}^{\infty}\frac{\lambda^i}{i!}\left\{h(i) - \mathcal{P}_\lambda(h)\right\}. \tag{8.20}$$

It is unique except at $k=0$. However, the value $g(0)$ does not enter into our calculations afterwards.

We consider now the sum of independent random variables. Let $Y_i\,(i = 1, \cdots, n)$ be independent non-negative integer-valued r.v. with positive expectations λ_i and let Y_i^* have the Y_i-Poisson zero biased distribution. Assume that Y_i and Y_i^* are mutually independent. Denote by $V = Y_1 + \cdots + Y_n$ and $\lambda_V = \mathsf{E}(V)$. Let I be a random index independent of $(\bar{Y}, \bar{Y}^*)$ satisfying $P(I = i) = \lambda_i/\lambda_V$. Then $V^{(I)} + Y_I^*$ has the V-Poisson zero biased distribution where $V^{(i)} = V - Y_i$.

For any integer $l \geq 1$, assume that Y and Y_i have to up $(l+1)$-order moments. Then

$$\mathsf{E}(|Y^* - Y|^l) = \frac{1}{\lambda}\,\mathsf{E}\left(Y|Y^s - 1|^l\right),\ \mathsf{E}(|V^* - V|^l) = \frac{1}{\lambda_V}\sum_{i=1}^n \mathsf{E}\left(Y_i|Y_i^s - 1|^l\right).$$

Finally, recall that Chen has established $\|\Delta g_h\|_{\sup} \leq 6\|h\|_{\sup}\min\left(\lambda^{-\frac{1}{2}}, 1\right)$ with which we obtain the following zero order estimation

$$|\,\mathsf{E}\{h(V)\} - \mathcal{P}_{\lambda_V}(h)| \leq 6\|h\|_{\sup}\min\left(\frac{1}{\sqrt{\lambda_V}}, 1\right)\sum_{i=1}^n \mathsf{E}\left(Y_i|Y_i^s - 1|\right). \tag{8.21}$$

There also exist other estimations of error bound (see e.g. Barbour and Eagleson (1983)). However we here are more interested in the order than the constant of the error.

8.2.4 First-Order Poisson Approximation

We now present the first-order Poisson approximation following the same idea as in the normal case. Firstly, recall the zero-order approximation formula. If V is a random variable taking non-negative integers with expectation λ_V, then we may approximate $\mathsf{E}\{h(V)\}$ by a Poisson function

$$\mathcal{P}_{\lambda_V}(h) = \sum_{m=0}^n \frac{\lambda_V^m}{m!}e^{-\lambda_V}h(m).$$

The Poisson approximation is efficient under some conditions, for example, when $V \sim B(n,p)$ and $np < 10$. We shall improve the Poisson approximation by presenting a corrector term as above. We remark that due to the property that a Poisson distributed random variable takes non-negative integer values, the variables Y_i's in Theorem 8.2 are discrete integer random variables.

PROPOSITION 8.2 *Let $Y_1, \ldots, Y_n$ be independent random variables taking non-negative integer values such that $\mathsf{E}(Y_i^3)$ $(i = 1, \ldots, n)$ exist. Let*

$V = Y_1 + \cdots + Y_n$ with expectation $\lambda_V = \mathsf{E}(V)$ and variance $\sigma_V^2 = \mathsf{Var}(V)$. Then, for any bounded function h defined on $\mathbb{N}_+$, the Poisson approximation $\mathcal{P}_{\lambda_V}(h)$ of $\mathsf{E}\{h(V)\}$ has the corrector:

$$C_h^{\mathcal{P}} = \frac{\sigma_V^2 - \lambda_V}{2}\mathcal{P}_{\lambda_V}(\Delta^2 h) \tag{8.22}$$

where $\mathcal{P}_\lambda(h) = \mathsf{E}\{h(\Lambda)\}$ with $\Lambda \sim \mathcal{P}(\lambda)$ and $\Delta h(x) = h(x+1) - h(x)$. The corrected approximation error is bounded by

$$\begin{aligned}
&\left| \mathsf{E}\{h(V)\} - \mathcal{P}_{\lambda_V}(h) - \lambda_V \mathcal{P}_{\lambda_V}\{\Delta g_h(x+1)\}\, \mathsf{E}(Y_I^* - Y_I) \right| \\
&\le 2\|\Delta g_h\|_{\sup} \sum_{i=1}^n \lambda_i\, \mathsf{E}\left\{ |Y_i^* - Y_i| \big(|Y_i^* - Y_i| - 1\big) \right\} \\
&\quad + 6\|\Delta g_h\|_{\sup} \Big\{ \sum_{i=1}^n \mathsf{E}(Y_i |Y_i^s - 1|) \Big\}^2 \\
&\quad + \mathsf{Var}\{\Delta g_h(V+1)\}^{\frac{1}{2}} \Big\{ \sum_{i=1}^n \lambda_i^2 \mathsf{Var}(Y_i^* - Y_i) \Big\}^{\frac{1}{2}}.
\end{aligned}$$

Proof:
Let us first recall the discrete Taylor formula. For any integers x and any positive integer $k \ge 1$,

$$g(x+k) = g(x) + k\Delta g(x) + \sum_{j=0}^{k-1} (k-1-j)\Delta^2 g(x+j).$$

Similar as in the Gaussian case, we apply the above formula to right-hand side of $\mathsf{E}\{h(V)\} - \mathcal{P}_{\lambda_V}(h) = \lambda_V\, \mathsf{E}\{g_h(V^*+1) - g_h(V+1)\}$ and we shall make decompositions. Since $V^* - V$ is not necessarily positive, we take expansion around $V^{(i)}$ for the following three terms respectively and obtain

$$\begin{aligned}
&\mathsf{E}\left\{ g_h(V^*+1) - g_h(V+1) - \Delta g_h(V+1)(V^* - V) \right\} = \sum_{i=1}^n \frac{\lambda_i}{\lambda_V} \cdot \\
&\Big[\mathsf{E}\left\{ g_h(V^{(i)}+1) + Y_i^* \Delta g_h(V^{(i)}+1) + \sum_{j=0}^{Y_i^*-1} (Y_i^* - 1 - j)\Delta^2 g_h(V^{(i)} + 1 + j) \right\} \\
&\quad - \mathsf{E}\left\{ g_h(V^{(i)}+1) + Y_i \Delta g_h(V^{(i)}+1) + \sum_{j=0}^{Y_i-1} (Y_i - 1 - j)\Delta^2 g_h(V^{(i)} + 1 + j) \right\} \\
&\quad - \mathsf{E}\left\{ \Delta g_h(V^{(i)}+1)(Y_i^* - Y_i) + \sum_{j=0}^{Y_i-1} (Y_i^* - Y_i)\Delta^2 g_h(V^{(i)} + 1 + j) \right\} \Big]
\end{aligned}$$

which implies that the remaining term is bounded by

$$\begin{aligned}&\Big|\,\mathsf{E}\left\{g_h(V^*+1)-g_h(V+1)-\Delta g_h(V+1)(V^*-V)\right\}\Big|\\&\le\|\Delta^2 g_h\|_{\sup}\sum_{i=1}^n\frac{\lambda_i}{\lambda_V}\Big[\,\mathsf{E}\Big\{\binom{Y_i^*}{2}+\binom{Y_i}{2}\Big\}+\mathsf{E}\left\{|Y_i(Y_i^*-Y_i)|\right\}\Big].\end{aligned}$$

We then make decomposition

$$\begin{aligned}&\mathsf{E}\left\{\Delta g_h(V+1)(V^*-V)\right\}\\&=\mathcal{P}_{\lambda_V}\{\Delta g_h(x+1)\}\,\mathsf{E}(Y_I^*-Y_I)+\mathrm{Cov}\left\{Y_I^*-Y_I,\Delta g_h(V+1)\right\}\\&\quad+\big[\,\mathsf{E}\{\Delta g_h(V+1)\}-\mathcal{P}_{\lambda_V}\{\Delta g_h(x+1)\}\big]\,\mathsf{E}(Y_I^*-Y_I).\end{aligned}\tag{8.23}$$

Similar as in the Gaussian case, the first term of (8.23) is the candidate of the corrector. For the second term, we use again the technique of conditional expectation and obtain

$$\mathrm{Cov}\left\{\Delta g_h(V+1),Y_I^*-Y_I\right\}\le\frac{1}{\lambda_V}\mathsf{Var}\big\{\Delta g_h(V+1)\big\}^{\frac12}\Big\{\sum_{i=1}^n\lambda_i^2\mathsf{Var}(Y_i^*-Y_i)\Big\}^{\frac12}.$$

For the last term of (8.23), we have by the zero order estimation

$$\big[\,\mathsf{E}\{\Delta g_h(V+1)\}-\mathcal{P}_{\lambda_V}\{\Delta g_h(x+1)\}\big]\,\mathsf{E}(Y_I^*-Y_I)\ \le\ \frac{6\|\Delta g_h\|_{\sup}}{\lambda_V}\times\Big\{\sum_{i=1}^n\mathsf{E}(Y_i|Y_i^s-1|)\Big\}^2.$$

It remains to observe that $\mathcal{P}_{\lambda_V}\{\Delta g_h(x+1)\}=\frac12\mathcal{P}_{\lambda_V}(\Delta^2 h)$ and let the corrector to be

$$C_h^{\mathcal{P}}=\frac{\lambda_V}{2}\mathcal{P}_{\lambda_V}(\Delta^2 h)\,\mathsf{E}(Y_I^*-Y_I).$$

Combining all these terms, we obtain

$$\begin{aligned}&\big|\,\mathsf{E}\{h(V)\}-\mathcal{P}_{\lambda_V}(h)-C_h^{\mathcal{P}}\big|\\&\le\|\Delta^2 g_h\|_{\sup}\sum_{i=1}^n\lambda_i\,\mathsf{E}\Big\{|Y_i^*-Y_i|\big(|Y_i^*-Y_i|-1\big)\Big\}\\&+\mathsf{Var}\big\{\Delta g_h(V+1)\big\}^{\frac12}\Big\{\sum_{i=1}^n\lambda_i^2\mathsf{Var}(Y_i^*-Y_i)\Big\}^{\frac12}\\&+6\|\Delta g_h\|_{\sup}\Big\{\sum_{i=1}^n\mathsf{E}(Y_i|Y_i^s-1|)\Big\}^2.\end{aligned}\tag{8.24}$$

□

The Poisson corrector $C_h^{\mathcal{P}}$ is of similar form with the Gaussian one and contains two terms as well: one term depends on the moments of Y_i and the other is a Poisson expectation.

Since Y_i's are $\mathbb{N}_+$-valued random variables, they can represent directly the default indicators $\mathbf{1}_{\{\tau_i \leq t\}}$. This fact limits however the recovery rate to be identical or proportional for all credits. We now consider the order of the corrector. Suppose that λ_V does not depend on n to ensure that $\mathcal{P}_{\lambda_V}(\Delta^2 h)$ is of constant order. Then in the homogeneous case, the conditional default probability $p \sim \mathcal{O}(1/n)$. For the percentage conditional losses, as in the Gaussian case, the corrector is of order $\mathcal{O}(1/n)$ with the coefficient $1/n$.

Since $\Delta^2 h(x) = \mathbf{1}_{\{x=k-1\}}$ for the call function, its Poisson approximation corrector is given by

$$C_h^{\mathcal{P}} = \frac{\sigma_V^2 - \lambda_V}{2(\lfloor k \rfloor - 1)!} e^{-\lambda_V} \lambda_V^{\lfloor k \rfloor - 1} \tag{8.25}$$

where $\lfloor k \rfloor$ is the integer part of k. The corrector vanishes when the expectation and the variance of the sum variable V are equal. The difficulty here is that the call function is not bounded. However, we can prove that Theorem 8.2 holds for any function of polynomial increasing speed El Karoui and Jiao (2007).

8.3 Numerical Tests

Before exploring real life applications, we would like in this section to perform some basic testing of the preceding formulae. In the sequel, we consider the call value $\mathsf{E}\{(l-k)_+\}$ where $l = n^{-1}\sum_{i=1}^n (1-R_i)\xi_i$ and the ξ_i's are independent Bernoulli random variables with success probability equal to p_i.

8.3.1 Validity Domain of the Approximations

We begin by testing the accuracy of the corrected Gauss and Poisson approximations for different values of $np = \sum_{i=1}^n p_i$ in the case $R_i = 0$, $n = 100$ and for different values of k such that $0 \leq k \leq 1$. The benchmark value is obtained through the recursive methodology well known by the practitioners which computes the loss distribution by reducing the portfolio size by one name at each recursive step.

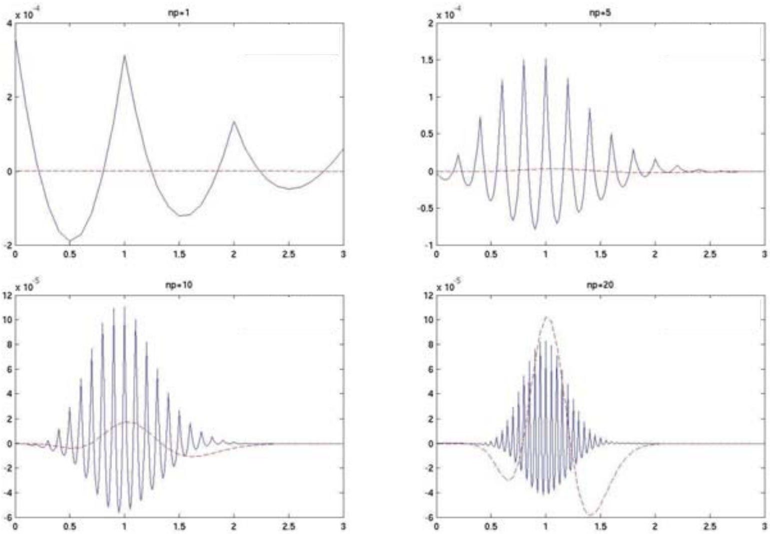

Figure 8.1. Gauss and Poisson approximation errors for various values of np as a function of the strike over the expected loss, with line curve for Gaussian errors and dotted curve for Poisson errors. `XFGgperror`

In Figure 8.1 are plotted the differences between the corrected Gauss approximation and the benchmark (Error Gauss) and the corrected Poisson approximation and the benchmark (Error Poisson) for different values of np as a function of the call strike over the expected loss. Note that when the tranche strike equals the expected loss, the normalized strike value in the Gaussian case equals zero due to the centered random variables, which means that the correction vanishes. We observe in Figure 8.1 that the Gaussian error is maximal around this point.

We observe on these graphs that the Poisson approximation outperforms the Gaussian one for approximately $np < 15$. On the contrary, for large values of np, the Gaussian approximation is the best one. Because of the correction, the threshold between the Gauss-Poisson approximation is higher than the classical one $np \approx 10$. In addition, the threshold may be chosen rather flexibly around 15. Combining the two approximations, the minimal error of the two approximations is relatively larger in the overlapping area when np is around 15. However, we obtain satisfactory results even in this case. In all the graphs presented, the error of the mixed approximation is inferior than

1 bp.

Our tests are made with inhomogeneous p_i's obtained as

$$p_i = p\exp(\sigma W_i - 0.5\sigma^2)$$

(log-normal random variable with expectation p and volatility σ) where W_i is a family of independent standard normal random variables and values of σ ranging from 0% to 100%. Qualitatively, the results were not affected by the heterogeneity of the p_i's.

Observe that there is oscillation in the Gaussian approximation error, while the Poisson error is relatively smooth. This phenomenon is related to the discretization impact of discrete laws.

As far as a unitary computation is concerned (one call price), the Gaussian and Poisson approximation perform much better than the recursive methodology: we estimate that these methodologies are 200 times faster. To be fair with the recursive methodology one has to recall that by using it we obtain not only a given call price but the whole loss distribution which correspond to about 100 call prices. In that case, our approximations still outperform the recursive methodology by a factor 2.

8.3.2 Stochastic Recovery Rate - Gaussian Case

We then consider the case of stochastic recovery rate and check the validity of the Gauss approximation in this case. Following the standard in the industry (Moody's assumption), we will model the R_i's as independent beta random variables with expectation 50% and standard deviation 26%.

An application of Theorem 8.1 is used so that the first order corrector term takes into account the first three moments of the random variables R_i. To describe the obtained result let us first introduce some notations. Let $\mu_{R_i}, \sigma^2_{R_i}$ and $\gamma^3_{R_i}$ be the first three centered moments of the random variable R_i, namely

$$\mu_{R_i} = \mathsf{E}(R_i), \quad \sigma^2_{R_i} = \mathsf{E}\{(R_i - \mu_{R_i})^2\}, \quad \gamma^3_{R_i} = \mathsf{E}\{(R_i - \mu_{R_i})^3\}.$$

We also define $X_i = n^{-1}(1 - R_i)\xi_i - \mu_i$ where $\mu_i = n^{-1}(1 - \mu_{R_i})p_i$ and $p_i = \mathsf{E}(\xi_i)$. Let W be $\sum_{i=1}^n X_i$. We have

$$\sigma^2_W = \mathsf{Var}(W) = \sum_{i=1}^n \sigma^2_{X_i} \quad \text{where} \quad \sigma^2_{X_i} = \frac{p_i}{n^2}\Big\{\sigma^2_{R_i} + (1 - p_i)(1 - \mu_{R_i})^2\Big\}.$$

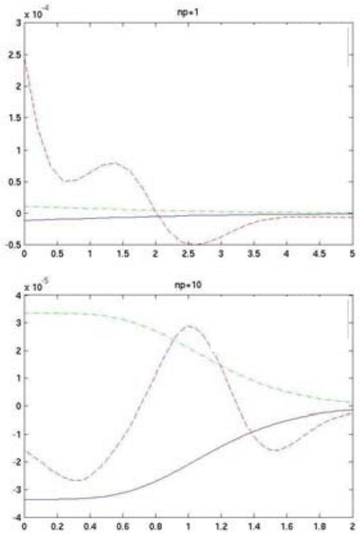

Figure 8.2. Gaussian approximation errors in the stochastic recovery case for various values of np as a function of the strike over the expected loss, compared with upper and lower 95% confidence interval bounds of Monte Carlo 1,000,000 simulations. XFGstoerror

Finally, if $\widetilde{k} = k - \sum_{i=1}^{n} \mu_i$, we have the following approximation

$$\mathsf{E}\{(l-k)_+\} \approx \Phi_{\sigma_W}(\cdot - k)_+ + \frac{1}{6}\frac{1}{\sigma_W^2}\sum_{i=1}^{n} \mathsf{E}\big(X_i^3\big)\widetilde{k}\phi_{\sigma_W}(\widetilde{k})$$

where

$$\mathsf{E}(X_i^3) = \frac{p_i}{n^3}\Big\{(1-\mu_{R_i})^3\big(1-p_i\big)(1-2p_i) + 3(1-p_i)(1-\mu_{R_i})\sigma_{R_i}^2 - \gamma_{R_i}^3\Big\}.$$

The benchmark is obtained using standard Monte Carlo integration with 1,000,000 simulations. We display, in Figure 8.2, the difference between the approximated call price and the benchmark as a function of the strike over the expected loss. We also consider the lower and upper 95% confidence interval for the Monte Carlo results. As in the standard case, one observes that the greater the value of np the better the approximation. Furthermore, the stochastic recovery brings a smoothing effect since the conditional loss no longer follows a binomial law.

The Poisson approximation, due to constraint of integer valued random variables, can not treat directly the stochastic recovery rates. We can however take the mean value of R_i's as the uniform recovery rate especially for low value of np without improving the results except for very low strike (equal to a few bp).

8.3.3 Sensitivity Analysis

We are finally interested in calculating the sensitivity with respect to p_j. As for the Greek of the classical option theory, direct approximations using the finite difference method implies large errors. We hence propose the following procedure.

Let $l_t^j = \frac{1-R}{n}\mathbf{1}_{\{\tau_j \le t\}}$. Then for all $j = 1, \cdots, n$,

$$(l_t - k)_+ = \mathbf{1}_{\{\tau_j \le t\}}\Big(\sum_{i \ne j} l_t^i + \frac{1-R}{n} - k\Big)_+ + \mathbf{1}_{\{\tau_j > t\}}\Big(\sum_{i \ne j} l_t^i - k\Big)_+.$$

As a consequence, we may write

$$\begin{aligned}\mathsf{E}\{(l_t - k)_+ | U\} = F(p_j, U)\,\mathsf{E}\Big\{\Big(\sum_{i \ne j} l_t^i + \frac{1-R}{n} - k\Big)_+ \Big| U\Big\} \\ + \big\{1 - F(p_j, U)\big\}\,\mathsf{E}\Big\{\Big(\sum_{i \ne j} l_t^i - k\Big)_+ \Big| U\Big\}.\end{aligned}$$

Since the only term which depends on p_j is the function $F(p_j, U)$, we obtain that $\partial_{p_j} C(t, k)$ can be calculated as

$$\int_0^1 \mathrm{d}u \partial_1 F(p_j, u)\,\mathsf{E}\Big[\Big\{\sum_{i \ne j} l_t^i + \omega_j(1 - R_j) - k\Big\}_+ - \Big(\sum_{i \ne j} l_t^i - k\Big)_+ \Big| U = u\Big] \quad (8.26)$$

where we compute the call spread using the mixed approximation for the partial total loss.

We test this approach in the case where $R_i = 0$ on a portfolio of 100 names such that one fifth of the names has a default probability of 25 bp, 50 bp, 75 bp, 100 bp and 200 bp respectively for an average default probability of 90 bp. We compute call prices derivatives with respect to each individual name probability according to the formula (8.26) and we benchmark this result by the sensitivities given by the recursive methodology.

In Figure 8.3, we plot these derivatives for a strike value of 3% computed using the recursive and the approximated methodology. Our finding is that

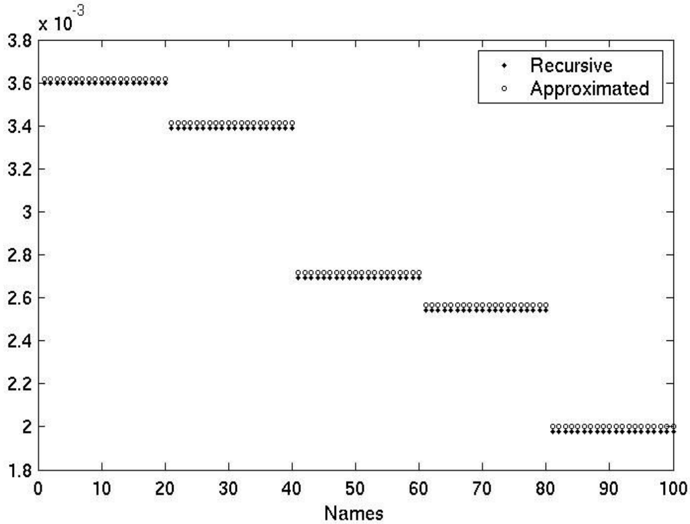

Figure 8.3. Sensitivity with respect to individual default probability by the approximated and the recursive methodology, for 5 types of 100 total names. XFGsensibility

in all tested cases (strike ranging from 3% to 20%), the relative errors on these derivatives are less than 1% except for strike higher than 15%, in which case the relative error is around 2%. Note however that in this case, the absolute error is less than 0.1 bp for derivatives whose values are ranging from 2 bp to 20 bp. We may remark that the approximated methodology always overvalues the derivatives value. However in the case of a true mezzanine tranche this effect will be offset. We consider these results as very satisfying.

8.4 Real Life Applications

After recalling the main mathematical results, we use them on two real life applications: valuation of single tranche CDOs and computing VaR figures in a timely manner.

8.4.1 Gaussian Approximation

Let μ_i and σ_i be respectively the expectation and standard deviation of the random variable $\chi_i = n^{-1}(1-R)\mathbf{1}_{\{\tau_i \le t\}}$. Let $X_i = \chi_i - \mu_i$ and $W = \sum_{i=1}^{n} X_i$, so that the expectation and standard deviation of the random variable W are 0 and $\sigma_W = \sqrt{\sum_{i=1}^n \sigma_i^2}$ respectively. Let also p_i be the default probability of issuer i. We want to calculate

$$C(t,k) = \mathsf{E}\{(l_t - k)_+\} = \mathsf{E}\{(W - \widetilde{k})_+\}$$

where $\widetilde{k} = k - \sum_{i=1}^n \mu_i$.

Assuming that the random variables X_i's are mutually independent, the result of Theorem 8.1 may be stated in the following way

$$C(t,k) \approx \int_{-\infty}^{+\infty} \mathrm{d}x\, \phi_{\sigma_W}(x)(x - \widetilde{k})_+ + \frac{1}{6}\frac{1}{\sigma_W^2}\sum_{i=1}^n \mathsf{E}(X_i^3)\widetilde{k}\phi_{\sigma_W}(\widetilde{k}) \qquad (8.27)$$

where $\mathsf{E}(X_i^3) = \frac{(1-R)^3}{n^3} p_i(1-p_i)(1-2p_i)$. The first term on the right-hand side of (8.27) is the Gaussian approximation that can be computed in closed form thanks to Bachelier formula whereas the second term is a correction term that accounts for the non-normality of the loss distribution.

In the sequel, we will compute the value of the call option on a loss distribution by making use of the approximation (8.27). In the conditionally independent case, one can indeed write

$$\mathsf{E}(l_t - k)_+ = \int \mathrm{P}_U(\mathrm{d}u)\, \mathsf{E}\{(l_t - k)_+ | U = u\}$$

where U is the latent variable describing the general state of the economy. As the default time are conditionally independent upon the variable U, the integrand may be computed in closed form using (8.27).

We note finally that in the real life test, we model U in a non-parametric manner such that the base correlation skew of the market can be reproduced.

8.4.2 Poisson Approximation

Recall that $\mathcal{P}_\lambda$ is the Poisson measure of intensity λ. Let $\lambda_i = p_i$ and $\lambda_V = \sum_{i=1}^n \lambda_i$ where now $V = \sum_{i=1}^n Y_i$ with $Y_i = \mathbf{1}_{\{\tau_i \le t\}}$. We want to calculate

$$C(t,k) = \mathsf{E}\{(l_t - k)_+\} = \mathsf{E}\{(n^{-1}(1-R)V - k)_+\}.$$

Recall that the operator Δ is such that $(\Delta f)(x) = f(x+1) - f(x)$. We also let the function h be defined by $h(x) = \{n^{-1}(1-R)x - k\}_+$.

Assuming that the random variables Y_i's are mutually independent, we may write according to the results of theorem 8.2 that

$$C(t,k) \approx \mathcal{P}_{\lambda_V}(h) - \frac{1}{2}\Big(\sum_{i=1}^{n} \lambda_i^2\Big)\mathcal{P}_{\lambda_V}(\Delta^2 h) \tag{8.28}$$

where

$$\mathcal{P}_{\lambda_V}(\Delta^2 h) = n^{-1}(1-R)\mathrm{e}^{-\lambda_V}\frac{\lambda_V^{\lfloor m\rfloor -1}}{(\lfloor m\rfloor - 1)!}$$

where $m = nk/(1-R)$. The formula (8.28) may be used to compute the unconditional call price in the same way as in the preceding subsection.

8.4.3 CDO Valuation

In this subsection, we finally use both Gaussian and Poisson first order approximations to compute homogeneous single tranche CDO value and break even as described in formula (8.1). As this formula involves conditioning on the latent variable U, we are either in the validity domain of the Poisson approximation or in the validity domain of the Gaussian approximation. Taking into account the empirical facts underlined in Section 8.3, we choose to apply the Gaussian approximation for the call value as soon as $\sum_i F(p_i, u) > 15$ and the Poisson approximation otherwise. All the subsequent results are benchmarked using the recursive methodology.

Our results for the quoted tranches are gathered in the following table. Level represents the premium leg for the spread of 1 bp and break even is the spread of CDO in (8.1).

In the following table are gathered the errors on the break even expressed in bp. We should note that in all cases, the error is less than 1.15 bp which is below the market uncertainty that prevails on the bespoke CDO business. We observe furthermore that the error is maximal for the tranche 3%-6% which correspond to our empirical finding (see Figure 8.1) that the approximation error is maximal around the expected loss of the portfolio (equal here to 4.3%).

Trying to understand better these results, we display now in the following two tables the same results but for equity tranches.

Attach	Detach	Output	REC	Approx.
0%	3%	Default Leg	2.1744%	2.1752%
		Level	323.2118%	323.2634%
		Break Even	22.4251%	22.4295%
3%	6%	Default Leg	0.6069%	0.6084%
		Level	443.7654%	443.7495%
		Break Even	4.5586%	4.5702%
6%	9%	Default Leg	0.1405%	0.1404%
		Level	459.3171%	459.3270%
		Break Even	1.0197%	1.0189%
9%	12%	Default Leg	0.0659%	0.0660%
		Level	462.1545%	462.1613%
		Break Even	0.4754%	0.4758%
12%	15%	Default Leg	0.0405%	0.0403%
		Level	463.3631%	463.3706%
		Break Even	0.2910%	0.2902%
15%	22%	Default Leg	0.0503%	0.0504%
		Level	464.1557%	464.1606%
		Break Even	0.1549%	0.1552%
0%	100%	Default Leg	3.1388%	3.1410%
		Level	456.3206%	456.3293%
		Break Even	1.1464%	1.1472%

Table 8.1. Break even values for the quoted tranches, by recursive method and our approximation method respectively

	Error
0-3	0.44
3-6	1.15
6-9	- 0.08
9-12	0.04
12-15	- 0.08
15-22	0.02
0-100	0.08

Table 8.2. Break even errors for the quoted tranches compared to the recursive method, expressed in bp

Attach	Detach	Output	REC	Mixte
0%	3%	DL	2.1744%	2.1752%
		Level	323.2118%	323.2634%
		BE	22.4251%	22.4295%
0%	6%	DL	2.7813%	2.7836%
		Level	383.4886%	383.5114%
		BE	12.0878%	12.0969%
0%	9%	DL	2.9218%	2.9240%
		Level	408.7648%	408.7853%
		BE	7.9422%	7.9476%
0%	12%	DL	2.9877%	2.9900%
		Level	422.1122%	422.1302%
		BE	5.8984%	5.9025%
0%	15%	DL	3.0282%	3.0303%
		Level	430.3624%	430.3788%
		BE	4.6909%	4.6940%
0%	22%	DL	3.0785%	3.0807%
		Level	441.1148%	441.1280%
		BE	3.1723%	3.1744%
0%	100%	DL	3.1388%	3.1410%
		Level	456.3206%	456.3293%
		BE	1.1464%	1.1472%

Table 8.3. Break even values for the equity tranches, by recursive method and our approximation method respectively

	Error
0-3	0.44
0-6	0.92
0-9	0.55
0-12	0.41
0-15	0.31
0-22	0.21
0-100	0.08

Table 8.4. Break even errors for the equity tranches compared to the recursive method, expressed in bp

8.4.4 Robustness of VaR Computation

In this section, we consider the VaR computation for a given CDOs book and show that the use of the Gaussian first order approximation as in subsection

8.4.1 can speed up substantially credit derivatives VaR computation without loss of numerical accuracy. We restrict our attention on the Gaussian approximation as we want to be able to consider non-homogeneous reference portfolio. We study the approximation effect on VaR computation by using a stylized portfolio which strikes and maturities are distributed such that the resulting book is reasonably liquid and diversified.

Our finding is that we may safely use this approximation without a significant loss of accuracy for our stylized portfolio and this could lead, according to our estimation, to a reduction of 90% of VaR computation time as compared with the recursive methodology. The production of the VaR in due time for financial institution will then still be possible even if its business on single tranche increases steadily.

To test the robustness of the proposed approximation in VaR computation, we decide to study the accuracy (as compared by a full recursive valuation) of differences of the form

$$\Delta^{\omega}(T,K) = \mathrm{E}^{\omega}\{(l_T - K)_+\} - \mathrm{E}^{\omega_0}\{(l_T - K)_+\}$$

for various (spreads and correlation) VaR scenarios ω randomly generated. Here ω_0 denotes the initial scenario.

Generating VaR Scenarios

We aim here at generating by a Monte Carlo procedure a family of scenarios for spreads and the base correlation that we will assume constant in this set of tests.

We choose the following dynamic for the daily variation of the spreads of the common reference portfolio

$$\frac{\Delta s_i}{s_i} = 50\%\big(\sqrt{30\%}\varepsilon + \sqrt{70\%}\varepsilon_i\big)\sqrt{\Delta t}$$

where $\varepsilon, \varepsilon_1, \ldots, \epsilon_M$ are independent standard Gaussian random variables and $\Delta t 1/252$. In other words, we assume a joint log-normal dynamic with volatility 50% and correlation 30%.

We then assume that the shocks on the base correlation are normally distributed with initial value 30% and annual volatility of 15%.

In the sequel and for our testing, we will use a sample of 1000 such scenarios of spreads and correlation daily moves.

Stylized Portfolio Description

We start from the stylized distribution of a single tranche CDO portfolio. The resulting position is chosen so that is is reasonably liquid and diversified in term of maturity, strike and credit risk.

Each strike (expressed in expected loss unit) and maturity will be assigned a positive and a negative weight according to the corresponding notional in position. Hence, we come up with two positive normalized (=unity total mass) measures μ_+ and μ_- that reflects the book repartition in terms of strike (expressed in expected loss) and maturity. We also let $\mu = \mu_+ - \mu_-$ and $\widetilde{\mu} = |\mu|/2 = (\mu_+ + \mu_-)/2$.

We give below an example to explain more precisely. Let us consider, for instance, a protection buyer CDO position with maturity T, with expected loss $\mathsf{E}(l_T)$, with notional N and strikes A and B expressed in percentage. We also define $a(T) = A/\,\mathsf{E}(l_T)$ and $b(T) = B/\,\mathsf{E}(l_T)$. Using the following approximate formulas for the payout of the default and premium legs

$$\begin{aligned}\text{Default Leg} &= N \times \big[\{l_T - a(T)\,\mathsf{E}(l_T)\}_+ - \{l_T - b(T)\,\mathsf{E}(l_T)\}_+\big],\\ \text{Premium Leg} &= N \times \text{Spread} \times \frac{T}{2} \times \\ &\quad \big[(B-A) - \{l_{T/2} - a(T/2)\,\mathsf{E}(l_{T/2})\}_+ + \{l_{T/2} - b(T/2)\,\mathsf{E}(l_{T/2})\}_+\big],\end{aligned}$$

we observe that this deal will contribute for a positive amount of N on the point $\{a(T), T\}$, a negative amount of $-N$ on the point $\{b(T), T\}$, a positive amount of $N \times \text{Spread} \times T/2$ on the point $\{a(T/2), T/2\}$ and a negative amount of $-N \times \text{Spread} \times T/2$ on the point $\{b(T/2), T/2\}$.

Error Computation

Let $\Delta^{\omega}_{GA}(T,K)$ and $\Delta^{\omega}_{REC}(T,K)$ be the value of the difference

$$\mathrm{E}^{\omega}\{(l_T - K)_+\} - \mathrm{E}^{\omega_0}\{(l_T - K)_+\}$$

as given respectively by the Gaussian approximation and a full recursive valuation.

We are interested in different types of errors that will allow us to assess the robustness of the proposed approximation for VaR computation purposes. The algebraic average error (see Figure 8.4) arising from the use of the approximation on the book level and expressed in spread term may be defined as

$$\text{Algebraic Average Error}(\omega) \int \frac{\mu(\mathrm{d}k, \mathrm{d}T)}{T} \big\{\Delta^{\omega}_{GA}(T, kE_T) - \Delta^{\omega}_{REC}(T, kE_T)\big\}$$

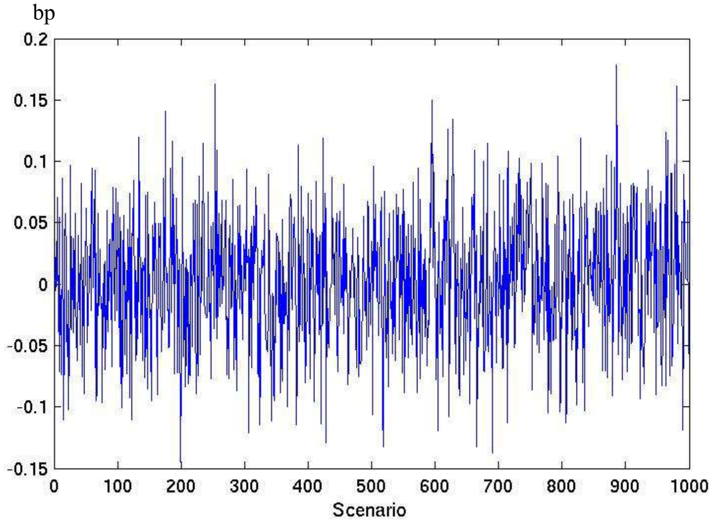

Figure 8.4. Algebraic Average Error of VaR per Scenario, expressed in bp. XFGalgerror

where $E_T = \mathsf{E}(l_T)$. The maximum algebraic average error on the book in spread term is defined as

$$\text{Max Algebraic Error} = \max_{\omega} |\text{Algebraic Average Error}(\omega)| \,.$$

Note that this way of computing the error allows the offset of individual errors due to the book structure. It is reasonable to take these effects into account when one tries to degrade numerical computation for VaR computation purposes. However, we will also compute the more stringent absolute average error (see Figure 8.5) on the book in spread term which is defined as

$$\text{Absolute Average Error}(\omega) \int \frac{\widetilde{\mu}(\mathrm{d}k, \mathrm{d}T)}{T} \big|\Delta^{\omega}_{GA}(T, kE_T) - \Delta^{\omega}_{REC}(T, kE_T)\big|.$$

The maximum absolute average error on the book in spread term is then defined as

$$\text{Max Absolute Error} = \max_{\omega} \text{Absolute Average Error}(\omega).$$

Our main results are

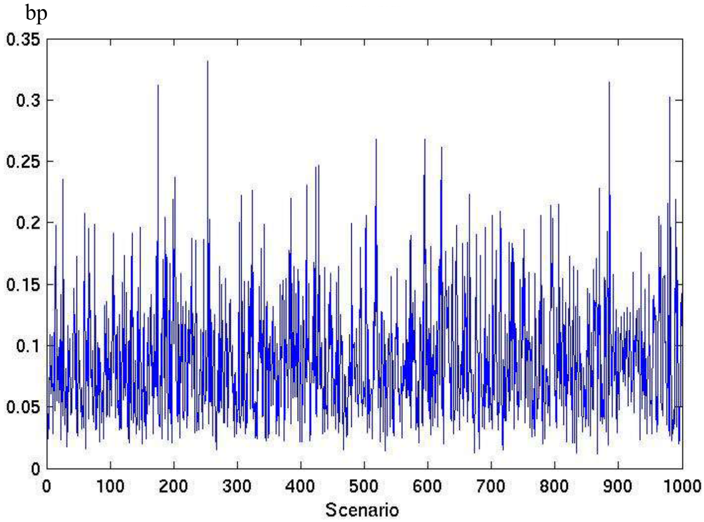

Figure 8.5. Absolute Average Error of VaR per Scenario, expressed in bp. XFGabsolute

$$\text{Max Algebraic Error} = 0.1785 \text{ bp},$$

$$\text{Max Absolute Error} = 0.3318 \text{ bp}.$$

As expected the maximum algebraic error is half the maximum absolute error as we allow the offsetting of the error due to the book structure.

These results are quite satisfying and justify the use of this approach for VaR computations in an industrial setting.

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